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# MAXIMAL QUASI-NORMED EXTENSION OF QUASI-NORMED LATTICES

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*To Professor A. B. Shabat*
*on occasion of his* $80^{th}$ *birthday*

The purpose of this article is to extend the Abramovich's construction of a maximal normed extension of a normed lattice to quasi-Banach setting. It is proved that the maximal quasi-normed extension $X^{\varkappa}$ of a Dedekind complete quasi-normed lattice $X$ with the weak $\sigma$-Fatou property is a quasi-Banach lattice if and only if $X$ is intervally complete. Moreover, $X^{\varkappa}$ has the Fatou and the Levi property provided that $X$ is a Dedekind complete quasi-normed space with the Fatou property. The possibility of applying this construction to the definition of a space of weakly integrable functions with respect to a measure taking values from a quasi-Banach lattice is also discussed, since the duality based definition does not work in the quasi-Banach setting.

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## 1. Introduction

For over recent 25 years the spaces of integrable functions with respect to a measure taking values in a Banach (quasi-Banach) lattice have been a field of increased interest. The spaces of integrable and weakly integrable functions with respect to a vector measure possess interesting order and metric properties and have been studied intensively by many authors. They find applications in important problems such as the representation of abstract quasi-Banach lattices as spaces of integrable functions, the study of the optimal domain of linear operators, domination and factorization of operators, spectral integration etc., see [4, 5, 7, 18, 20] and the references therein.

A key role in the theory is played by the space $L^1_w(\mu)$ of *weakly integrable* functions with respect to a measure $\mu$ with values in a Banach space, see the survey paper by Curbera and Ricker [5] and the book by Okada, Ricker and Sánches Pérez [18]. However, in the context of quasi-Banach spaces, when the conjugate space may turn out to be trivial, the duality based definition of $L^1_w(\mu)$ does not work, so we need to find a suitable substitute for $L^1_w(\mu)$.

In the case of a measure taking values from a quasi-Banach lattice two natural candidates for the space of weakly integrable function were indicated in [10]. The first one arises as the domain of the smallest extension of the integration operator (see Aliprantis and Burkinshaw [3, Theorem 1.30]), and the second one is based on the construction of the maximal normed extension introduced by Abramovich in [1]. The approach based on the smallest extension of the integration operator is presented in [11].

In order to realize the second possibility, it is necessary to extend Abramovich's construction to quasi-Banach setting, as done in this article. In Section 2 we sketch the needed information concerning quasi-Banach lattices and prove some Riesz–Fischer type completeness theorems for quasi-normed lattices; next, we gave a characterization of order continuous quasi-Banach lattices. In Section 3 we examine the construction of the maximal quasi-normed extension introduced by Abramovich [1] for Banach lattices. It is proved that the maximal quasi-normed extension $X^{\varkappa}$ of a Dedekind complete quasi-normed lattice $X$ with the weak $\sigma$-Fatou property is a quasi-Banach lattice if and only if $X$ is intervally complete. Moreover, $X^{\varkappa}$ has the Fatou and the Levi property provided that $X$ is a Dedekind complete quasi-normed space with the Fatou property.

We use the standard notation and terminology of Aliprantis and Burkinshaw [3] and Meyer-Nieberg [17] for the theory of vector and Banach lattices (see also Abramovich and Aliprantis [2], Luxemburg and Zaanen [13]). Throughout the text we assume that all vector spaces are defined over the field of reals and all vector lattices are Archimedean. We let := denote the assignment by definition, while $\mathbb{N}$ and $\mathbb{R}$ symbolize the naturals and the reals.

## 2. Quasi-Banach Lattices

In this section, we briefly sketch the needed information concerning quasi-Banach lattices. In particular, we give some simple results on the completeness and order continuity of quasi-Banach lattices for which we have not found references.

Definition 2.1. A *quasi-normed space* is a pair $(X, \|\cdot\|)$ where $X$ is a real vector space and $\|\cdot\|$ is a *quasi-norm*, a function from $X$ to $\mathbb{R}$ such that the following conditions hold:

(1) $\|x\| \geqslant 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$.

(3) There exists a constant $C \geqslant 1$ with $\|x + y\| \leqslant C(\|x\| + \|y\|)$ for all $x, y \in X$.

If, in addition, for some $0 < p \leqslant 1$ the inequality

(4) $\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$ holds for all $x, y \in X$,

then $\|\cdot\|$ is called a *p-norm* and $(X, \|\cdot\|)$ is called a *p-normed space*.

The best constant $C$ in 2.1 (3) is called the *quasi-triangle constant*, or *quasi-norm multiplier*, or *modulus of concavity* of the quasi norm. Note that $\|\sum_{k=1}^{n} x_k\| \leqslant \sum_{k=1}^{n} C^k \|x_k\|$ for all $x_1, \dots, x_n \in X$.

Two quasi-norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if there is a constant $A \geqslant 1$ such that $A^{-1}\|x\| \leqslant \|x\|' \leqslant A\|x\|$ for all $x \in X$. By the Aoki–Rolewicz theorem (see [8]), each quasi-norm is equivalent to some $p$-norm for some $0 < p \leqslant 1$.

**Theorem 2.2** (Aoki–Rolewicz). *Let $(X, \|\cdot\|)$ be a quasi-normed space with the quasi-triangle constant $C \geqslant 1$ and $p = (1 + \log_2 C)^{-1}$. Define $\|\cdot\|_p : X \to \mathbb{R}$ as*

$$\|x\|_p := \inf\left\{ \left( \sum_{k=1}^{n} \|x_k\|^p \right)^{\frac{1}{p}} : \ x = \sum_{k=1}^{n} x_k, \ n \in \mathbb{N} \right\} \quad (x \in X).$$

*Then $0 < p \leqslant 1$, $\|\cdot\|_p$ is a p-norm, and $\|x\|_p \leqslant \|x\| \leqslant 2^{1/p}\|x\|_p$ for all $x \in X$.*

⊲ See Maligranda [15, Theorem 1.2], Pietsch [19, 6.2.5]. ⊳

Thus, we may assume unless otherwise is mentioned that a quasi-Banach space is equipped with a $p$-norm for some $0 < p \leqslant 1$.

A topological vector space $X$ is said to be *locally bounded* if it has a bounded neighborhood of zero. A quasi-normed space is a locally bounded topological vector space if we take the sets $\{x \in X : \|x\| \leqslant \varepsilon\}$ $(0 < \varepsilon \in \mathbb{R})$ for a base of neighborhoods of zero. Moreover, this topology may be induced by metric $d(x, y) := |||x - y|||^p$ $(x, y \in X)$ where $|||\cdot|||$ is an equivalent $p$-norm. Conversely, Hyers [6] proved that the topology of a locally bounded topological vector space $X$ can be deduced from a quasi-norm, which may be obtained as the Minkowski functional of a bounded balanced neighborhood $B$ of zero:

$$\|x\| := \|x\|_B := \inf\left\{0 < \lambda \in \mathbb{R} :\ x \in \lambda B\right\} \quad (x \in X).$$

A quasi-norm may be discontinuous in its own topology [19, 6.1.9]. However, every quasi-norm is equivalent to a continuous one, since a $p$-norm is continuous.

Definition 2.3. A *quasi-Banach space* (*$p$-normed space*) is a quasi-normed space which is complete in its metric uniformity.

**Theorem 2.4.** *A quasi-normed space $X := (X, \|\cdot\|)$ with a triangle constant $C \geqslant 1$ is complete (and hence a quasi-Banach space) if and only if for every series $(x_k)$ in $X$ such that $\sum_{k=1}^{\infty} C^k \|x_k\| < \infty$ there exists $\sum_{k=1}^{\infty} x_k \in X$ and*

$$\left\| \sum_{k=1}^{\infty} x_k \right\| \leqslant \sum_{k=1}^{\infty} C^{k+1} \|x_k\|.$$

◁ See Maligranda [15, Theorem 1.1]. ▷

The basic results of the Banach space theory such as open mapping theorem and the closed graph theorem (for linear operators) are valid also in the context of quasi-Banach spaces, see [9].

Definition 2.5. A quasi-Banach (quasi-normed, $p$-Banach) space $(X, \|\cdot\|)$ is called a *quasi-Banach lattice* (respectively, *quasi-normed lattice*, *$p$-Banach lattice*) if, in addition, $X$ is a vector lattice and $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for all $x, y \in X$.

**Lemma 2.6.** *In any quasi-normed lattice $X$ lattice operations are continuous and the positive cone is closed. Moreover, if an increasing (decreasing) net $(x_\alpha)_{\alpha \in A}$ is quasi-norm convergent to $x \in X$, then $x = \sup_{\alpha \in A} x_\alpha$ $(x = \inf_{\alpha \in A} x_\alpha)$.*

◁ This can be ensured just as in the case of Banach lattice using monotonicity of the quasi-norm and quasi-triangle inequality. ▷

It follows from Lemma 2.6 that the completion of a quasi-normed lattice $X$ is a quasi-Banach lattice including $X$ as a vector sublattice. Along similar lines, it can also be proved that Amemiya's result on completeness of normed lattices is true in the context of quasi-normed spaces: *a quasi-normed lattice $X$ is complete if and only if every increasing Cauchy sequence in $X$ is convergent.* This fact in combination with Theorem 2.4 leads to the following result.

**Theorem 2.7.** *For a quasi-normed space $X := (X, \|\cdot\|)$ with a triangle constant $C \geqslant 1$ the following assertions are equivalent:*

*(1) $X$ is a quasi-Banach lattice.*

*(2) For every series $(x_k)$ in $X_+$ such that $\sum_{k=1}^{\infty} C^k \|x_k\| < \infty$ there exists $x \in X$ with $x = \sum_{k=1}^{\infty} x_k$.*

*(3) For every series $(x_k)$ in $X_+$ such that $\sum_{k=1}^{\infty} C^k \|x_k\| < \infty$ there exists $x \in X$ with $x = o\text{-}\sum_{k=1}^{\infty} x_k := \sup_{n \in \mathbb{N}} \sum_{k=1}^{n} x_k$.*

◁ See [12, Theorem 2.7]. ▷

Definition 2.8. A quasi-Banach lattice $(X, \|\cdot\|)$ (as well as the quasi-norm $\|\cdot\|$) is said to be *order continuous*, if $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$ for any net $(x_\alpha)_{\alpha\in A}$ in $X$. If arbitrary nets are replaced by sequences, one speak of *order $\sigma$-continuity*.

**Theorem 2.9.** *For a quasi-Banach lattice $X$ the following are equivalent:*

(1) *$X$ is order continuous.*

(2) *Every increasing order bounded sequence in $X_+$ is convergent.*

(3) *$X$ is Dedekind $\sigma$-complete and order $\sigma$-continuous.*

$\lhd$ See [12, Theorem 2.10]. $\rhd$

Definition 2.10. A quasi-Banach lattice $(X, \|\cdot\|)$ is said to have the *weak Fatou property* (respectively *weak $\sigma$-Fatou property*) if there exists $K > 0$ (called the *weak Fatou constant*) such that for every increasing net $(x_\alpha)$ (respectively sequence $(x_n)$) with the supremum $x \in X$ we have $\|x\| \leqslant K \sup_\alpha \|x_\alpha\|$ (respectively $\|x\| \leqslant K \sup_n \|x_n\|$). If $K = 1$ then $\|x\| = \sup_\alpha \|x_\alpha\|$ and in this situation $X$ is said to have the *Fatou property* (respectively *$\sigma$-Fatou property*).

Definition 2.11. Say that a quasi-normed lattice $(X, \|\cdot\|)$ has the *Levi property* (respectively *$\sigma$-Levi property*) if $\sup_\alpha x_\alpha$ (respectively $\sup_n x_n$) exists for every increasing net $(x_\alpha)$ (respectively sequence $(x_n)$) in $X_+$ provided that $\sup_\alpha \|x_\alpha\| < \infty$ (respectively $\sup_n \|x_n\| < \infty$). A *quasi-KB-space* is an order continuous quasi-normed lattice with the Levi property.

**Proposition 2.12.** *Suppose that $X$ is a quasi-normed lattice with the Levi property. Then $X$ is a Dedekind complete quasi-Banach lattice with the weak Fatou property.*

$\lhd$ The fact that a quasi-normed lattice with the Levi property has also the weak Fatou property is the only thing that needs verification. The proof is similar to that of Proposition 2.4.19 in Meyer-Nieberg [17].

Assume that $X$ has the Levi property but lacks the weak Fatou property. Then for every $n \in \mathbb{N}$ there exists an increasing net $(y_{n,\alpha})_{\alpha\in \mathrm{A}(n)}$ in $X_+$ such that $y_n = \sup_{\alpha\in\mathrm{A}(n)} y_{n,\alpha}$ exists and

$$\|y_n\| \geqslant n\tau, \quad \tau = C^n n^2 \sup_{\alpha\in\mathrm{A}(n)} \|y_{n,\alpha}\| \quad (n \in \mathbb{N}),$$

where $C \geqslant 1$ is the triangle constant of $X$. Putting $\bar{y}_n := y_n/\tau$, $\bar{y}_{n,\alpha} := y_{n,\alpha}/\tau$ we arrive at the following relations:

$$\bar{y}_n = \sup_{\alpha\in\mathrm{A}(n)} \bar{y}_{n,\alpha}, \quad \|\bar{y}_n\| \geqslant n, \quad \|\bar{y}_{n,\alpha}\| \leqslant C^{-n} n^{-2} \quad (n \in \mathbb{N}).$$

Let $(x_\gamma)$ stands for the net of finite suprema of elements in $\{\bar{y}_{n,\alpha} : \ n \in \mathbb{N},\ \alpha \in \mathrm{A}(n)\}$. If $x_\gamma = \bar{y}_{n_1,\alpha_1} \vee \dots \vee \bar{y}_{n_k,\alpha_k}$ with $\alpha_j \in \mathrm{A}(n_j)$, then

$$\|x_\gamma\| \leqslant \|\bar{y}_{n_1,\alpha_1} + \dots + \bar{y}_{n_k,\alpha_k}\| \leqslant \sum_{j=1}^{k} C^j \|\bar{y}_{n_j,\alpha_j}\| \leqslant \sum_{j=1}^{k} C^j C^{-n_j} n_j^{-2} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

By hypothesis, $x = \sup_\gamma x_\gamma$ exists and satisfies $x \geqslant \bar{y}_n$ for all $n \in \mathbb{N}$. Consequently, $\|x\| \geqslant n$ for all $n \in \mathbb{N}$, a contradiction. $\rhd$

## 3. Maximal Quasi-Normed Extension

Consider a quasi-normed lattice $(X, \|\cdot\|)$ with the quasi-triangle constant $C$. Let $X^\delta$ stand for the Dedekind completion of $X$, so that $X$ is identified with a majorizing order dense sublattice of $X^\delta$, while $X^\delta$ itself is a Dedekind complete vector lattice. Define a function $\|\cdot\|_\delta : X^\delta \to \mathbb{R}$ as

$$\|\bar{x}\|_\delta := \inf\big\{\|x\| :\ x \in X_+,\ |\bar{x}| \leqslant x\big\} \quad (\bar{x} \in X^\delta).$$

Clearly, $\|x\| = \|x\|_\delta$ for all $x \in X$ and $\|\bar{x}\|_\delta < \infty$ for each $\bar{x} \in X^\delta$, since $X$ is majorizing sublattice. Positive homogeneity and monotonicity of $\|\cdot\|_\delta$ are obvious. Moreover, if $|\bar{x}| \leqslant x$ and $|\bar{y}| \leqslant y$ for some $x, y \in X$ and $\bar{x}, \bar{y} \in X^\delta$, then $|\bar{x} + \bar{y}| \leqslant x + y$ and $\|\bar{x} + \bar{y}\|_\delta \leqslant \|x + y\| \leqslant C(\|x\| + \|y\|)$ and hence $\|\bar{x}+\bar{y}\|_\delta \leqslant C(\|\bar{x}\|_\delta + \|\bar{y}\|_\delta)$. It follows that $(X_\delta, \|\cdot\|_\delta)$ is a quasi-normed lattice with the same quasi-triangle constant.

**Lemma 3.1.** *If $(X, \|\cdot\|)$ is a quasi-Banach lattice with a triangle constant $C$ or a $p$-Banach lattice, then so is $(X^\delta, \|\cdot\|_\delta)$.*

◁ Assume that $\sum_{k=1}^{\infty} C^k \|\bar{x}_k\|_\delta < \infty$ for a sequence $(\bar{x}_k)$ in $X^\delta_+$. Pick $x_k \in X_+$ such that $\bar{x}_k \leqslant x_k$ and $\|x_k\| \leqslant \|\bar{x}_k\|_\delta + 1/(2C)^k$. Then

$$\sum_{k=1}^{n} C^k \|x_k\| \leqslant \sum_{k=1}^{n} C^k \|\bar{x}_k\|_\delta + \sum_{k=1}^{n} \frac{1}{2^k}$$

and hence $\sum_{k=1}^{\infty} C^k \|x_k\| < \infty$. By Theorem 2.6 $x := o\text{-}\sum_{k=1}^{\infty} x_k$ exists in $X$. Consequently, $o\text{-}\sum_{k=1}^{\infty} \bar{x}_k$ exists in $X^\delta$, since $\sum_{k=1}^{n} \bar{x}_k \leqslant x$ for all $n \in \mathbb{N}$. ▷

Assume now that $(X, \|\cdot\|)$ is a Dedekind complete quasi-normed lattice with a quasi-triangle constant $C$. Identify $X$ with an order dense ideal in its universal completion $X^u$. Define a function $\|\cdot\|_\varkappa : X^u \to \mathbb{R} \cup \{+\infty\}$ by putting

$$\|\hat{x}\|_\varkappa := \sup\big\{\|x\| : x \in X,\ 0 \leqslant x \leqslant |\hat{x}|\big\} \quad (\hat{x} \in X^u).$$

Observe that $\|x\| = \|x\|_\varkappa$ for all $x \in X$. Denote $X^\varkappa := \{\hat{x} \in X^u :\ \|\hat{x}\|_\varkappa < \infty\}$. If $0 \leqslant u \leqslant |\hat{x} + \hat{y}| \leqslant |\hat{x}| + |\hat{y}|$ for some $\hat{x}, \hat{y} \in X^\varkappa$ and $u \in X$, then there exist $x, y \in X$ with $0 \leqslant x \leqslant |\hat{x}|$, $0 \leqslant y \leqslant |\hat{y}|$, and $u = x + y$. It follows that $\|u\| \leqslant C(\|x\| + \|y\|) \leqslant C(\|\hat{x}\|_\varkappa + \|\hat{y}\|_\varkappa)$ and thus $\|\hat{x} + \hat{y}\|_\varkappa \leqslant C(\|\hat{x}\|_\varkappa + \|\hat{y}\|_\varkappa)$. Similarly, $\|\cdot\|_\varkappa$ is a $p$-norm, whenever $\|\cdot\|$ is. Taking into account obvious monotonicity and positive homogeneity of $\|\cdot\|_\varkappa$, we see that $(X^\varkappa, \|\cdot\|_\varkappa)$ is a quasi-normed lattice with the quasi-triangle constant $C$ and, if $\|\cdot\|$ is a $p$-norm, so is $\|\cdot\|_\varkappa$.

Definition 3.2. A *maximal quasi-normed extension* of a quasi-normed lattice $(X, \|\cdot\|)$ is the pair $(X^{\delta\varkappa}, \|\cdot\|_{\delta\varkappa})$ with $X^{\delta\varkappa} := (X^\delta)^\varkappa$ and

$$\|\hat{x}\|_{\delta\varkappa} := \sup\Big\{\inf\{\|x\| :\ x \in X,\ |\bar{x}| \leqslant x\} :\ \bar{x} \in X^\delta,\ 0 \leqslant \bar{x} \leqslant |\hat{x}|\Big\} \quad (\hat{x} \in X^{\delta\varkappa}).$$

Observe that if $X$ is Dedekind complete then $X^{\delta\varkappa} = X^\varkappa$ and $\|\cdot\|_{\delta\varkappa} = \|\cdot\|_\varkappa$.

**Lemma 3.3.** *If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are quasi-normed lattices, $Y$ is an order dense ideal in $X^u$ containing $X$, and $\|x\|_X = \|x\|_Y$ for all $x \in X$, then $Y \subset X^\varkappa$.*

◁ This is an immediate consequence of the definition. ▷

Definition 3.4. A quasi-normed lattice $X$ is called *intervally complete* if every order interval of $X$ is complete or, in other words, every order bounded Cauhcy sequence of $X$ is convergent to an element of $X$.

It can be easily seen that each intervally complete quasi-normed lattice is an order dense ideal of its own metric completion and every order ideal of any quasi-Banach lattice is an intervally complete quasi-normed lattice. Thus, the class of intervally complete quasi-normed lattices coincides with the class of order dense ideals of quasi-Banach lattices.

**Lemma 3.5.** *Intervally complete quasi-normed lattice is uniformly complete.*

$\lhd$ Let $(x_n)$ be a uniformly Cauchy sequence, that is, there exist $e \in X_+$ and a sequence of reals $(\varepsilon_n)$ such that $\lim_n \varepsilon_n = 0$ and $|x_{n+k} - x_n| \leqslant \varepsilon_n e$ for all $n, k \in \mathbb{N}$. Then $\|x_{n+k} - x_n\| \leqslant \varepsilon_n \|e\|$ and $(x_n)$ is Cauchy in $(X, \|\cdot\|)$. Moreover, $|x_{k+1}| \leqslant |x_1| + \varepsilon_1 e$ for all $k \in \mathbb{N}$. By hypothesis, there exists $x = \lim_n x_n$ in $(X, \|\cdot\|)$. Passage to the limit in $|x_{n+k} - x_n| \leqslant \varepsilon_n e$ with $k \to \infty$ yields $|x - x_n| \leqslant \varepsilon_n e$ for all $n \in \mathbb{N}$, whence $X$ is uniformly complete. $\rhd$

**Lemma 3.6.** *Let $\tilde{X}$ be the metric completion of an intervally complete quasi-normed lattice $X$. Then $\tilde{X}$ is Dedekind complete if and only if so is $X$.*

$\lhd$ If $\tilde{X}$ is Dedekind complete then so is $X$, since $X$ is an order dense ideal of $\tilde{X}$. Assume that a quasi-normed lattice $X$ is intervally complete and Dedekind complete and prove $\tilde{X}$ is Dedekind complete. It was proved by Veksler [21, 22] that an Archimedean vector lattice is Dedekind complete if and only if it is uniformly complete and has the projection property. By Lemma 3.5 it suffices to show that $\tilde{X}$ has the projection property. Consider an element $x \in \tilde{X}$ and a band $\tilde{B}$ in $\tilde{X}$ and pick a sequence $(x_n)$ in $X$ converging to $x$. Observe, that $B := \tilde{B} \cap X$ is a band of $X$ and $B^\perp = \tilde{B}^\perp \cap X$, since $X$ is an order dense ideal in $\tilde{X}$. If $\pi$ stands for the band projection in $X$ onto $B$, then $\pi' := I_X - \pi$ is the band projection onto $B^\perp$. The sequences $(\pi x_n)$ and $(\pi' x_n)$ are Cauchy, as so is $(x_n)$, hence they converge to some $u \in \tilde{X}$ and $u' \in \tilde{X}$, respectively. Clearly, $u \in \tilde{B}$, $u' \in \tilde{B}^\perp$, and $x = u + u'$. $\rhd$

**Lemma 3.7.** *A quasi-normed lattice $X$ is intervally complete if and only if every increasing order bounded Cauchy sequence in $X_+$ is quasi-norm convergent.*

$\lhd$ The proof given in [23, Theorem 1.1] for normed lattices works in the quasi-normed setting. $\rhd$

**Lemma 3.8.** *Let $X$ be a universally complete vector lattice and $(x_\alpha)_{\alpha \in \mathrm{A}}$ an increasing net in $X_+$. Then there exists a band projection $\pi$ on $X$ such that $\sup_\alpha \pi x_\alpha$ exists in $X$, while for the complementary band projection $\pi' := I_X - \pi$ we have $N\pi' e = \sup_\alpha \pi'(x_\alpha \wedge Ne)$ for all $N \in \mathbb{N}$ and $e \in X_+$.*

$\lhd$ There is no loss of generality in assuming that $X = C_\infty(Q)$ with extremally compact space $Q$. (Recall that the symbol $C_\infty(Q)$ denotes the universally complete vector lattice of all continuous functions $f : Q \to [-\infty, \infty]$ for which the open set $\{q \in Q : -\infty < f(q) < \infty\}$ is dense in $Q$.) Let $(x_\alpha)$ be an increasing net in $C_\infty(Q)$ and define two functions $\bar{x}, x : Q \to [0, \infty]$ by

$$\bar{x}(q) = \sup\{x_\alpha(q) : \alpha \in \mathrm{A}\} \quad (q \in Q),$$
$$x(q) := \inf_{U \in \mathscr{N}(q)} \sup_{q' \in U} \bar{x}(q') \quad (q \in Q),$$

where $\mathscr{N}(q)$ is a basis of neighborhoods of $q$. Then $\bar{x}$ is lower semicontinuous and $x$ is continuous, see [24, Lemma V.1.2 and Theorem V.1.1]. Consider an open set $Q_0 := \{q \in Q : x(q) < \infty\}$ and observe that its closure $\bar{Q}_0$ is clopen. Now, let $\pi$ stands for the band projection of $C_\infty(Q)$ corresponding to $\bar{Q}_0$ and $\pi x$ stands for the function coinciding with $x$ on $\bar{Q}_0$ and vanishing on $Q_1 := Q \setminus \bar{Q}_0$. Evidently, $\pi x \in C_\infty(Q)$ and $\pi x = \sup_\alpha \pi x_\alpha$, see [24, Theorem V.2.1]. At the same time $x(q) = \infty$ for all $q \in Q_1$, so that $\bar{x}(q) = \infty$ for all $q \in Q_1 \setminus A$ where $A$ is a meager subset of $Q_1$. The latter implies that $Ne(q) = \sup_\alpha x_\alpha(q) \wedge Ne(q)$ for all $q \in Q_1 \setminus A$, whence the desired equation $N\pi' e = \sup_\alpha \pi'(x_\alpha \wedge Ne)$ follows. $\rhd$

**Lemma 3.9.** *Let $X$ be a quasi-normed lattice $X$ with the weak $\sigma$-Fatou property. If $X$ is intervally complete and Dedekind complete, then its maximal quasi-normed extension $X^\varkappa$ is intervally complete.*

$\lhd$ Take an increasing order bounded Cauchy sequence $(\hat{x}_n)$ in $X^{\varkappa}_{+}$. Since $X^{\varkappa}$ is Dedekind complete, there exists $\hat{x} = \sup_n \hat{x}_n$. Prove that $(\hat{x}_n)$ converges to $\hat{x}$.

We may assume without loss of generality that $A := \sum_{n=1}^{\infty} C^n n \|\hat{x}_{n+1} - \hat{x}_n\|_{\varkappa} < \infty$. Applying Lemma 3.8 to the increasing sequence $(\hat{z}_n)$ with $\hat{z}_n := \sum_{k=1}^{n} k(\hat{x}_{k+1} - \hat{x}_k)$ yields a band projection $\pi$ on $X^u$ such that $\hat{z} := \sup_n \pi\hat{z}_n$ exists in $X^u$ and for $\pi' := I_{X^u} - \pi$ we have $N\pi'e = \sup_n \pi'(\hat{z}_n \wedge Ne)$ for all $N \in \mathbb{N}$ and $e \in X$, $0 \leqslant e \leqslant \hat{z}$. Making use of the weak $\sigma$-Fatou property and monotonicity of the quasi-norm we deduce

$$N\|\pi'e\| \leqslant K \sup_m \|\pi'(\hat{z}_m \wedge Ne)\| \leqslant K \sup_m \|\hat{z}_m\|_{\varkappa} \leqslant KA < \infty.$$

It follows that $\pi'e = 0$ for all $e \in X$ and hence $\pi'\hat{z} = 0$, since $X$ is order dense ideal in $X^u$. Thus, $\pi = I_{X^u}$ and $\hat{z} = \sup_n \hat{z}_n \in X^u$. To ensure that $\hat{z} \in X^{\varkappa}$ it suffices to check that $\|x\| \leqslant A$ for an arbitrary element $x \in X$ with $0 \leqslant x \leqslant \hat{z}$. For any such $x$ put $y_n := \hat{z}_n \wedge x$ and observe that $(y_n)$ is an increasing sequence in $X_+$ with $x = \sup_n y_n$. Moreover, $(y_n)$ is Cauchy, since for arbitrary $n, l \in \mathbb{N}$ we can estimate:

$$\|y_{n+l} - y_n\| = \|\hat{z}_{n+l} \wedge x - \hat{z}_n \wedge x\|_{\varkappa} \leqslant \|\hat{z}_{n+l} - \hat{z}_n\|_{\varkappa}$$
$$= \sum_{k=n+1}^{n+l} C^k k \|\hat{x}_{k+1} - \hat{x}_k\|_{\varkappa} \leqslant \sum_{k=n+1}^{\infty} C^k k \|\hat{x}_{k+1} - \hat{x}_k\|_{\varkappa} \to 0$$

as $n \to \infty$. The interval comleteness of $X$ implies that the sequence $(y_n)$ is convergent in $X$, so that $\lim_n y_n = \sup_n y_n = x$ by Lemma 2.6. Observe now that $\|x\| \leqslant A$, since $\|y_n\| \leqslant \|\hat{z}_n\|_{\varkappa} \leqslant A$ and $\|x\| = \lim_n \|y_n\| \leqslant A$, whence $\hat{z} \in X^{\varkappa}$.

Now we are able to show that $(\hat{x}_n)$ converges to $\hat{x}$. First note that $\hat{x} - \hat{x}_n = o\text{-}\sum_{k=n}^{\infty}(\hat{x}_{k+1} - \hat{x}_k)$, and consequently

$$n(\hat{x} - \hat{x}_n) \leqslant o\text{-}\sum_{k=n}^{\infty} k(\hat{x}_{k+1} - \hat{x}_k) \leqslant \hat{z}.$$

It follows that $0 \leqslant \hat{x} - \hat{x}_n \leqslant (1/n)\hat{z}$ and $\|\hat{x} - \hat{x}_n\|_{\varkappa} \leqslant (1/n)\|\hat{z}\|_{\varkappa} \to 0$. Appealing to Lemma 3.7 completes the proof. $\rhd$

**Theorem 3.10.** *Let $(X, \|\cdot\|_X)$ be a Dedekind complete quasi-normed lattice with the weak $\sigma$-Fatou property. The maximal quasi-normed extension $(X^{\varkappa}, \|\cdot\|_{\varkappa})$ is a quasi-Banach lattice if and only if $X$ is intervally complete.*

$\lhd$ The necessity is immediate from the fact that $X$ is an order dense ideal of $X^{\varkappa}$. To prove the sufficiency observe that the metric completion $(Y, \|\cdot\|_Y)$ of $(X^{\varkappa}, \|\cdot\|_{\varkappa})$ is Dedekind complete by Lemma 3.6. At the same time $X^{\varkappa}$ is order dense ideal of $Y$, since $X^{\varkappa}$ is intervally complete by Lemma 3.9 and an intervally complete quasi-normed lattice is an order dense ideal of its metric completion. Thus, $X \subset Y \subset (X^{\varkappa})^u = X^u$ and $\|x\| = \|x\|_Y$ for all $x \in X$ so that $Y \subset X^{\varkappa}$ by Lemma 3.3. It follows that $Y = X^{\varkappa}$ and $X^{\varkappa}$ is complete. $\rhd$

It is evident that if $X$ has the Levi property then $X = X^{\varkappa}$ but the converse is false, see [1, Examples 2 and 5]. The next result asserts that the maximal quasi-normed extension with the weak Fatou property has the Levi property.

**Theorem 3.11.** *Let $X$ be a Dedekind complete quasi-normed lattice. Then the maximal quasi-normed extension $X^{\varkappa}$ has the Levi property if and only if $X$ has the weak Fatou property.*

$\lhd$ Let $X$ be a Dedekind complete quasi-normed lattice with the weak Fatou constant $K$. Take an increasing net $(\hat{x}_\alpha)$ in $X^{\varkappa}$ with $B := \sup_\alpha \|\hat{x}_\alpha\|_{\varkappa} < \infty$. By Lemma 3.8 there exists a band projection $\pi$ on $X^{\varkappa}$ such that $\hat{x} = \sup_\alpha \pi\hat{x}_\alpha$ exists in $X^u$ and for every $N \in \mathbb{N}$ and

$e \in X_+$ we have $N\pi^\perp e = \sup_\alpha \pi^\perp(\hat{x}_\alpha \wedge Ne)$. Making use of the weak Fatou property we deduce $N\|\pi^\perp e\| \leqslant K \sup_\alpha \|\pi^\perp(\hat{x}_\alpha \wedge Ne)\| \leqslant K \sup_\alpha \|\hat{x}_\alpha\|_\varkappa = BK$ and $\pi^\perp = 0$, since $N$ and $e$ are arbitrary. It follows that $\pi$ is the identity operator and $\hat{x} = \sup_\alpha \hat{x}_\alpha$. Show that $\hat{x} \in X^\varkappa$. If $x \in X$ and $0 \leqslant x \leqslant \hat{x}$ then $x \wedge \hat{x}_\alpha \in X$ and $(x \wedge \hat{x}_\alpha)$ is an increasing net with the supremum $x$. By the weak Fatou property we have $\|x\| \leqslant K \sup_\alpha \|x \wedge \hat{x}_\alpha\| \leqslant K \sup_\alpha \|\hat{x}_\alpha\|_\varkappa = KB$. It follows that $\sup\{\|x\| : x \in X,\ 0 \leqslant x \leqslant \hat{x}\} \leqslant KB$ and $\hat{x} \in X^\varkappa$.

To prove the converse, it suffices to observe that if $X^\varkappa$ has the Levi property, then $X$ has the weak Fatou property by Proposition 2.12. $\rhd$

**Proposition 3.12.** *Let $X$ be a Dedekind complete quasi-normed lattice. Then the maximal quasi-normed extension $X^\varkappa$ has the Fatou property if only if $X$ has the Fatou property.*

$\lhd$ The necessity is obvious. To prove the sufficiency take an increasing net $(\hat{x}_\alpha)$ in $X^\varkappa_+$ such that $\hat{x} = \sup_\alpha \hat{x}_\alpha$ for some $\hat{x} \in X^\varkappa_+$. Pick an arbitrary $x \in X_+$ with $0 \leqslant x \leqslant \hat{x}$ and note that $\hat{x}_\alpha \wedge x$ is an increasing set in $X_+$ and $\sup_\alpha \hat{x}_\alpha \wedge x = x$. In virtue of the Fatou property we have $\|x\| = \sup_\alpha \|\hat{x}_\alpha \wedge x\| \leqslant \sup_\alpha \|\hat{x}_\alpha\|_\varkappa$. Hence, $\|x\| \leqslant \sup_\alpha \|\hat{x}_\alpha\|_\varkappa \leqslant \|\hat{x}\|_\varkappa$ for all $x \in X_+$ with $0 \leqslant x \leqslant \hat{x}$. The latter implies that $\|\hat{x}\|_\varkappa = \sup_\alpha \|\hat{x}_\alpha\|_\varkappa$. $\rhd$

**Corollary 3.13.** *Let $X$ be a Dedekind complete quasi-normed lattice. If $X$ has the Fatou property then the maximal quasi-normed extension $X^\varkappa$ has the Fatou and the Levi property.*

$\lhd$ The proof follows immediately from Theorem 3.11 and Proposition 3.12. $\rhd$

## 4. Concluding remarks

Remark 4.1. The maximal normed extension of a Dedekind complete normed lattice was introduced and the Theorem 3.10 was proved in Abramovich [1, Definition on p. 8 and Theorem 3]. Lemmas 3.6 and 3.7 for normed lattices can be seen in Veksler [22, Lemma 2] and [23, Theorem 1.1], respectively.

Remark 4.2. In the case of normed lattices Theorem 3.10 is true without the weak $\sigma$-Fatou property, see Abramovich [1]. We do not know whether or not the assumption about the weak $\sigma$-Fatou property is superfluous in Theorem 3.10.

Remark 4.3. Let $X$ be a quasi-Banach lattice and $(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ a vector measure space with a localizable measure $\mu : \mathscr{R} \to X_+$ which is countable additive in the sense of order or quasi-norm convergence depending on the context, see [10, 11]. The Bartle–Dunford–Schwartz type integration and the purely order based Kantorovich–Wright integration with respect to $\mu$ provide two quasi-Banach lattices of integrable functions, $L^1_\tau(\mu)$ and $L^1_o(\mu)$, respectively, see [10]. Moreover, the vector lattice $L^0(\mu)$ (of equivalence classes) of $\mu$-a.e. finite $\mathscr{R}^{\mathrm{loc}}$-measurable real-valued functions is a universal completion of both quasi-Banach lattices $L^1_o(\mu)$ and $L^1_\tau(\mu)$. According to Definition 3.2 we can construct maximal quasi-normed extensions $(L^1_{o\varkappa}(\mu), \|\cdot\|_{o\varkappa})$ and $(L^1_{\tau\varkappa}(\mu), \|\cdot\|_{\tau\varkappa})$ of $L^1_o(\mu)$ and $L^1_\tau(\mu)$, respectively. By virtue of Theorem 3.10 $L^1_{\tau\varkappa}(\mu)$ is a quasi-Banach lattice and an order dense ideal in $L^0(\mu)$. Moreover, $L^1_{\tau\varkappa}(\mu)$ has the Fatou and Levi properties by Corollary 3.13, since $L^1_\tau(\mu)$ is order continuous.

Remark 4.4. Similarly, $L^1_{o\varkappa}(\mu)$ is a quasi-normed lattice and order dense ideal in $L^0(\mu)$, but $L^1_{o\varkappa}(\mu)$ is metrically complete under the additional assumption that $L^1_o(\mu)$ has the weak $\sigma$-Fatou property. We do not know whether $L^1_{o\varkappa}(\mu)$ is metrically complete (and hence a quasi-Banach lattice) without this additional assumption coming from Theorem 3.10.

Definition 4.5. An $\mathscr{R}^{\mathrm{loc}}$-measurable function $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is called *weakly integrable with respect to $\mu$* or *weakly $\mu$-integrable* if

$$\|f\|_\mu := \sup_{x^* \in B^*_+} \int |f| \, d|x^*\mu| < +\infty,$$

where $|x^*\mu| : \mathscr{R}^{\mathrm{loc}} \to [0, \infty]$ variation of $x^*\mu$ and $B_+^*$ the positive part of the unit ball in $X^*$. A weakly integrable function $f$ is *integrable with respect to* $\mu$ if for each $A \in \mathscr{R}^{\mathrm{loc}}$ there exists a vector denoted by $\int_A f\,d\mu \in X$, such that

$$x^*\Bigg(\int\limits_A f\,d\mu\Bigg) = \int\limits_A f\,dx^*\mu \ \text{ for all } \quad x^* \in X^*.$$

Denote by $L_w^1(\mu)$ the space of (equivalence classes) all weakly $\mu$-integrable function equipped with the norm $\|\cdot\|_\mu$ and let $L^1(\mu)$ stand for the subspace of $L_w^1(\mu)$ consisting of (equivalence classes) all $\mu$-integrable functions. Note that if $\|f\|_\mu < \infty$ then $|f| < \infty$ $\mu$-a.e. Thus, $L_w^1(\mu)$ and $L^1(\mu)$ can be considered as subspaces of $L^0(\mu)$.

**Theorem 4.6.** *Let $X$ be a Banach lattice and $(\Omega, \mathscr{R}, \mu)$ a vector measure space with $\mathscr{R}$-decomposable measure $\mu : \mathscr{R} \to X_+$. Then $L_w^1(\mu)$ and $L_{\tau\varkappa}^1(\mu)$ coincide as Banach lattices.*

Remark 4.7. In Theorem 4.6 $\mathscr{R}$-decomposability of measure $\mu$ provides $L_w^1(\mu)$ with the Levi and Fatou properties (see [4, Theorem 5.8]), while $L_{\tau\varkappa}^1(\mu)$ always has these properties. Without $\mathscr{R}$-decomposability assumption it may happen that $L_w^1(\mu) \neq L_{\tau\varkappa}^1(\mu)$. Similar questions for the space of order integrable functions $L_o^1(\mu)$ and the corresponding maximal quasi-Banach extension $L_{o\varkappa}^1(\mu)$ remain open.

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# О МАКСИМАЛЬНОМ КВАЗИНОРМИРОВАННОМ РАСШИРЕНИИ КВАЗИНОРМИРОВАННЫХ ВЕКТОРНЫХ РЕШЕТОК

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Цель работы — распространить конструкцию Абрамовича максимального нормированного расширения нормированной решетки на класс квазинормированных решеток. Установлено, что максимальное квазинормированное расширение $X^{\varkappa}$ порядково полной квазинормированной решетки $X$ со слабым счетным свойством Фату является квазибанаховой решеткой в том и только в том случае, когда $X$ интервально полна. Боле того, $X^{\varkappa}$ обладает свойствами Леви и Фату, если только $X$ — порядково полная квазинормированная решетка со свойством Фату. Обсуждается также возможность применения этой конструкции к определению пространства слабо интегрируемых функций относительно меры со значениями в квазибанаховой решетке, не прибегая к двойственности (которая может оказаться тривиальной).

**Ключевые слова:** квазинормированная решетка, максимальное квазинормированное расширение, свойство Фату, свойство Леви, векторная мера, слабо интегрируемые функции.