Vladikavkaz Mathematical Journal
2018, Volume 20, Issue 3, P. 78–86

УДК 517.98
DOI 10.23671/VNC.2018.3.18031

SOME ESTIMATES FOR THE GENERALIZED FOURIER TRANSFORM ASSOCIATED WITH THE CHEREDNIK–OPDAM OPERATOR ON $\mathbb{R}$

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Abstract. In the classical theory of approximation of functions on $\mathbb{R}^+$, the modulus of smoothness are basically built by means of the translation operators $f \to f(x+y)$. As the notion of translation operators was extended to various contexts (see [2] and [3]), many generalized modulus of smoothness have been discovered. Such generalized modulus of smoothness are often more convenient than the usual ones for the study of the connection between the smoothness properties of a function and the best approximations of this function in weight functional spaces (see [4] and [5]). In [1], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator. In this paper, we also discuss this subject. More specifically, we prove some estimates (similar to those proved in [1]) in certain classes of functions characterized by a generalized continuity modulus and connected with the generalized Fourier transform associated with the differential-difference operator $T^{(\alpha,\beta)}$ in $L^2_{\alpha,\beta}(\mathbb{R})$. For this purpose, we use a generalized translation operator.

Key words: Cherednik–Opdam operator, generalized Fourier transform, generalized translation.
Mathematical Subject Classification (2010): 34K99, 42A63.

1. Introduction

In [1], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator.
In this paper, we prove some estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the generalized Fourier transform associated to $T^{(\alpha,\beta)}$ in $L^2_{\alpha,\beta}(\mathbb{R})$ analogs of the statements proved in [1, 2–4]. For this purpose, we use a generalized translation operator.

In section 2, we give some definitions and preliminaries concerning the generalized Fourier transform. Some estimates are proved in section 3.

2. Preliminaries

In this section, we develop some results from harmonic analysis related to the differential-difference operator $T^{(\alpha,\beta)}$. Further details can be found in [5] and [6]. In the following we fix parameters α, β subject to the constraints $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ and $\alpha > -\frac{1}{2}$.

Let $\rho = \alpha + \beta + 1$ and $\lambda \in \mathbb{C}$. The Opdam hypergeometric functions $G_\lambda^{(\alpha,\beta)}$ on $\mathbb{R}$ are eigenfunctions $T^{(\alpha,\beta)} G_\lambda^{(\alpha,\beta)}(x) = i\lambda G_\lambda^{(\alpha,\beta)}(x)$ of the differential-difference operator

$$T^{(\alpha,\beta)} f(x) = f'(x) + \big[(2\alpha+1)\coth x + (2\beta+1)\tanh x\big]\frac{f(x)-f(-x)}{2} - \rho f(-x)$$

that are normalized such that $G_\lambda^{(\alpha,\beta)}(0) = 1$. In the notation of Cherednik one would write $T^{(\alpha,\beta)}$ as

$$T(k_1+k_2)f(x) = f'(x) + \left\{\frac{2k_1}{1+e^{-2x}} + \frac{4k_2}{1-e^{-4x}}\right\}\big(f(x)-f(-x)\big) - (k_1+2k_2)f(x),$$

with $\alpha = k_1 + k_2 - \frac{1}{2}$ and $\beta = k_2 - \frac{1}{2}$. Here k_1 is the multiplicity of a simple positive root and k_2 the (possibly vanishing) multiplicity of a multiple of this root. By [5] or [6], the eigenfunction $G_\lambda^{(\alpha,\beta)}$ is given by

$$G_\lambda^{(\alpha,\beta)}(x) = \varphi_\lambda^{\alpha,\beta}(x) - \frac{1}{\rho - i\lambda}\frac{\partial}{\partial x}\varphi_\lambda^{\alpha,\beta}(x) = \varphi_\lambda^{\alpha,\beta}(x) + \frac{\rho}{4(\alpha+1)}\sinh(2x)\varphi_\lambda^{\alpha+1,\beta+1}(x),$$

where $\varphi_\lambda^{\alpha,\beta}(x) = {}_2F_1\left(\frac{\rho+i\lambda}{2}; \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 x\right)$ is the classical Jacobi function.

Lemma 2.1 [7]. *The following inequalities are valid for Jacobi functions* $\varphi_\lambda^{\alpha,\beta}(x)$

(i) $|\varphi_\lambda^{\alpha,\beta}(x)| \leqslant 1$;

(ii) $1 - \varphi_\lambda^{\alpha,\beta}(x) \leqslant x^2(\lambda^2 + \rho^2)$.

Denote $L^2_{\alpha,\beta}(\mathbb{R})$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{2,\alpha,\beta} \;=\; \left(\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x)\,dx\right)^{1/2} < +\infty,$$

where

$$A_{\alpha,\beta}(x) = (\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

The generalized Fourier transform of $f \in C_c(\mathbb{R})$ (the space of continuous functions on $\mathbb{R}$ with compact support) is defined by

$$\mathscr{H} f(\lambda) = \int_{\mathbb{R}} f(x) G_\lambda^{(\alpha,\beta)}(-x) A_{\alpha,\beta}(x)\,dx \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

The inverse transform is given as

$$\mathscr{H}^{-1} g(x) = \int_{\mathbb{R}} g(\lambda) G_\lambda^{(\alpha,\beta)}(x)\left(1 - \frac{\rho}{i\lambda}\right)\frac{d\lambda}{8\pi|c_{\alpha,\beta}(\lambda)|^2},$$

here

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma\big(\frac{1}{2}(\rho+i\lambda)\big)\Gamma\big(\frac{1}{2}(\alpha-\beta+1+i\lambda)\big)}.$$

The corresponding Plancherel formula was established in [5], to the effect that

$$\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x)\,dx \;=\; \int_0^{+\infty} \left(|\mathscr{H} f(\lambda)|^2 + |\mathscr{H} \check{f}(\lambda)|^2\right)\,d\sigma(\lambda),$$

where $\check{f}(x) := f(-x)$ and $d\sigma$ is the measure given by

$$d\sigma(\lambda) = \frac{d\lambda}{16\pi|c_{\alpha,\beta}(\lambda)|^2}.$$

According to [6] there exists a family of signed measures $\mu_{x,y}^{(\alpha,\beta)}$ such that the product formula

$$G_\lambda^{(\alpha,\beta)}(x)G_\lambda^{(\alpha,\beta)}(y) = \int_{\mathbb{R}} G_\lambda^{(\alpha,\beta)}(z)\,d\mu_{x,y}^{(\alpha,\beta)}(z),$$

holds for all $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, where

$$d\mu_{x,y}^{(\alpha,\beta)}(z) = \begin{cases} \mathscr{K}_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)\,dz, & xy \neq 0; \\ \\ d\delta_x(z), & y = 0; \\ d\delta_y(z), & x = 0 \end{cases}$$

and

$$\mathscr{K}_{\alpha,\beta}(x,y,z) = M_{\alpha,\beta}|\sinh x \times \sinh y \times \sinh z|^{-2\alpha} \int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}$$
$$\times \left[1 - \sigma_{x,y,z}^\chi + \sigma_{x,z,y}^\chi + \sigma_{z,y,x}^\chi + \frac{\rho}{\beta+\frac{1}{2}} \coth x \times \coth y \times \coth z (\sin\chi)^2\right] (\sin\chi)^{2\beta} d\chi,$$

if $x, y, z \in \mathbb{R}\backslash\{0\}$ satisfy the triangular inequality $\|x-y\| < |z| < |x|+|y|$, and $\mathscr{K}_{\alpha,\beta}(x,y,z) = 0$ otherwise. Here

$$\sigma_{x,y,z}^\chi = \begin{cases} \frac{\cosh x + \cosh y - \cosh z \cos\chi}{\sinh x \sinh y}, & xy \neq 0; \\ 0, & xy = 0 \end{cases} \qquad (\forall\, x, y, z \in \mathbb{R},\ \chi \in [0, 1])$$

and

$$g(x,y,z,\chi) = 1 - \cosh^2 x - \cosh^2 y \times \cosh^2 z + 2\cosh x \times \cosh y \times \cosh z \times \cos\chi.$$

Lemma 2.2 [6]. *For all $x, y \in \mathbb{R}$, we have*
(i) $\mathscr{K}_{\alpha,\beta}(x,y,z) = \mathscr{K}_{\alpha,\beta}(y,x,z)$;
(ii) $\mathscr{K}_{\alpha,\beta}(x,y,z) = \mathscr{K}_{\alpha,\beta}(-x,z,y)$;
(iii) $\mathscr{K}_{\alpha,\beta}(x,y,z) = \mathscr{K}_{\alpha,\beta}(-z,y,-x)$.

The product formula is used to obtain explicit estimates for the generalized translation operators

$$\tau_x^{(\alpha,\beta)} f(y) = \int_{\mathbb{R}} f(z)\,d\mu_{x,y}^{(\alpha,\beta)}(z).$$

It is known from [6] that

$$\mathscr{H}\tau_x^{(\alpha,\beta)} f(\lambda) = G_\lambda^{(\alpha,\beta)}(x)\mathscr{H}f(\lambda), \tag{1}$$

for $f \in C_c(\mathbb{R})$.

For $\alpha > -\frac{1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n+\alpha+1)}, \quad x \in \mathbb{R}.$$

In the terms of $j_\alpha(x)$, we have (see [8])

$$\sqrt{hx}J_\alpha(hx) = O(1), \quad hx \geqslant 0, \tag{2}$$

where $J_\alpha(x)$ is Bessel function of the first kind, which is related to $j_\alpha(x)$ by the formula

$$j_\alpha(x) = \frac{2^\alpha\Gamma(\alpha+1)}{x^\alpha}J_\alpha(x). \tag{3}$$

Lemma 2.3 [9]. *Let $\alpha \geqslant \beta \geqslant \frac{-1}{2}$, $\alpha \neq \frac{-1}{2}$. Then for $|\nu| \leqslant \rho$, there exists a positive constant c_0 such that*

$$\left|1 - \varphi^{\alpha,\beta}_{\lambda+i\nu}(x)\right| \geqslant c_0|1 - j_\alpha(\lambda x)|.$$

For $f \in L^2_{\alpha,\beta}(\mathbb{R})$, we define the finite differences of first and higher order as follows:

$$\Delta^1_h f = \Delta_h f = \left(\tau^{(\alpha,\beta)}_h + \tau^{(\alpha,\beta)}_{-h} - 2I\right)f,$$

$$\Delta^k_h f = \Delta_h\left(\Delta^{k-1}_h f\right) = \left(\tau^{(\alpha,\beta)}_h + \tau^{(\alpha,\beta)}_{-h} - 2I\right)^k f, \quad k = 2, 3, \ldots,$$

where I is the unit operator in the space $L^2_{\alpha,\beta}(\mathbb{R})$.

The generalized modulus of continuity of a function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ is defined by

$$\omega(f,\delta)_{2,\alpha,\beta} = \sup_{0<h\leqslant\delta}\left\|\Delta^k_h f\right\|_{2,\alpha,\beta}, \quad \delta > 0.$$

3. Main Result

The goal of this work is to prove some estimates for the integral

$$J^2_N(f) = \int_N^{+\infty}\left(|\mathscr{H}f(\lambda)|^2 + |\mathscr{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda),$$

in certain classes of functions in $L^2_{\alpha,\beta}(\mathbb{R})$.

Lemma 3.1. *If $f \in C_c(\mathbb{R})$, then*

$$\mathscr{H}\check{\tau}^{(\alpha,\beta)}_x f(\lambda) = G^{(\alpha,\beta)}_\lambda(-x)\mathscr{H}\check{f}(\lambda). \tag{4}$$

$\lhd$ For $f \in C_c(\mathbb{R})$, we have

$$\mathscr{H}\check{\tau}^{(\alpha,\beta)}_x f(\lambda) = \int_{\mathbb{R}}\tau^{(\alpha,\beta)}_x f(-y)G^{(\alpha,\beta)}_\lambda(-y)A_{\alpha,\beta}(y)\,dy = \int_{\mathbb{R}}\tau^{(\alpha,\beta)}_x f(y)G^{(\alpha,\beta)}_\lambda(y)A_{\alpha,\beta}(y)\,dy$$

$$= \int_{\mathbb{R}}\left[\int_{\mathbb{R}} f(z)\mathscr{K}_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)\,dz\right]G^{(\alpha,\beta)}_\lambda(y)A_{\alpha,\beta}(y)\,dy$$

$$= \int_{\mathbb{R}} f(z)\left[\int_{\mathbb{R}} G^{(\alpha,\beta)}_\lambda(y)\mathscr{K}_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(y)\,dy\right]A_{\alpha,\beta}(z)\,dz.$$

Since $\mathscr{K}_{\alpha,\beta}(x,y,z)=\mathscr{K}_{\alpha,\beta}(-x,z,y)$, it follows from the product formula that

$$\mathscr{H}\check{\tau}_x^{(\alpha,\beta)}f(\lambda)=G_\lambda^{(\alpha,\beta)}(-x)\int\limits_{\mathbb{R}} f(z)G_\lambda^{(\alpha,\beta)}(z)A_{\alpha,\beta}(z)\,dz$$

$$=G_\lambda^{(\alpha,\beta)}(-x)\int\limits_{\mathbb{R}} f(-z)G_\lambda^{(\alpha,\beta)}(-z)A_{\alpha,\beta}(z)\,dz=G_\lambda^{(\alpha,\beta)}(-x)\mathscr{H}\check{f}(\lambda).\ \rhd$$

Lemma 3.2. *For $f\in L^2_{\alpha,\beta}(\mathbb{R})$, then*

$$\left\|\Delta_h^k f\right\|^2_{2,\alpha,\beta}=2^{2k}\int\limits_0^{+\infty}\left|\varphi_\lambda^{\alpha,\beta}(h)-1\right|^{2k}\left(|\mathscr{H}f(\lambda)|^2+|\mathscr{H}\check{f}(\lambda)|^2\right)\,d\sigma(\lambda).$$

$\lhd$ From formulas (1) and (4), we have

$$\mathscr{H}\big(\Delta_h^k f\big)(\lambda)=\Big(G_\lambda^{(\alpha,\beta)}(h)+G_\lambda^{(\alpha,\beta)}(-h)-2\Big)^k\mathscr{H}(f)(\lambda)$$

and

$$\mathscr{H}(\check{\Delta}_h^k f)(\lambda)=\Big(G_\lambda^{(\alpha,\beta)}(-h)+G_\lambda^{(\alpha,\beta)}(h)-2\Big)^k\mathscr{H}(\check{f})(\lambda).$$

Since

$$G_\lambda^{(\alpha,\beta)}(h)=\varphi_\lambda^{\alpha,\beta}(h)+\frac{\rho}{4(\alpha+1)}\sinh(2h)\varphi_\lambda^{\alpha+1,\beta+1}(h),$$

and $\varphi_\lambda^{\alpha,\beta}$ is even, then

$$\mathscr{H}\big(\Delta_h^k f\big)(\lambda)=2^k\Big(\varphi_\lambda^{\alpha,\beta}(h)-1\Big)^k\mathscr{H}(f)(\lambda)$$

and

$$\mathscr{H}\big(\check{\Delta}_h^k f\big)(\lambda)=2^k\Big(\varphi_\lambda^{\alpha,\beta}(h)-1\Big)^k\mathscr{H}(\check{f})(\lambda).\ \rhd$$

Now by Plancherel Theorem, we have the result.

Theorem 3.1. *Given k and $f\in L^2_{\alpha,\beta}(\mathbb{R})$. Then there exist a constant $c>0$ such that, for all $N>0$,*

$$J_N(f)=O\big(\omega\big(f,cN^{-1}\big)_{2,\alpha,\beta}\big).$$

$\lhd$ Firstly, we have

$$J_N^2(f)\leqslant\int\limits_N^{+\infty}|j_\alpha(\lambda h)|\,d\mu(\lambda)+\int\limits_N^{+\infty}|1-j_\alpha(\lambda h)|\,d\mu(\lambda),\tag{5}$$

with $d\mu(\lambda)=(|\mathscr{H}f(\lambda)|^2+|\mathscr{H}\check{f}(\lambda)|^2)d\sigma(\lambda)$. The parameter $h>0$ will be chosen in an instant.

In view of formulas (2) and (3), there exist a constant $c_1>0$ such that

$$|j_\alpha(\lambda h)|\leqslant c_1(\lambda h)^{-\alpha-\frac{1}{2}}.$$

Then

$$\int\limits_N^{+\infty}|j_\alpha(\lambda h)|\,d\mu(\lambda)\leqslant c_1(hN)^{-\alpha-\frac{1}{2}}J_N^2(f).$$

Choose a constant c_2 such that the number $c_3 = 1 - c_1 c_2^{-\alpha-\frac{1}{2}}$ is positive.
Setting $h = c_2/N$ in the inequality (5), we have

$$c_3 J_N^2(f) \leqslant \int_N^{+\infty} |1 - j_\alpha(\lambda h)| \, d\mu(\lambda). \tag{6}$$

By Hölder inequality and Lemma 2.3 the second term in (6) satisfies

$$\begin{aligned}
\int_N^{+\infty} |1 - j_\alpha(\lambda h)| \, d\mu(\lambda) &= \int_N^{+\infty} |1 - j_\alpha(\lambda h)| \times 1 \, d\mu(\lambda) \\
&\leqslant \Bigg(\int_N^{+\infty} |1 - j_\alpha(\lambda h)|^{2k} \, d\mu(\lambda) \Bigg)^{1/2k} \Bigg(\int_N^{+\infty} d\mu(\lambda) \Bigg)^{1-1/2k} \\
&\leqslant \Bigg(\int_N^{+\infty} |1 - j_\alpha(\lambda h)|^{2k} \, d\mu(\lambda) \Bigg)^{1/2k} (J_N(f))^{2-1/k} \\
&\leqslant \frac{1}{c_0} \Bigg(\int_N^{+\infty} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^{2k} \, d\mu(\lambda) \Bigg)^{1/2k} (J_N(f))^{2-1/k}.
\end{aligned}$$

From Lemma 3.2, we conclude that

$$\int_N^{+\infty} \left|1 - \varphi_\lambda^{\alpha,\beta}(h)\right|^{2k} d\mu(\lambda) \leqslant \left\|\Delta_h^k f\right\|_{2,\alpha,\beta}^2.$$

Therefore

$$\int_N^{+\infty} \left|1 - j_\alpha(\lambda h)\right| d\mu(\lambda) \leqslant \frac{1}{c_0} \left\|\Delta_h^k f\right\|_{2,\alpha,\beta}^{1/k} (J_N(f))^{2-1/k}.$$

For $h = c_2/N$, we obtain

$$c_3 J_N^2(f) \leqslant \frac{1}{c_0} \, \omega\Big(f, \frac{c_2}{N}\Big)_{2,\alpha,\beta}^{1/k} (J_N(f))^{2-1/k}.$$

Consequently by raising both sides to the power k and simplifying by $(J_N(f))^{2k}$ we finally obtain

$$c_0^k c_3^k J_N(f) \leqslant \omega\Big(f, \frac{c}{N}\Big)_{2,\alpha,\beta}$$

for all $N > 0$. The theorem is proved with $c = c_2$. $\rhd$

Theorem 3.3. *Let $f \in L^2_{\alpha,\beta}(\mathbb{R})$. Then, for all $N > 0$,*

$$\omega\big(f, N^{-1}\big)_{2,\alpha,\beta} = O\left(N^{-2k} \left(\sum_{l=0}^{N} (l+1)^{4k-1} J_l^2(f) \right)^{1/2} \right).$$

◁ From Lemma 3.2, we have

$$\left\|\Delta_h^k f\right\|_{2,\alpha,\beta}^2 = 2^{2k} \int_0^{+\infty} \left|\varphi_\lambda^{\alpha,\beta}(h) - 1\right|^{2k} \left(|\mathscr{H} f(\lambda)|^2 + |\mathscr{H} \check{f}(\lambda)|^2\right) \, d\sigma(\lambda).$$

This integral is divided into two

$$\int_0^{+\infty} = \int_0^N + \int_N^{+\infty} = I_1 + I_2,$$

where $N = [h^{-1}]$. We estimate them separately.

From (i) of Lemma 2.1, we have the estimate

$$I_2 \leqslant c_4 \int_N^{+\infty} \left(|\mathscr{H} f(\lambda)|^2 + |\mathscr{H} \check{f}(\lambda)|^2\right) \, d\sigma(\lambda) = c_4 J_N^2(f).$$

Now, we estimate I_1. From formula (ii) of Lemma 2.1, we have

$$\begin{aligned} I_1 &\leqslant h^{4k} \int_0^N (\lambda + \rho)^{4k} \left(|\mathscr{H} f(\lambda)|^2 + |\mathscr{H} \check{f}(\lambda)|^2\right) \, d\sigma(\lambda) \\ &= h^{4k} \sum_{l=0}^{N-1} \int_l^{l+1} (\lambda + \rho)^{4k} \left(|\mathscr{H} f(\lambda)|^2 + |\mathscr{H} \check{f}(\lambda)|^2\right) \, d\sigma(\lambda) \\ &\leqslant h^{4k} \sum_{l=0}^{N-1} (l + \rho + 1)^{4k} \left(J_l^2(f) - J_{l+1}^2(f)\right). \end{aligned}$$

From the inequality $l + \rho + 1 \leqslant (\rho + 1)(l + 1)$ we conclude

$$I_1 \leqslant (\rho + 1)^{4k} h^{4k} \sum_{l=0}^{N-1} a_l \left(J_l^2(f) - J_{l+1}^2(f)\right)$$

with $a_l = (l + 1)^{4k}$.

For all integers $m \geqslant 1$, the Abel transformation shows

$$\begin{aligned} \sum_{l=0}^{m} a_l \left(J_l^2(f) - J_{l+1}^2(f)\right) &= a_0 J_0^2(f) + \sum_{l=1}^{m} (a_l - a_{l-1}) J_l^2(f) - a_m J_{m+1}^2(f) \\ &\leqslant a_0 J_0^2(f) + \sum_{l=1}^{m} (a_l - a_{l-1}) J_l^2(f), \end{aligned}$$

because $a_m J_{m+1}^2(f) \geqslant 0$.

Hence

$$I_1 \leqslant (\rho + 1)^{4k} N^{-4k} \left(J_0^2(f) + \sum_{l=1}^{N-1} \left((l + 1)^{4k} - l^{4k}\right) J_l^2(f)\right),$$

since $N \leqslant 1/h$. Moreover by the finite increments theorem, we have $(l+1)^{4k} - l^{4k} \leqslant 4k(l+1)^{4k-1}$. Then

$$I_1 \leqslant (\rho+1)^{4k} N^{-4k} \left(J_0^2(f) + 4k \sum_{l=1}^{N-1} (l+1)^{4k-1} J_l^2(f) \right).$$

Combining the estimates for I_1 and I_2 gives

$$\|\Delta_h^k f\|_{2,\alpha,\beta}^2 = O\left(N^{-4k} \sum_{l=0}^{N} (l+1)^{4k-1} J_l^2(f) \right),$$

which implies

$$\omega(f, N^{-1})_{2,\alpha,\beta} = O\left(N^{-2k} \left(\sum_{l=0}^{N} (l+1)^{4k-1} J_l^2(f) \right)^{1/2} \right),$$

and this ends the proof. ▷

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Received February 24, 2016
Final version January 19, 2018

Владикавказский математический журнал
2018, Том 20, Выпуск 3, С. 78–86

НЕКОТОРЫЕ ОЦЕНКИ ДЛЯ ОБОБЩЕННОГО ПРЕОБРАЗОВАНИЯ ФУРЬЕ, АССОЦИИРОВАННОГО С ОПЕРАТОРОМ ЧЕРЕДНИКА — ОПДАМА

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Аннотация. В классической теории приближения функций на $\mathbb{R}^+$, модуль гладкости в основном строится посредством операторов сдвига $f(\cdot) \mapsto f(\cdot + y)$. Поскольку понятие оператора сдвига было расширено в различных направлениях (см. [2] и [3]), были обнаружено много других обобщенных модулей гладкости. Часто при изучения взаимосвязи свойств гладкости функции и наилучшего приближения этой функции в весовых функциональных пространствах такие обобщенные модули гладкости оказываются более удобными, чем обычные (см. [4] и [5]). В работе [1] Абилов и др. для преобразования Фурье в пространстве квадратично интегрируемых функций доказали с использованием оператора сдвига две полезные оценки на некоторых классах функций, характеризуемых обобщенным модулем непрерывности. В данной статье мы также обсуждаем этот вопрос. Более конкретно, мы доказываем некоторые оценки (аналогичные доказанным в [1]) в классах функций, характеризуемых обобщенным модулем непрерывности и связанных с обобщенным преобразование Фурье, ассоциированное с дифференциально-разностным оператором $T^{(\alpha,\beta)}$ в пространстве $L^2_{\alpha,\beta}(\mathbb{R})$. Для этой цели мы используем обобщенный оператор сдвига.

Ключевые слова: оператор Чередника — Опдама, обобщенное преобразование Фурье обобщенный сдвиг.

Mathematical Subject Classification (2000): 34K99, 42A63.