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UNBOUNDED ORDER CONVERGENCE AND THE GORDON THEOREM[#]

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Dedicated to Professor E. I. Gordon
on occasion of his 70th birthday

Abstract. The celebrated Gordon's theorem is a natural tool for dealing with universal completions of Archimedean vector lattices. Gordon's theorem allows us to clarify some recent results on unbounded order convergence. Applying the Gordon theorem, we demonstrate several facts on order convergence of sequences in Archimedean vector lattices. We present an elementary Boolean-Valued proof of the Gao–Grobler–Troitsky–Xanthos theorem saying that a sequence x_n in an Archimedean vector lattice X is *uo*-null (*uo*-Cauchy) in X if and only if x_n is *o*-null (*o*-convergent) in X^u. We also give elementary proof of the theorem, which is a result of contributions of several authors, saying that an Archimedean vector lattice is sequentially *uo*-complete if and only if it is σ-universally complete. Furthermore, we provide a comprehensive solution to Azouzi's problem on characterization of an Archimedean vector lattice in which every *uo*-Cauchy net is *o*-convergent in its universal completion.

Key words: unbounded order convergence, universally complete vector lattice, Boolean valued analysis.
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1. Introduction

Throughout the paper, we let X stand for a vector lattice, and all vector lattices are assumed to be real and Archimedean. We refer to [1, 2] for the unexplained terminology and facts on vector lattices and start with recalling some definitions and results. A vector lattice X is said to be *Dedekind* (*σ-Dedekind*) *complete* if each nonempty order bounded (countable) subset of X has a supremum. A Dedekind complete (σ-Dedekind complete) vector lattice X is said to be *universally* (*σ-universally*) *complete* if each nonempty pairwise disjoint (countable)

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subset of X_+ has a supremum. Clearly, each universally complete vector lattice has a weak unit. It is well known that X possesses Dedekind and universal completions unique up to lattice isomorphism which are denoted by X^δ and X^u. We will always suppose that $X \subseteq X^\delta \subseteq X^u$, whereas X^δ is an ideal in X^u.

A sublattice Y of X is said to be *regular* if $y_\alpha \downarrow 0$ in Y implies $y_\alpha \downarrow 0$ in X; while Y is *order dense* in X if for every $0 \neq x \in X_+$ there exists $y \in Y$ satisfying $0 < y \leqslant x$. Obviously, the ideals and order dense sublattices are regular. In what follows, we will freely use the regularity of X in X^u. Note also that X is *atomic* iff X is lattice isomorphic to an order dense sublattice of $\mathbb{R}^C$ (cf. [1, Theorem 1.78]).

A net $(x_\alpha)_{\alpha\in A}$ in X o-converges to x if there exists a net $(z_\gamma)_{\gamma\in\Gamma}$ in X satisfying $z_\gamma \downarrow 0$ and, for each $\gamma \in \Gamma$, there is $\alpha_\gamma \in A$ with $|x_\alpha - x| \leqslant z_\gamma$ for all $\alpha \geqslant \alpha_\gamma$. In this case we write $x_\alpha \xrightarrow{o} x$. This definition is used for instance in [2, 3]. Sometimes (in particular, see [1, 4, 5]) the slightly different definition of o-convergence appears: $(x_\alpha)_{\alpha\in A}$ o-converges to $x \in X$ if there is a net $(z_\alpha)_{\alpha\in A}$ such that $z_\alpha \downarrow 0$ and $|x_\alpha - x| \leqslant z_\alpha$ for all α. These two definitions agree in the case of order bounded nets in Dedekind complete vector lattices (cf. [3, Remark 2.2]). The article [6] contains a more details discussion of the definitions of o-convergence. By [7, Theorem 1] (cf. also [8, Theorem 2]), o-convergence in X is topological iff X is finite dimensional.

A net x_α in X is said to be *uo-convergent* to x if $|x_\alpha - x| \wedge y \xrightarrow{o} 0$ for every $y \in X_+$. We write $x_\alpha \xrightarrow{uo} x$. Following Nakano [9], uo-convergence is investigated as a generalization of almost everywhere convergence (see [3, 4, 10–18] and references therein). Note that o-convergence agrees with eventually order bounded uo-convergence. Furthermore, uo-convergence passes freely between X, X^δ, and X^u [3, Theorem 3.2]. It was shown in [3, Corollary 3.5] that if e is a weak unit of X then $x_\alpha \xrightarrow{uo} x \Leftrightarrow |x_\alpha - x| \wedge e \xrightarrow{o} 0$. By [3, Corollary 3.12] every uo-null sequence in X is o-null in X^u. This is untrue for arbitrary nets. By Theorem 4 below, or independently, by [18, Proposition 15.2], all uo-null nets in X are o-null in X^u if only if $\dim(X) < \infty$. Although uo-convergence is not topological in many important cases (e. g., in $L_1[0,1]$ and in $C[0,1]$), it is topological in atomic vector lattices; see [7, Theorem 2].

A net x_α is said to be *o-Cauchy* (*uo-Cauchy*) if the double net $(x_\alpha - x_\beta)_{(\alpha,\beta)}$ o-converges (uo-converges) to 0. Clearly, every o-Cauchy net is uo-Cauchy. In a Dedekind complete vector lattice with a weak unit e, a net x_α is uo-Cauchy iff $\inf_\alpha \sup_{\beta,\gamma\geqslant\alpha} |x_\beta - x_\gamma| \wedge e = 0$ [13, Lemma 2.7]. It is well known that completeness with respect to o-convergence is equivalent to Dedekind completeness. By [3, Corollary 3.12], a sequence in X is *uo-Cauchy* in X iff it is o-convergent in X^u. As showed in Theorem 4, there is no net-version of the latter fact unless X is finite-dimensional. It was proved in [16, Theorem 3.9] (see also [15, Theorem 28]) that X is σ-universally complete iff X is sequentially uo-complete. In [15, Theorem 17], it was demonstrated that uo-completeness is equivalent to universal completeness. Thus, there is no need in any special investigation of (sequential) uo-completion.

The (always complete) Boolean algebra $\mathfrak{B}(X)$ of all bands of X is called the *base* of X. If X has the projection property (e.g., if X is Dedekind complete), then $\mathfrak{B}(X)$ can be identified with the Boolean algebra $\mathfrak{P}(X)$ of all band projections in X and, if X has also a weak unit e, both $\mathfrak{B}(X)$ and $\mathfrak{P}(X)$ can be identified with the Boolean algebra $\mathfrak{C}(e)$ of all fragments of e (cf. [2, Theorem 1.3.7 (1)]).

2. Boolean-Valued Analysis and Unbounded Order Convergence

The classical Gordon's discovery [19, Theorem 2] (expressing the immanent connection between vector lattices and Boolean-valued analysis) reads shortly as follows: *Each universally complete vector lattice is an interpretation of the reals $\mathcal{R}$ in an appropriate Boolean-valued* *model* $V^{(B)}$. Furthermore, each Archimedean vector lattice is an order dense ideal of the descent of $\mathcal{R}$ within $V^{(B)}$. These facts are combined as follows (see [2, Theorems 8.1.2 and 8.1.6]):

Theorem 1 (Gordon's Theorem). *Let X be an Archimedean vector lattice, while $B = \mathfrak{B}(X)$ and $\mathcal{R}$ is the reals in the Boolean-valued model $V^{(B)}$. Then $\mathcal{R}{\downarrow}$ is a universally complete vector lattice including X as an order dense sublattice. Moreover,*

$$bx \leqslant by \iff b \leqslant [\![x \leqslant y]\!] \quad (\forall\, b \in B);(\forall\, x, y \in \mathcal{R}{\downarrow}).$$

By the Gordon Theorem, the universal completion X^u of an Archimedean vector lattice X is the descent $\mathcal{R}{\downarrow}$ of the reals $\mathcal{R}$ in $V^{(\mathfrak{B}(X))}$, and the uniqueness of X^u up to an order isomorphism follows from the uniqueness of $\mathcal{R}$ in $V^{(\mathfrak{B}(X))}$ (see [2, 8.1.7]).

In [20] Kantorovich introduced Dedekind complete vector lattices and propounded his famous Heuristic Transfer Principle: *The members of every Dedekind complete vector lattice are generalized reals* (see [5] for further historical notes). This Kantorovich's motto was justified by the Gordon Theorem [19] published 42 years later in the same journal. The aim of the present paper, published another 42 years after [19], is to provide another illustration of the fruitfulness of the Gordon Theorem in exploring the theory of *uo*-convergence. To some extent, Archimedean vector lattices are commonly presented in the repertoire of the *Boolean-valued orchestra*, where the musicians are *complete Boolean algebras* and the orchestra director is the *reals*. To our knowledge, the present paper is a first attempt to apply Theorem 1 to *uo*-convergence. For the unexplained terminology and techniques of Boolean-valued analysis we refer the reader to [2, 5, 19, 21–25].

Let us turn to *uo*-convergence in X. Passing to $X^u = \mathcal{R}{\downarrow}$, which has the weak unit $\mathbf{1}$, $[\![\mathbf{1}$ is the multiplicative unit of $\mathcal{R}]\!] = \mathbb{1}$ we have, by [3, Corollary 3.5],

$$x_\alpha \xrightarrow{uo} 0 \iff |x_\alpha| \wedge \mathbf{1} \xrightarrow{o} 0 \quad (x_\alpha \in X).$$

By [2, 8.1.4], for every net $s = (x_\alpha)_{\alpha\in A}$ in $\mathcal{R}{\downarrow}$, the *standard name* $A^\wedge$ of A in $V^{(B)}$ (see [2, p. 401]) is also directed and $(s{\uparrow}) : A^\wedge \to \mathcal{R}$ is a net in $\mathcal{R}$ (within $V^{(B)}$); moreover,

$$b \leqslant [\![\lim(s{\uparrow}) = x]\!] \iff o - \lim \chi(b) \circ s = \chi(b)x$$

for every $b \in B = \mathfrak{B}(X) = \mathfrak{P}(\mathcal{R}{\downarrow})$ and every $x \in \mathcal{R}{\downarrow}$ [2, 8.1.4 (3)]. Thus,

$$x_\alpha \xrightarrow{uo} x \Leftrightarrow o - \lim_A(|x_\alpha - x| \wedge \mathbf{1}) = 0 \Leftrightarrow \Big[\!\!\Big[\lim_{A^\wedge}(|x_\alpha - x| \wedge \mathbf{1}) = 0\Big]\!\!\Big] = \mathbb{1}. \tag{1}$$

In the case of a sequence, $A = \mathbb{N}$, $A^\wedge = \mathbb{N}^\wedge = \mathcal{N}$ [25, p. 330]), and hence

$$\begin{gathered} x_n \xrightarrow{uo} 0 \text{ in } X \iff x_n \xrightarrow{uo} 0 \text{ in } \mathcal{R}{\downarrow} \iff \Big[\!\!\Big[\lim_{\mathcal{N}\ni n\to\infty}(|x_n| \wedge \mathbf{1}) = 0\Big]\!\!\Big] = \mathbb{1} \\ \iff [\![\lim |x_n| = 0]\!] = \mathbb{1} \iff x_n \xrightarrow{o} 0 \text{ in } \mathcal{R}{\downarrow} = X^u. \end{gathered} \tag{2}$$

Similarly,

$$\begin{gathered} x_n \text{ is uo-Cauchy in } X \iff x_n \text{ is uo-Cauchy in } \mathcal{R}{\downarrow} \\ \iff o - \lim_{k,m\to\infty} |x_k - x_m| \wedge \mathbf{1} = 0 \iff \Big[\!\!\Big[\lim_{\mathcal{N}^2=(\mathbb{N}\times\mathbb{N})^\wedge\ni(k,m)\to\infty}(|x_k - x_m| \wedge \mathbf{1}) = 0\Big]\!\!\Big] = \mathbb{1} \\ \iff \Big[\!\!\Big[\lim_{\mathcal{N}\ni k,m\to\infty} |x_k - x_m| = 0\Big]\!\!\Big] = \mathbb{1} \iff [\![x_n \text{ is Cauchy in } \mathcal{R}]\!] = \mathbb{1} \\ \iff [\![(\exists\, z \in \mathcal{R}) \lim x_n = z]\!] = \mathbb{1} \\ \iff [\![\lim x_n = z]\!] = \mathbb{1}, \text{ for some } z \in \mathcal{R}{\downarrow}\,; \iff x_n \xrightarrow{o} z \in \mathcal{R}{\downarrow} = X^u. \end{gathered} \tag{3}$$

The last equivalence in (3) is actually due to Gordon [19, Theorem 4] (see also [22]). Clearly, (3) implies that X^u is always sequentially *uo*-complete. The equivalences of (2) are exactly the first part of the following theorem (see [3, Corollary 3.12]), whereas (3) is its second part.

Theorem 2 (Gao–Grobler–Troitsky–Xanthos). *A sequence x_n in an Archimedean vector lattice X is uo-null in X iff x_n is o-null in X^u; while x_n is uo-Cauchy in X iff x_n is o-convergent in X^u.*

The presented proof of Theorem 2 is based on the fundamental fact that the standard name $\mathbb{N}^\wedge$ of the naturals is the naturals $\mathcal{N}$ in $V^{(B)}$. It seems to be the main obstacle in obtaining the net versions of this theorem which are indeed impossible due to Theorem 4.

The following theorem, stated and proved in [16, Theorem 3.9] and [15, Theorem 28], is a result of contributions of several authors (cf. also [3, Theorem 3.10], [3, Proposition 5.7], and [13, Proposition 2.8]).

Theorem 3. *X is sequentially uo-complete iff X σ-universally complete.*

$\lhd$ For the "if part" we remark firstly that the fact that every (sequentially) *uo*-complete vector lattice is (σ-) Dedekind complete is already contained in the proof of [3, Proposition 5.7]. Now, the (σ-) lateral completeness of a (sequentially) *uo*-complete vector lattice follows from the *o*-summability of every (countable) order bounded disjoint family in a (σ-) Dedekind complete vector lattice (cf. [2, 1.3.4]).

The "only if part" is exactly [3, Theorem 3.10]. $\rhd$

It could be illustrative to present some Boolean-valued proof of Theorem 3 as well as a Boolean-valued proof of Azouzi's Theorem [15, Theorem 17] which yields the equivalence of *uo*-completeness and universal completeness.

We conclude our paper with the following theorem which provides, among other things, an answer to Azouzi's question [15, Problem 23].

Theorem 4. *Let X be an Archimedean vector lattice. Then the following are equivalent*:

(1) $\dim(X) < \infty$;
(2) *every uo-Cauchy net in X is eventually order bounded in X^u*;
(3) *every uo-Cauchy net in X is o-convergent in X^u*;
(4) *every uo-null net in X is o-null in X^u*;
(5) *every uo-null net in X is eventually order bounded in X^u*;
(6) *every uo-convergent net in X is eventually order bounded in X^u*;
(7) *every uo-convergent net in X is eventually order bounded in X*;
(8) *every uo-convergent net in X o-converges in X^u to the same limit*;
(9) *every uo-convergent net in X^u o-converges in X^u to the same limit.*

Before proving the theorem, we include the following modification of [13, Example 2.6]. Given a nonempty subset $A \subset X$, pr_A stands for the band projection in X^u onto the band in X^u generated by A.

Example 1. In any infinite-dimensional Archimedean vector lattice X there exists a *uo*-null net which is not eventually order bounded in X^u.

As $\dim(X) = \infty$, there is a sequence e_n of pairwise disjoint positive nonzero elements of X. Let $\mathbb{N}^2$ be the coordinatewise directed set of pairs of naturals. A net in X is defined via $x_{(n,m)} = (n \vee m) \cdot e_{n\wedge m}$. Since $\{x_{(n,m)} : (n, m) \in \mathbb{N}^2\} \subseteq B_{\{e_k : k\in\mathbb{N}\}}$ and

$$\lim_{(n,m)\to\infty} pr_{\{e_k\}}(x_{(n,m)}) = \lim_{(n,m)\to\infty} (n \vee m) pr_{\{e_k\}}(e_{n\wedge m}) = 0 \quad (\forall\, k \in \mathbb{N}),$$

then $x_{(n,m)} \xrightarrow{uo} 0$ as $(n, m) \to \infty$ (e. g., it can be seen by use of [3, Corollary 3.5.] for a weak unit u in X^u s.t. $u \wedge e_k = e_k$ for all k). If $x_{(n,m)}$ is eventually order bounded by some $y \in X^u$, then for some $(n_0, m_0) \in \mathbb{N}^2$ we have $y \geqslant x_{(n,m)} \quad (\forall\,(n,m) \geqslant (n_0, m_0))$. Since $n \wedge m_0 = m_0$ and $(n, m_0) \geqslant (n_0, m_0)$ for $n \geqslant n_0 \vee m_0$, then

$$y \geqslant x_{(n,m_0)} = (n \vee m_0) \cdot e_{n \wedge m_0} = (n \vee m_0) \cdot e_{m_0} = n \cdot e_{m_0} > 0 \quad (\forall\, n \geqslant n_0 \vee m_0)$$

which is impossible. Therefore, the net $x_{(n,m)}$ is not eventually order bounded in X^u.

$\lhd$ Proof of Theorem 4. (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) $\Leftrightarrow$ (6), and (7) $\Rightarrow$ (6) are trivial.

(2) $\Rightarrow$ (3): Suppose x_α is *uo*-Cauchy in X. Then x_α is *uo*-Cauchy in X^u by [3, Theorem 3.2], because X is regular in X^u. It follows from [15, Theorem 17] that $x_\alpha \xrightarrow{uo} y$ for some $y \in X^u$. Since x_α is eventually order bounded in X^u by the assumption, then $x_\alpha \xrightarrow{o} y$.

(3) $\Rightarrow$ (4) follows since every *uo*-null net is *uo*-Cauchy, *o*-convergent implies *uo*-convergent, and the *uo*-limit of any *uo*-convergent net is unique.

(5) $\Rightarrow$ (1) is Example 1.

(6) $\Rightarrow$ (7) follows from the equivalence (6) $\Leftrightarrow$ (1) because (1) $\Rightarrow$ (7) is obvious.

(1) $\Leftrightarrow$ (8) follows from the equivalence (1) $\Leftrightarrow$ (4), since (8) is equivalent to the fact that every *uo*-null net in X is *o*-null in X^u.

(1) $\Leftrightarrow$ (9) follows from (1) $\Leftrightarrow$ (8) since $(X^u)^u = X^u$ and $\dim(X) < \infty$ iff $\dim(X^u) < \infty$. $\rhd$

While preparing this paper, we became aware of the still unpublished work [18] by Taylor which provides the construction [18, Proposition 15.2] similar to Example 1. The equivalence (1) $\Leftrightarrow$ (8) of Theorem 4 is also contained in [18, Corollary 15.3].

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НЕОГРАНИЧЕННАЯ ПОРЯДКОВАЯ СХОДИМОСТЬ И ТЕОРЕМА ГОРДОНА

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Аннотация. Знаменитая теорема Гордона является естественным инструментом для построения универсального пополнения архимедовой векторной решетки. Она позволяет нам уточнить некоторые недавние результаты о неограниченной порядковой сходимости. Применяя теорему Гордона, мы демонстрируем несколько фактов о сходимость последовательностей. В частности, приводится элементарное доказательство теоремы Гао — Гроблера — Троицкого — Хантоса о том, что последовательность в архимедовой векторной решетке *uo*-сходится к нулю (соответственно, является *uo*-фундаментальной) тогда и только тогда когда она порядково сходится к нулю (соответственно, является порядково сходящейся) в универсальном пополнении этой решетки. В статье дается простое доказательство известной теоремы о том, что архимедова векторная решетка секвенциально *uo*-полна тогда и только тогда когда она σ-универсально полна. Кроме того в статье дается полное решение недавней проблемы Азози о конечномерности всякой архимедовой векторной решетки в которой любая *uo*-фундаментальная последовательность порядково сходится в универсальном пополнении этой решетки.

Ключевые слова: неограниченная порядковая сходимость, расширенное пространство Канторовича, булевозначный анализ.

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