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HANKEL DETERMINANT OF THIRD KIND FOR CERTAIN SUBCLASS OF MULTIVALENT ANALYTIC FUNCTIONS

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Abstract. The objective of this paper is to obtain an upper bound (not sharp) to the third order Hankel determinant for certain subclass of multivalent (p-valent) analytic functions, defined in the open unit disc E. Using the Toeplitz determinants, we may estimate the Hankel determinant of third kind for the normalized multivalent analytic functions belongng to this subclass. But, using the technique adopted by Zaprawa [1], i. e., grouping the suitable terms in order to apply Lemmas due to Hayami [2], Livingston [3] and Pommerenke [4], we observe that, the bound estimated by the method adopted by Zaprawa is more refined than using upon applying the Toeplitz determinants.

Key words: p-valent analytic function, upper bound, third Hankel determinant, positive real function.

Mathematical Subject Classification (2010): 30C45, 30C50.

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1. Introduction

Let A_p (p is a fixed integer $\geqslant 1$) denotes the class of functions f of the form

$$f(z) = z^p \sum_{n=0}^{\infty} a_{p+n} z^n, \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture also called as Coefficient conjecture, which states that for a univalent function its n^{th}-Taylor's coefficient is bounded by n (see [5]). The bounds for the coefficients of these functions give information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f given in (1.1) (when $p = 1$), for $q, n \in \mathbb{N}$ was defined by Pommerenke [6] as follows and has been extensively studied by many authors:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

One can easily observe that the Fekete–Szegö functional is $H_2(1)$. In recent years, the research on Hankel determinants has focused on the estimation of $|H_2(2)|$, where

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2,$$

known as the second Hankel determinant obtained for $q = 2$ and $n = 2$ in (1.2). Many authors obtained upper bound to the functional $|a_2a_4 - a_3^2|$ for various subclasses of univalent and multivalent analytic functions. The exact (sharp) estimates of $|H_2(2)|$ for the subclasses of S namely, bounded turning, starlike and convex functions denoted by $\mathscr{R}$, S^* and $\mathscr{K}$ respectively in the open unit disc E, that is, functions satisfying the conditions $\operatorname{Re} f'(z) > 0$, $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ and $\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$ were proved by Janteng et al. [7, 8] and obtained the bounds as 4/9, 1, and 1/8 respectively. For the class $S^*(\psi)$ of Ma-Minda starlike functions, the exact bound of the second Hankel determinant was obtained by Lee et al. [9]. Choosing $q = 2$ and $n = p + 1$ in (1.2), we obtain the second Hankel determinant for the p-valent function (see [10]), namely

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2.$$

The case $q = 3$ appears to be much more difficult than the case $q = 2$. Very few papers have been devoted to the third Hankel determinant denoted by $H_3(1)$, obtained by choosing $q = 3$ and $n = 1$ in (1.2). Babalola [11] is the first one, who tried to estimate an upper bound to $|H_3(1)|$ for the classes $\mathscr{R}$, S^* and $\mathscr{K}$. Following this paper, Raza and Malik [12] obtained an upper bound for the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. Sudharsan et al. [13] derived an upper bound to $H_3(1)$ for a subclass of analytic functions. Bansal et al. [14] modified the upper bound for $|H_3(1)|$ for some of the classes estimated by Babalola [11] to some extent. Recently, Zaprawa [1] improved the results obtained by Babalola [11]. Further, Orhan and Zaprawa [15] obtained an upper bound for third Hankel determinant for the classes S^* and $\mathscr{K}$ functions of order alpha. Very recently, Kowalczyk et al. [16] estimated sharp upper bound to $|H_3(1)|$ for the class of convex functions $\mathscr{K}$ and showed as $|H_3(1)| \leqslant \frac{4}{135}$, which is far better than the bound obtained by Zaprawa [1]. Lecko et al. [17] calculated sharp bound for Hankel determinant of the third kind for starlike functions of order 1/2. For our discussion in this paper, we consider $H_3(p)$ for the values $q = 3$ and $n = p$ in (1.2), called as Hankel determinant of third order for the p-valent function given in (1.1), namely

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix} \qquad (a_p = 1).$$

Expanding the determinant, we have

$$H_3(p) = \big[a_p(a_{p+2}a_{p+4} - a_{p+3}^2) + a_{p+1}(a_{p+2}a_{p+3} - a_{p+1}a_{p+4}) + a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2)\big], \quad (1.3)$$

equivalently

$$H_3(p) = H_2(p+2) + a_{p+1}J_{p+1} + a_{p+2}H_2(p+1),$$

where $J_{p+1} = (a_{p+2}a_{p+3} - a_{p+1}a_{p+4})$ and $H_2(p+2) = (a_{p+2}a_{p+4} - a_{p+3}^2)$.

Motivated by the results obtained by different authors mentioned above and who are working in this direction (see [18, 19]), in particular the result obtained by Zaprawa [1] in finding an upper bound to the Hankel determinant of third kind for the subclass $\mathscr{R}$ of S, consisting of functions whose derivative has a positive real part (also called as bounded turning functions), introduced by Alexander in 1915 and a systematic study of properties of these functions was conducted by MacGregor [20], who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part. In the present paper, we are making an attempt to obtain an upper bound to $|H_3(p)|$, for the function f given in (1.1), when it belongs to certain subclass of analytic functions, defined as follows.

Definition 1.1. A function $f \in A_p$ is said to be in the class $I_p(\beta)$ (β is real) (see [21]), if it satisfies the condition

$$\operatorname{Re}\left\{(1-\beta)\frac{f(z)}{z^p} + \beta\frac{f'(z)}{pz^{p-1}}\right\} > 0, \quad z \in E - \{0\}. \tag{1.4}$$

1. Choosing $\beta = 1$ and $p = 1$, we obtain $I_1(1) = \mathscr{R}$.

2. Selecting $\beta = 1$, we get $I_p(1) = \mathscr{R}_p$, denotes the class of multivalent bounded turning functions.

In proving our result, we require a few sharp estimates in the form of Lemmas valid for functions with positive real part.

Let $\mathscr{P}$ denote the class of functions consisting of g, such that

$$g(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.5}$$

which are analytic in E and $\operatorname{Re} g(z) > 0$ for $z \in E$. Here g is called the Caratheodòry function [22].

Lemma 1.1 [2]. *If $g \in \mathscr{P}$, then the sharp estimate $|c_k - \mu c_k c_{n-k}| \leqslant 2$, holds for $n, k \in \mathbb{N}$, with $n > k$ and $\mu \in [0,1]$.*

Lemma 1.2 [3]. *If $g \in \mathscr{P}$, then the sharp estimate $|c_k - c_k c_{n-k}| \leqslant 2$, holds for $n, k \in \mathbb{N}$, where $n > k$.*

Lemma 1.3 [4]. *If $g \in \mathscr{P}$ then $|c_k| \leqslant 2$, for each $k \geqslant 1$ and the inequality is sharp for the function $g(z) = \frac{1+z}{1-z}$, $z \in E$.*

In order to obtain our result, we referred to the classical method devised by Libera and Zlotkiewicz [23, 24], used by several authors in the literature.

2. Main Result

Theorem 2.1. *If $f \in I_p(\beta)$ ($\beta \geqslant 1$ is real) with $p \in \mathbb{N}$, then*

$$|H_3(p)| \leqslant \left[\frac{4p^2\left(6p^6 + 60p^5\beta + 227p^4\beta^2 + 426p^3\beta^3 + 437p^2\beta^4 + 252p\beta^5 + 68\beta^6\right)}{(p+\beta)^2(p+2\beta)^3(p+3\beta)^2(p+4\beta)}\right].$$

$\triangleleft$ For the function $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in I_p(\beta)$, by virtue of Definition 1.1, there exists an analytic function $g \in \mathscr{P}$ in the open unit disc E with $g(0) = 1$ and $\operatorname{Re} g(z) > 0$ such that

$$(1-\beta)\frac{f(z)}{z^p} + \beta\frac{f'(z)}{pz^{p-1}} = g(z) \Leftrightarrow \left[(1-\beta)pf(z) + \beta f'(z) = pz^p g(z)\right]. \tag{2.1}$$

Replacing f' and g with their series expressions in (2.1), upon simplification, we obtain

$$a_{p+n} = \frac{pc_n}{p+n\beta}, \quad n, p \in \mathbb{N}. \tag{2.2}$$

Substituting the values of a_{p+1}, a_{p+2}, a_{p+3} and a_{p+4} from (2.2) in the functional given in (1.3), it simplifies to

$$\begin{aligned} \left|H_3(p)\right| = p^2 \Bigg[& \frac{c_2 c_4}{(p+2\beta)(p+4\beta)} - \frac{pc_2^3}{(p+2\beta)^3} - \frac{c_3^2}{(p+3\beta)^2} \\ & - \frac{pc_1^2 c_4}{(p+\beta)^2(p+4\beta)} + \frac{2pc_1c_2c_3}{(p+\beta)(p+2\beta)(p+3\beta)} \Bigg]. \end{aligned} \tag{2.3}$$

On grouping the terms in (2.3), in order to apply Lemmas, we have

$$\begin{aligned} \left|H_3(p)\right| = p^2 \Bigg[& \frac{pc_4(c_2 - c_1^2)}{(p+\beta)^2(p+4\beta)} - \frac{1}{(p+3\beta)^2} c_3 \left\{ c_3 - \frac{6pc_1c_2}{(p+\beta)(p+2\beta)} \right\} \\ & + \frac{pc_2(c_4 - c_2^2)}{(p+2\beta)^3} - \frac{2p^2 c_2(c_4 - c_1c_3)}{(p+\beta)(p+2\beta)(p+3\beta)^2} \\ & + \frac{(p^6 + 6p^5\beta + 3p^4\beta^2 - 30p^3\beta^3 - 36p^2\beta^4 + 24p\beta^5 + 36\beta^6)c_2c_4}{(p+\beta)^2(p+2\beta)^3(p+3\beta)^2(p+4\beta)} \Bigg]. \end{aligned} \tag{2.4}$$

Applying the triangle inequality in (2.4), we obtain

$$\begin{aligned} \left|H_3(p)\right| \leqslant p^2 \Bigg[& \frac{p|c_4||c_2 - c_1^2|}{(p+\beta)^2(p+4\beta)} + \frac{1}{(p+3\beta)^2} |c_3| \left| c_3 - \frac{6pc_1c_2}{(p+\beta)(p+2\beta)} \right| \\ & + \frac{p|c_2||c_4 - c_2^2|}{(p+2\beta)^3} + \frac{2p^2|c_2||c_4 - c_1c_3|}{(p+\beta)(p+2\beta)(p+3\beta)^2} \\ & + \frac{(p^6 + 6p^5\beta + 3p^4\beta^2 - 30p^3\beta^3 - 36p^2\beta^4 + 24p\beta^5 + 36\beta^6)|c_2||c_4|}{(p+\beta)^2(p+2\beta)^3(p+3\beta)^2(p+4\beta)} \Bigg]. \end{aligned} \tag{2.5}$$

Upon using the Lemmas given in 1.2, 1.3 and 1.4 in the inequality (2.5), it reduces to

$$\begin{aligned} \Big|H_3(p)\Big| \leqslant 4p^2 \Bigg[& \frac{p}{(p+\beta)^2(p+4\beta)} \\ & + \frac{1}{(p+3\beta)^2} + \frac{p}{(p+2\beta)^3} + \frac{2p^2}{(p+\beta)(p+2\beta)(p+3\beta)^2} \\ & + \frac{(p^6 + 6p^5\beta + 3p^4\beta^2 - 30p^3\beta^3 - 36p^2\beta^4 + 24p\beta^5 + 36\beta^6)c_2c_4}{(p+\beta)^2(p+2\beta)^3(p+3\beta)^2(p+4\beta)} \Bigg]. \end{aligned} \tag{2.6}$$

Further simplification, we obtain

$$|H_3(p)| \leqslant \left[\frac{4p^2(6p^6 + 60p^5\beta + 227p^4\beta^2 + 426p^3\beta^3 + 437p^2\beta^4 + 252p\beta^5 + 68\beta^6)}{(p+\beta)^2(p+2\beta)^3(p+3\beta)^2(p+4\beta)} \right]. \tag{2.7}$$

This completes the proof of our Theorem. ⊳

REMARK 2.1. Choosing $p = 1$ and $\beta = 1$ in the inequality (2.7), it coincides with the result obtained by Zaprawa [1].

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ОПРЕДЕЛИТЕЛЬ ГАНКЕЛЯ ТРЕТЬЕГО РОДА ДЛЯ НЕКОТОРОГО ПОДКЛАССА МНОГОВАЛЕНТНЫХ АНАЛИТИЧЕСКИХ ФУНКЦИЙ

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Аннотация. Целью данной статьи является получение (не точной) верхней границы для определителя Ганкеля третьего порядка для некоторого подкласса многовалентных (p-валентных) аналитических функций, определенных на открытом единичном диске E. Используя определители Теплица, мы можем оценить определитель Ганкеля третьего рода для нормированных многовалентных аналитических функций, принадлежащих этому подклассу. Однако, используя технику, принятую Сапpавой [1], т. е. группируя подходящие члены для применения лемм Хаями [2], Ливингстона [3] и Померенке [4], мы видим, что оценка методом Сапpавы точнее, чем при применении определителей Теплица.

Ключевые слова: p-валентная аналитическая функция, верхняя граница, третий определитель Ганкеля, положительная вещественная функция.

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