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GRAND MORREY TYPE SPACES#

S. G. Samko[1,2] and S. M. Umarkhadzhiev[2,3]

[1] University of Algarve, Faro 8005-139, Portugal;
[2] Kh. Ibragimov Complex Institute of the Russian Academy of Sciences,
21 a Staropromyslovskoe Hwy, Grozny 364051, Russia;
[3] Academy of Sciences of Chechen Republic, 13 Esambaev Av., Grosny 364024, Russia

E-mail: ssamko@ualg.pt, umsalaudin@gmail.com

Dedicated to the 75-th anniversary of Professor S. S. Kutateladze

Abstract. The so called grand spaces nowadays are one of the main objects in the theory of function spaces. Grand Lebesgue spaces were introduced by T. Iwaniec and C. Sbordone in the case of sets Ω with finite measure $|\Omega| < \infty$, and by the authors in the case $|\Omega| = \infty$. The latter is based on introduction of the notion of grandizer. The idea of "grandization" was also applied in the context of Morrey spaces. In this paper we develop the idea of grandization to more general Morrey spaces $L^{p,q,w}(\mathbb{R}^n)$, known as Morrey type spaces. We introduce grand Morrey type spaces, which include mixed and partial grand versions of such spaces. The mixed grand space is defined by the norm

$$\sup_{\varepsilon,\delta} \varphi(\varepsilon,\delta) \sup_{x\in E} \left(\int\limits_0^\infty w(r)^{q-\delta} b(r)^{\frac{\delta}{q}} \left(\int\limits_{|x-y|<r} \big|f(y)\big|^{p-\varepsilon} a(y)^{\frac{\varepsilon}{p}}\, dy \right)^{\frac{q-\delta}{p-\varepsilon}} \frac{dr}{r} \right)^{\frac{1}{q-\varepsilon}}$$

with the use of two grandizers a and b. In the case of grand spaces, partial with respect to the exponent q, we study the boundedness of some integral operators. The class of these operators contains, in particular, multidimensional versions of Hardy type and Hilbert operators.

Key words: Morrey type space, grand space, grand Morrey type space, grandizer, partial grandization, mixed grandization, homogeneous kernel, Hardy type operator, Hilbert operator.

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1. Introduction

Last decades there were widely investigated the so called grand Lebesgue spaces, introduced in [1]; see for instance, [2–5] and references therein, where such spaces and operators on them were studied in the case of finite measure underlying set. An approach to grand Lebesgue spaces on sets of infinite measure was suggested and developed in [6–10].

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That idea of "grandization" was also applied in the context of Morrey spaces defined by the norm

$$\|f\|_{L^{p,\varphi}(\Omega)} = \sup_{x\in\Omega} \left\| \left(\frac{1}{\varphi(r)} \int\limits_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} \right\|_{L^\infty(0,\, \mathrm{diam}\, \Omega)}, \tag{1}$$

we refer for instance to [11–14], see also references therein.

Our goal is to extend the notion of grandization to the spaces $L^{p,q,w}(\mathbb{R}^n)$, with the norm of the type (1), where L^∞-norm is replaced by L^q-norm, $1 \leqslant q < \infty$, see precise definitions in Section 2. Such spaces are usually referred to as Morrey type spaces. These spaces were introduced and studied in [15] and [16]. In the case $w(r) = r^{-\lambda}$, $\lambda > 0$, these space first appeared, though as an episode, in [17, p. 44]. For further studies of operators on $L^{p,q,w}(\mathbb{R}^n)$-spaces we refer, for instance, to [18, 19] and references therein, see also the surveying papers [20, 21] and [22].

We study various approaches to the grandization of Morrey type spaces, with respect to the exponents p and q. This includes partial grandization and mixed grandization. To this end, we deal with "grandizers" $a(y)$ and $b(r)$ in the corresponding variables, see Definitions 3.1, 3.2, 3.3.

We find conditions on the grandizers a and b, which ensure embedding of Morrey type spaces into the introduced grand Morrey type spaces.

In the case of partial grandization with respect the exponent q, we study, in grand Morrey type spaces, the boundedness of a certain class of integral operators K with a kernel homogeneous of degree $-n$. This class includes, in particular, multidimensional versions of Hardy type and Hilbert operators. Within the frameworks of generalized Morrey spaces, corresponding to the case $q = \infty$, a more general class of operators with homogeneous kernel was studied in [23].

We first study such operators in Morrey type spaces (not grand ones) and obtain sufficient conditions and also some necessary conditions for their boundedness. In fact, we obtain a result stronger than just boundedness: we estimate the norms $\|Kf\|_{L^{p,q,w}(\mathbb{R}^n)}$ via similar one-dimensional norms of spherical means of f. Then we apply the obtained results on the boundedness to grand Morrey type spaces.

In application to the Hardy operators with power weights, the obtained conditions have a form of criterion when $w(r) = r^{-\lambda}$, $\lambda > 0$.

The paper is organized as follows. Section 2 contains necessary definitions. In Section 3.1 we discuss varions ways of grandization of Morrey type spaces $L^{p,q,w}(\mathbb{R}^n)$. In Section 3.2 we provide conditions on grandizers ensuring embedding of Morrey type spaces into grand Morrey type spaces. In Section 4.1 we study the operators K in the Morrey type spaces $L^{p,q,w}(\mathbb{R}^n)$, and in Section 4.2 in grand Morrey type spaces $L_b^{p,q),w}(\mathbb{R}^n)$, both with application to the Hardy and Hilbert type operators.

2. Preliminaries

Following the known definitions, we introduce the spaces $L^{p,q,w}(\mathbb{R}^n)$, defined by the norm

$$\|f\|_{L^{p,q,w}(\mathbb{R}^n)} := \sup_{x\in E} \left(\int\limits_0^\infty w(r)^q \left(\int\limits_{|x-y|<r} |f(y)|^p \, dy \right)^{\frac{q}{p}} \frac{dr}{r} \right)^{\frac{1}{q}}, \tag{2}$$

where E is an arbitrary set in $\mathbb{R}^n$, $E \subseteq \mathbb{R}^n$, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, $w \in \Omega_q(\mathbb{R}_+)$,

$$\Omega_q(\mathbb{R}_+) := \left\{ w : w \text{ is a weight and } \int\limits_t^\infty \frac{w(r)^q}{r}\, dr < \infty \text{ for some } t > 0 \right\}.$$

In the cases $E = \{0\}$ and $E = \mathbb{R}^n$ we have the *local* and *global* Morrey type spaces, respectively. We do not indicate dependence of the space on the choice of the set E, since it is unessential for our consideration. Only in Section 4 we choose $E = \{0\}$.

In the special case $w(r) = r^{-\lambda}$, $\lambda > 0$, we also use the notation

$$L^{p,q,\lambda}(\mathbb{R}^n) := L^{p,q,w}(\mathbb{R}^n)\big|_{w=r^{-\lambda}}$$

without danger of confusion of notation.

For a function $w(r)$ defined on $\mathbb{R}_+$, we will use the notation

$$w^*(t) := \sup_{r\in\mathbb{R}_+} \frac{w(tr)}{w(r)} \quad \text{and} \quad w_*(t) := \inf_{x\in\mathbb{R}_+} \frac{w(tr)}{w(r)}, \quad t > 0.$$

Observe that $w_*\left(\frac{1}{t}\right) = \frac{1}{w^*(t)}$. Obviously $w^*(r) = w_*(r) = w(r)$, when $w(r) = r^{-\lambda}$, $\lambda \in \mathbb{R}$. However, in the case of piece-wise power function

$$w_{\lambda,\gamma}(r) = r^{-\lambda}(1+r)^{\lambda-\gamma} \sim \begin{cases} r^{-\lambda}, & r < 1, \\ r^{-\gamma}, & r > 1, \end{cases}$$

where $\lambda, \gamma \in \mathbb{R}$, we have a gap between w^* and w_*:

$$w^*_{\lambda,\gamma}(r) = \begin{cases} r^{-\max\{\lambda,\gamma\}}, & r < 1, \\ r^{-\min\{\lambda,\gamma\}}, & r > 1, \end{cases} \quad \text{and} \quad (w_{\lambda,\gamma})_*(r) = \begin{cases} r^{-\min\{\lambda,\gamma\}}, & r < 1, \\ r^{-\max\{\lambda,\gamma\}}, & r > 1, \end{cases} \tag{3}$$

see e. g. [24, p. 715].

3. Grand Morrey Type Spaces

3.1. Grandization of Morrey type spaces. Everywhere in the sequel, $a = a(y)$ and $b = b(r)$ are weights on $\mathbb{R}^n$ and $\mathbb{R}_+$, respectively.

Definition 3.1. Let

$$1 < p < \infty, \quad 1 < q < \infty, \quad w \in \Omega_q(\mathbb{R}_+) \tag{4}$$

and

$$\varphi \in L^\infty(R_{p,q}), \quad \varphi(\varepsilon,\delta) > 0 \text{ for } (\varepsilon,\delta) \in R_{p,q} \quad \text{and} \quad \lim_{(\varepsilon,\delta)\to(0,0)} \varphi(\varepsilon,\delta) = 0,$$

where $R_{p,q} := \{(\varepsilon,\delta) \in \mathbb{R}^2_+ : 0 < \varepsilon < p-1, 0 < \delta < q-1\}$.

We define the mixed grand Morrey type space $L^{p),q),w}_{a,b}(\mathbb{R}^n)$ as the space of functions with the finite norm

$$\|f\|_{L^{p),q),w}_{a,b}(\mathbb{R}^n)} := \sup_{(\varepsilon,\delta)\in R_{p,q}} \varphi(\varepsilon,\delta) \sup_{x\in E} \left(\int\limits_0^\infty w(r)^{q-\delta} b(r)^{\frac{\delta}{q}} \left(\int\limits_{|x-y|<r} |f(y)|^{p-\varepsilon} a(y)^{\frac{\varepsilon}{p}} dy \right)^{\frac{q-\delta}{p-\varepsilon}} \frac{dr}{r} \right)^{\frac{1}{q-\delta}}.$$

We also say that $L^{p),q),w}_{a,b}(\mathbb{R}^n)$ is *the mixed gandization* of the space $L^{p,q,w}(\mathbb{R}^n)$.

Note that mixed coordinate-wise grandization of mixed Lebesgue spaces was studied in [25].

Definition 3.2. We define partial grandizations $L_a^{p),q,w}(\mathbb{R}^n)$, $1 < p < \infty$, $1 \leqslant q < \infty$, and $L_b^{p,q),w}(\mathbb{R}^n)$, $1 \leqslant p < \infty$, $1 < q < \infty$, of the space $L^{p,q,w}(\mathbb{R}^n)$ as the spaces of functions with the finite norm

$$\|f\|_{L_a^{p),q,w}(\mathbb{R}^n)} := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon) \sup_{x\in E} \left(\int_0^\infty w(r)^q \left(\int_{|x-y|<r} |f(y)|^{p-\varepsilon} a(y)^{\frac{\varepsilon}{p}}\, dy \right)^{\frac{q}{p-\varepsilon}} \frac{dr}{r} \right)^{\frac{1}{q}},$$

where $\varphi \in L^\infty(0, p-1)$, $\varphi(\varepsilon) > 0$ and $\lim_{\varepsilon\to 0} \varphi(\varepsilon) = 0$, and

$$\|f\|_{L_b^{p,q),w}(\mathbb{R}^n)} := \sup_{0<\delta<q-1} \varphi(\delta) \sup_{x\in E} \left(\int_0^\infty w(r)^{q-\delta} b(r)^{\frac{\delta}{q}} \left(\int_{|x-y|<r} |f(y)|^p\, dy \right)^{\frac{q-\delta}{p}} \frac{dr}{r} \right)^{\frac{1}{q-\delta}},$$

where $\varphi \in L^\infty(0, q-1)$, $\varphi(\delta) > 0$ and $\lim_{\delta\to 0} \varphi(\delta) = 0$, respectively.

Definitions 3.1 and 3.2 may be generalized in the following direction. Let $U \subseteq R_{p,q}$ be an arbitrary measurable set of points in $R_{p,q}$, such that $(0,0)$ is a limiting point for U.

Definition 3.3. Let $1 < p < \infty$, $1 < q < \infty$, $\varphi \in L^\infty(U)$, $\varphi(\varepsilon,\delta) > 0$ for $(\varepsilon,\delta) \in U$ and $\lim_{U\ni(\varepsilon,\delta)\to(0,0)} \varphi(\varepsilon,\delta) = 0$. We define the U-grandization ${}^U L_{a,b}^{p),q),w}(\mathbb{R}^n)$ of the Morrey type space as the space of functions with finite norm

$$\|f\|_{{}^U L_{a,b}^{p),q),w}(\mathbb{R}^n)} := \sup_{(\varepsilon,\delta)\in U} \varphi(\varepsilon,\delta) \sup_{x\in E} \left(\int_0^\infty w(r)^{q-\delta} b(r)^{\frac{\delta}{q}} \left(\int_{|x-y|<r} |f(y)|^{p-\varepsilon} a(y)^{\frac{\varepsilon}{p}}\, dy \right)^{\frac{q-\delta}{p-\varepsilon}} \frac{dr}{r} \right)^{\frac{1}{q-\delta}}.$$

Under the choice $U = R_{p,q}$ we have the mixed grand Morrey type space introduced in Definition 3.1. Partial grandization from Definition 3.2 formally correspond to the case where $U = \{(\varepsilon,\delta) : 0 < \varepsilon < p-1, \delta = 0\}$ and $U = \{(\varepsilon,\delta) : \varepsilon = 0, 0 < \delta < q-1\}$.

In the sequel we use the notation

$$\mathfrak{N}(f;\varepsilon,\delta) := \sup_{x\in E} \left(\int_0^\infty w(r)^{q-\delta} b(r)^{\frac{\delta}{q}} \left(\int_{|x-y|<r} |f(y)|^{p-\varepsilon} a(y)^{\frac{\varepsilon}{p}}\, dy \right)^{\frac{q-\delta}{p-\varepsilon}} \frac{dr}{r} \right)^{\frac{1}{q-\delta}}$$

for brevity, assuming that p, q, w, a and b are fixed. In case of partial grandization we have $\mathfrak{N}(f;\varepsilon,0)$ and $\mathfrak{N}(f;0,\delta)$ for $b \equiv 1$ and $a \equiv 1$, respectively.

Lemma 3.1. I. *Let* $(\varepsilon_0,\delta_0) \in U_{\varepsilon_0,\delta_0} := \{(\varepsilon,\delta) \in U : 0 < \varepsilon < \varepsilon_0, 0 < \delta < \delta_0\}$, $a \in L^1(\mathbb{R}^n)$ *and* $b \in L^1\big(\mathbb{R}_+, \frac{dr}{r}\big)$, *then*

$$\sup_{(\varepsilon,\delta)\in U} \varphi(\varepsilon,\delta)\, \mathfrak{N}(f;\varepsilon,\delta) \leqslant C \sup_{(\varepsilon,\delta)\in U_{\varepsilon_0,\delta_0}} \varphi(\varepsilon,\delta)\, \mathfrak{N}(f;\varepsilon,\delta),$$

where $C = C\big(\varepsilon_0, \delta_0, \|a\|_{L^1(\mathbb{R}^n)}, \|b\|_{L^1(\mathbb{R}_+, \frac{dr}{r})}\big)$.

II. *In the case of partial grandization with respect to the variable* r, *similarly*

$$\sup_{0<\delta<q-1} \varphi(\delta)\, \mathfrak{N}(f;0,\delta) \leqslant C \sup_{0<\delta<\delta_0} \varphi(\delta)\, \mathfrak{N}(f;0,\delta),$$

where $C = C\big(\delta_0, \|b\|_{L^1(\mathbb{R}_+, \frac{dr}{r})}\big)$.

◁ We have to estimate

$$\sup_{(\varepsilon,\delta)\in U\setminus U_{\varepsilon_0,\delta_0}} \varphi(\varepsilon,\delta)\,\mathfrak{N}(f;\varepsilon,\delta) = \max\big\{S_1, S_{12}, S_2\big\},$$

where

$$S_1 := \sup_{(\varepsilon,\delta)\in U_1} \varphi(\varepsilon,\delta)\,\mathfrak{N}(f;\varepsilon,\delta), \quad S_2 := \sup_{(\varepsilon,\delta)\in U_2} \varphi(\varepsilon,\delta)\,\mathfrak{N}(f;\varepsilon,\delta),$$

$$S_{12} := \sup_{(\varepsilon,\delta)\in U_{12}} \varphi(\varepsilon,\delta)\,\mathfrak{N}(f;\varepsilon,\delta), \quad U_1 := \big\{(\varepsilon,\delta)\in U : \varepsilon<\varepsilon_0,\ \delta>\delta_0\big\},$$

$$U_2 := \big\{(\varepsilon,\delta)\in U : \varepsilon>\varepsilon_0,\ \delta<\delta_0\big\}, \quad U_{12} := \big\{(\varepsilon,\delta)\in U : \varepsilon>\varepsilon_0,\ \delta>\delta_0\big\}.$$

For S_{12}, we apply the Hölder inequalities in the variables y and r with the expends $\frac{p-\varepsilon_0}{p-\varepsilon}$ and $\frac{q-\delta_0}{q-\delta}$, respectively and obtain

$$S_{12} \leqslant \sup_{(\varepsilon,\delta)\in U_{12}} \varphi(\varepsilon,\delta)\big\|a\big\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p-\varepsilon}-\frac{1}{p-\varepsilon_0}} \big\|b\big\|_{L^1(\mathbb{R}_+,\frac{dr}{r})}^{\frac{1}{q-\delta}-\frac{1}{q-\delta_0}}$$

$$\times\left(\int_0^\infty w(r)^{q-\delta_0} b(r)^{\frac{\delta_0}{q}} \left(\int_{|x-y|<r} \big|f(y)\big|^{p-\varepsilon_0} a(y)^{\frac{\varepsilon_0}{p}}\,dy\right)^{\frac{q-\delta_0}{p-\varepsilon_0}} \frac{dr}{r}\right)^{\frac{1}{q-\delta_0}}$$

$$\leqslant \frac{\sup\limits_{(\varepsilon,\delta)\in U_{12}} \varphi(\varepsilon,\delta)\big\|a\big\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p-\varepsilon}-\frac{1}{p-\varepsilon_0}} \big\|b\big\|_{L^1(\mathbb{R}_+,\frac{dr}{r})}^{\frac{1}{q-\delta}-\frac{1}{q-\delta_0}}}{\sup\limits_{(\varepsilon,\delta)\in U_{\varepsilon_0,\delta_0}} \varphi(\varepsilon,\delta)} \sup_{(\varepsilon,\delta)\in U_{\varepsilon_0,\delta_0}} \varphi(\varepsilon,\delta)\,\mathfrak{N}(\varepsilon,\delta).$$

Estimation of S_1 and S_2 is easier via similar use of the Hölder inequality in one variable only. We omit details. ▷

3.2. Embedding of Morrey type spaces into grand Morrey type spaces.

Lemma 3.2. *If*

$$C_0 := \sup_{(\varepsilon,\delta)\in U} \varphi(\varepsilon,\delta) \sup_{x\in E} \int_0^\infty b(r) A(x,r)^{\frac{q}{p}\frac{\frac{q}{\delta}-1}{\frac{p}{\varepsilon}-1}} \frac{dr}{r} < \infty, \tag{5}$$

where $A(x,r) := \int_{|x-y|<r} a(y)dy$, *then*

$$L^{p,q,w}(\mathbb{R}^n) \hookrightarrow {}^U L_{a,b}^{p),q),w}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{{}^U L_{a,b}^{p),q),w}(\mathbb{R}^n)} \leqslant C_0 \|f\|_{L^{p,q,w}(\mathbb{R}^n)}. \tag{6}$$

◁ It suffices to apply the Hölder inequalities with the exponents $\frac{p}{p-\varepsilon}$ and $\frac{q}{q-\delta}$ in the inner and outer integrals in the definition of the norm in Definition 3.3. ▷

Theorem 3.1. *The conditions* $a \in L^1(\mathbb{R}^n)$ *and* $b \in L^1\big(\mathbb{R}_+, \frac{dr}{r}\big)$ *are sufficient for the embedding* (6) *for any choice of the set* U. *The condition* $b \in L^1\big(\mathbb{R}_+, \frac{dr}{r}\big)$ *is sufficient for embedding of* $L^{p,q,w}(\mathbb{R}^n)$ *into the partial grand space* $L_b^{p,q),w}(\mathbb{R}^n)$.

◁ It is easy to see that the change of $a(y)$ by $\lambda a(y)$, $\lambda = \text{const} > 0$, keeps the grand space (up to equivalents of norms). Consequently, we may assume that $\|a\|_{L^1(\mathbb{R}^n)} = 1$. Then $A(r) \leqslant 1$ and the statement follows from (5).

The embedding into the partial grand space follows from Lemma 3.1. ▷

REMARK 3.1. Note that the condition (5) does not assume that $a \in L^1(\mathbb{R}^n)$, $b \in L^1\big(\mathbb{R}_+, \frac{dr}{r}\big)$. In particular, in the case of "partial" grandization in the direction $\delta = \frac{q}{p}\varepsilon$, the embedding (6) holds, if $b(r)A(r)^{q/p} \in L^1\big(\mathbb{R}_+, \frac{dr}{r}\big)$.

By $AC_{\mathrm{loc}}(\mathbb{R}_+)$ we denote the class of functions $\psi : \mathbb{R}_+ \to \mathbb{R}$, absolutely continuous on every interval $[0, N]$, $N > 0$.

In Theorem 3.2 we impose the condition

$$\sup_{0<\delta<q-1} \varphi(\delta) \left(\int_0^\infty \frac{b(t)^{\frac{\delta}{q}}}{t}\, dt \right)^{\frac{1}{q-\delta}} < \infty \tag{7}$$

on the grandizer b.

Lemma 3.3. *Let $\varphi(\delta) \leqslant c\delta^{1/q}$. Grandizer b of the form*

$$b(r) = \frac{r^\mu}{(1+r)^\nu}\, \ell_{\tau,\sigma}(r),$$

where

$$\ell_{\tau,\sigma}(r) = \begin{cases} \ln^\tau \dfrac{e}{r}, & r < 1, \\ \ln^\sigma r, & r \geqslant 1, \end{cases}$$

satisfies the condition (7) *if $0 < \mu < \nu < \infty$ and $\tau, \sigma \in \mathbb{R}$.*

$\lhd$ Let first $\tau = \sigma = 0$. We have

$$\int_0^r \frac{b(r)^{\frac{\delta}{q}}}{r}\, dr = \int_0^\infty \frac{r^{\frac{\mu\delta}{q}-1}}{(1+r)^{\frac{\nu\delta}{q}}}\, dr = \frac{\Gamma\big(\frac{\mu\delta}{q}\big)\,\Gamma\big(\frac{(\nu-\mu)\delta}{q}\big)}{\Gamma\big(\frac{\nu\delta}{q}\big)} \sim \frac{c}{\delta}, \tag{8}$$

so that (7) is satisfied.

In the case of the presence of the logarithmic factor $\ell_{\tau,\sigma}(r)$, it suffices to observe that $\ell_{\tau,\sigma}(r)$ is dominated by $\frac{r^{\eta_1}}{(1+r)^{\eta_2}}$ with arbitrarily small exponents $\eta_1 > \eta_2 > 0$ and the estimate by $\frac{c}{\delta}$ in (8) does not depend on μ and ν, provided $0 < \mu < \nu$. $\rhd$

Theorem 3.2. *Let $1 \leqslant p < \infty$ and $1 < q < \infty$. Let the grandizer b satisfy the condition* (7). *The embedding $L^{p,q,w}(\mathbb{R}^n) \subset L_b^{p,q),w}(\mathbb{R}^n)$ is strict if $w(r)$ satisfies one of the following assumptions:*

i) w is decreasing, $\lim_{r\to 0} w(r) = \infty$ and $\frac{1}{w} \in AC_{\mathrm{loc}}(\mathbb{R}_+)$;

ii) there exist numbers ν_1 and ν_2, $0 < \nu_2 < \nu_1 \leqslant \infty$, such that $w(r)r^{\nu_1}$ is almost increasing and $w(r)r^{\nu_2}$ is almost decreasing.

The corresponding counterexample is $f(x) = f_0(|x|)$, where

$$f_0(r) = \left(r^{\frac{1-n}{p}} \frac{d}{dr} \left[\frac{1}{w(r)^p} \right] \right)^{\frac{1}{p}}$$

for the case i) and

$$f_0(r) = \frac{1}{r^{\frac{n}{p}} w(r)}$$

in the case ii).

$\lhd$ *The case i*). For $f = f_0(|x|)$ we have

$$\int\limits_{|y|<r} \left|f_0(|y|)\right|^p dy = C \int\limits_0^r \frac{d}{d\rho}\left[\frac{1}{w(\rho)^p}\right] d\rho, \quad C = \left|S^{n-1}\right|.$$

By the absolute continuity of $\frac{1}{w^p}$ we obtain that

$$\left(\int\limits_{|y|<r} \left|f_0(|y|)\right|^p dy\right)^{\frac{1}{p}} = \frac{C^{\frac{1}{p}}}{w(r)}.$$

Therefore,

$$\mathfrak{N}(f;0,\delta) = C^{\frac{1}{p}} \left(\int\limits_0^\infty \frac{b(r)^{\frac{\delta}{q}}}{r}\, dr\right)^{\frac{1}{q-\delta}}$$

from which it follows that $f \in L_b^{p,q),w}(\mathbb{R}^n)$, but $f \notin L^{p,q,w}(\mathbb{R}^n)$.

The case ii). In this case we have

$$\int\limits_{|y|<r} |f(y)|^p\, dy = C \int\limits_0^r \frac{d\rho}{\rho w(\rho)^p}.$$

From the assumptions in *ii*) it is easy to obtain that

$$\frac{c_1}{w(r)^p} \leqslant \int\limits_0^r \frac{d\rho}{\rho w(\rho)^p} \leqslant \frac{c_2}{w(r)^p}. \tag{9}$$

From the equivalence (9) we obtain that

$$\mathfrak{N}(f;0,\delta) \sim \left(\int\limits_0^\infty \frac{b(r)^{\frac{\delta}{q}}}{r}\, dr\right)^{\frac{1}{q-\delta}},$$

which completes the proof. $\rhd$

4. Operators with Homogenous Kernel

In this section we choose $E = \{0\}$ in the definition of the space $L^{p,q,w}(\mathbb{R}^n)$.

We consider integral operators

$$Kf(x) = \int\limits_{\mathbb{R}^n} \mathscr{K}(|x|,|y|) f(y)\, dy,$$

where the kernel is homogeneous of degree $-n$, i. e.

$$\mathscr{K}(t|x|, t|y|) = t^{-n}\mathscr{K}(|x|,|y|).$$

4.1. Operator K in Morrey type spaces. In the sequel we use the notation

$$\Pi_t f(x) := f(tx), \quad x \in \mathbb{R}^n,\ t > 0.$$

It is not hard to check that

$$\frac{w_*\left(\frac{1}{t}\right)}{t^{\frac{n}{p}}}\, \|f\|_{L^{p,q,w}(\mathbb{R}^n)} \leqslant \|\Pi_t f\|_{L^{p,q,w}(\mathbb{R}^n)} \leqslant \frac{w^*\left(\frac{1}{t}\right)}{t^{\frac{n}{p}}}\, \|f\|_{L^{p,q,w}(\mathbb{R}^n)}. \tag{10}$$

If $w(r) = r^{-\lambda}$, then $w^*(t) = w_*(t) = t^{-\lambda}$ and $\|\Pi_t\|_{L^{p,q,w}(\mathbb{R}^n)} = t^{\lambda - n/p}$.

In the theorem below we also use the notation

$$\varkappa^*(n) := \left|S^{n-1}\right| \int\limits_0^\infty s^{\frac{n}{p'}-1} \left|\mathscr{K}(1,s)\right| w^*\left(\frac{1}{s}\right) ds$$

and

$$\varkappa_*(n) := \left|S^{n-1}\right| \int\limits_0^\infty s^{\frac{n}{p'}-1} \left|\mathscr{K}(1,s)\right| w_*\left(\frac{1}{s}\right) ds,$$

where $|S^{n-1}|$ should be replaced by 1 in the one-dimensional case of $\mathbb{R}_+$.

The following one-dimensional theorem is an immediate consequence of (10).

Theorem 4.1. *Let $1 \leqslant p < \infty$, $1 \leqslant q < \infty$ and $w \in \Omega_q(\mathbb{R}_+)$. The condition $\varkappa^*(1) < \infty$ is sufficient for the boundedness of the operator*

$$Kf(x) = \int\limits_0^\infty \mathscr{K}(x,y) f(y)\, dy, \quad x \in \mathbb{R}_+,$$

where $\mathscr{K}(tx,ty) = t^{-1}\mathscr{K}(x,y)$, $t > 0$, in the space $L^{p,q,w}(\mathbb{R}_+)$ and

$$\|Kf\|_{L^{p,q,w}(\mathbb{R}_+)} \leqslant \varkappa^*(1) \|f\|_{L^{p,q,w}(\mathbb{R}_+)}. \tag{11}$$

◁ We have

$$Kf(x) = \int\limits_0^\infty \mathscr{K}(1,y) f(xy)\, dy.$$

Then by the Minkowsky inequality we obtain

$$\|Kf\|_{L^{p,q,w}(\mathbb{R}_+)} \leqslant \int\limits_0^\infty K(1,y) \|\Pi_y f\|_{L^{p,q,w}(\mathbb{R}_+)}\, dy,$$

whence (11) follows by (10). ▷

For the multi-dimensional case, in the next theorem we provide a statement stronger than just the boundedness in the space $L^{p,q,w}(\mathbb{R}^n)$. More precisely, we estimate the norm $\|Kf\|_{L^{p,q,w}(\mathbb{R}^n)}$ via one-dimensional norms of spherical means of f.

Theorem 4.2. *Let $1 \leqslant p < \infty$, $1 \leqslant q < \infty$ and $w \in \Omega_q(\mathbb{R}_+)$. If $\varkappa^*(n) < \infty$, $n \geqslant 1$, then*

$$\|Kf\|_{L^{p,q,w}(\mathbb{R}^n)} \leqslant \left|S^{n-1}\right|^{\frac{1}{p}} \varkappa^*(n) \left(\int\limits_0^\infty \frac{w(r)^q}{r} \left(\int\limits_0^r t^{n-1} |\Phi(t)|^p\, dt \right)^{\frac{q}{p}} dr \right)^{\frac{1}{q}}, \tag{12}$$

where

$$\Phi(t) = \frac{1}{|S^{n-1}|} \int\limits_{S^{n-1}} f(t\sigma)\, d\sigma.$$

$\lhd$ Note that Kf is a radial function for any f. It is easily seen that for any radial function $g(x) = G(|x|)$ one has

$$\|g\|_{L^{p,q,w}(\mathbb{R}^n)} = \left|S^{n-1}\right|^{\frac{1}{p}} \left\|G_p\right\|_{L^{p,q,w}(\mathbb{R}_+)}, \tag{13}$$

where $G_p(\rho) = \rho^{(n-1)/p} G(\rho)$.

Furthermore, passing to polar coordinates we have

$$Kf(x) = \left|S^{n-1}\right| \int\limits_0^\infty t^{\frac{n-1}{p'}} \mathscr{K}(|x|, t) \Phi_p(t)\, dt,$$

where

$$\Phi_p(t) = \frac{t^{\frac{n-1}{p}}}{|S^{n-1}|} \int\limits_{S^{n-1}} f(t\sigma)\, d\sigma.$$

Then by (13) we have

$$\|Kf\|_{L^{p,q,w}(\mathbb{R}^n)} = \left|S^{n-1}\right|^{1+\frac{1}{p}} \left\|K_1 \Phi_p\right\|_{L^{p,q,w}(\mathbb{R}_+)},$$

where

$$K_1\Phi_p(\rho) = \int\limits_0^\infty \mathscr{K}_1(\rho, t) \Phi_p(t)\, dt, \quad \mathscr{K}_1(\rho, t) = \rho^{\frac{n-1}{p}} t^{\frac{n-1}{p'}} \mathscr{K}(\rho, t).$$

The kernel $K_1(\rho, t)$ is homogeneous of degree -1, i. e. $K_1(s\rho, st) = s^{-1} K_1(\rho, t)$, $s > 0$. Therefore, we can apply Theorem 4.1 and obtain (12) after easy calculation. $\rhd$

Remark 4.1. The estimate (12) is stronger than the boundedness in $L^{p,q,w}(\mathbb{R}^n)$. Indeed,

$$\left(\int\limits_0^\infty \frac{w(r)^q}{r} \left(\int\limits_0^r t^{n-1} |\Phi(t)|^p\, dt \right)^{\frac{q}{p}} dr \right)^{\frac{1}{q}} \leqslant \left|S^{n-1}\right|^{-\frac{1}{p}} \left\|f\right\|_{L^{p,q,w}(\mathbb{R}^n)} \tag{14}$$

by Jensen inequality

$$\left(\frac{1}{|S^{n-1}|} \int\limits_{S^{n-1}} f(\rho\sigma)\, d\sigma \right)^p \leqslant \frac{1}{|S^{n-1}|} \int\limits_{S^{n-1}} |f(\rho\sigma)|^p\, d\sigma.$$

Clearly, the left-hand side in (14) may be finite when the right-hand side is infinite (e. g. when $f(x) = f_1(\rho) f_2(\sigma)$, $x = \rho\sigma$, $|\sigma| = 1$, with $f_2 \in L^1(S^{n-1})$, but $f_2 \notin L^p(S^{n-1})$).

In the necessity part of Theorem 4.3 we shall use the following minimizing sequence

$$f_\varepsilon(x) = \frac{\mu_\varepsilon(|x|)}{|x|^{\frac{n}{p}} w(|x|)}, \quad \text{where } \mu_\varepsilon(r) = \begin{cases} r^\varepsilon, & r < 1, \\ r^{-\varepsilon}, & r > 1, \end{cases} \quad \varepsilon > 0.$$

Lemma 4.1. *Let w satisfy the condition that $r^\delta w(r)$ is almost decreasing for some $\delta > 0$. Then*

$$\int_0^r \frac{\mu_{\varepsilon p}(t)}{tw(t)^p}\,dt \leqslant \frac{c\mu_{\varepsilon p}(r)}{w(r)^p}$$

for all $\varepsilon \in [0,\varepsilon_0]$, where $0 < \varepsilon_0 < \delta$ and $c = c(\varepsilon_0)$ does not depend on ε.

$\lhd$ The proof is straightforward. $\rhd$

Theorem 4.3. *Let $1 \leqslant p < \infty$, $1 \leqslant q < \infty$ and $w \in \Omega_q(\mathbb{R}_+)$. If $\varkappa^*(n) < \infty$, then*

$$\|Kf\|_{L^{p,q,w}(\mathbb{R}^n)} \leqslant \varkappa^*(n)\|f\|_{L^{p,q,w}(\mathbb{R}^n)}.$$

If $\mathscr{K}(|x|,|y|) \geqslant 0$ and w satisfies the assumption of Lemma 4.1, then the condition $\varkappa_*(n) < \infty$ is necessary for such a boundedness; in particular, when $w(r) = r^{-\lambda}$, $\lambda > 0$, the operator K is bounded if and only if

$$\int_0^\infty t^{\frac{n}{p'}+\lambda-1}\mathscr{K}(1,t)\,dt < \infty.$$

$\lhd$ Sufficiency of the condition $\varkappa^*(n) < \infty$ follows from Theorem 4.2 by Remark 4.1.

To prove the necessity, we choose $f(x) = f_\varepsilon(x)$. By using Lemma 4.1, it is easy to check that $f_\varepsilon(x) \in L^{p,q,w}(\mathbb{R}^n)$ for all $\varepsilon \in (0,\varepsilon_0]$.

We have

$$Kf_\varepsilon(x) = \int_{\mathbb{R}^n} \mathscr{K}(1,|y|)\, f_\varepsilon(|x|y)\,dy = \frac{1}{|x|^{\frac{n}{p}}}\int_{\mathbb{R}^n} \frac{\mathscr{K}(1,|y|)}{|y|^{\frac{n}{p}} w(|x|\cdot|y|)}\,\mu_\varepsilon(|x|\cdot|y|)\,dy$$
$$\geqslant \frac{1}{|x|^{\frac{n}{p}} w(|x|)}\int_{\mathbb{R}^n} \frac{\mathscr{K}(1,|y|)}{|y|^{\frac{n}{p}}}\,w_*(|y|)\mu_\varepsilon(|x|\cdot|y|)\,dy.$$

It is easy to check that $\mu_\varepsilon(r\rho) \geqslant \mu_\varepsilon(r)\mu_\varepsilon(\rho)$, so that

$$Kf_\varepsilon(x) \geqslant \varkappa_*(n,\varepsilon) f_\varepsilon(x),$$

where

$$\varkappa_*(n,\varepsilon) = \int_{\mathbb{R}^n} \frac{\mathscr{K}(1,|y|)}{|y|^{\frac{n}{p}}}\,w_*(|y|)\mu_\varepsilon(|y|)\,dy$$
$$= |S^{n-1}|\left(\int_0^1 \rho^{\frac{n}{p'}+\varepsilon-1}\mathscr{K}(1,\rho)w_*\left(\frac{1}{\rho}\right)d\rho + \int_1^\infty \rho^{\frac{n}{p'}-\varepsilon-1}\mathscr{K}(1,\rho)w_*\left(\frac{1}{\rho}\right)d\rho\right).$$

Hence

$$\|K\| \geqslant \varkappa_*(n,\varepsilon).$$

It remains to apply Fatou theorem when passin to the limit as $\varepsilon \to 0$. $\rhd$

In the corollary below we consider *the Hardy operators*

$$H^\alpha f(x) = |x|^{\alpha-n}\int_{|y|<|x|} \frac{f(y)}{|y|^\alpha}\,dy \quad \text{and} \quad \mathscr{H}^\beta f(x) = |x|^\beta \int_{|y|>|x|} \frac{f(y)}{|y|^{n+\beta}}\,dy$$

and *the Hilbert type operator*

$$\mathbb{H}f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x|^n + |y|^n}\,dy$$

as examples of the operator K.

Corollary 4.1. *The operators H^α and $\mathscr{H}^\beta$ are bounded in the space $L^{p,q,\lambda}(\mathbb{R}^n)$, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, $\lambda > 0$, if and only if $\alpha < \frac{n}{p'} + \lambda$ and $\beta > \lambda - \frac{n}{p}$, respectively, and $\|H^\alpha\| = \frac{|S^{n-1}|}{\frac{n}{p}+\lambda-\alpha}$ and $\|\mathscr{H}^\beta\| = \frac{|S^{n-1}|}{\frac{n}{p}+\beta-\lambda}$.*

The operator $\mathbb{H}$ is bounded in the space $L^{p,q,\lambda}(\mathbb{R}^n)$, $1 \leqslant q < \infty$, $\lambda > 0$, if and only if $1 \leqslant p < \frac{n}{\lambda}$ and

$$\|\mathbb{H}\| = \frac{|S^{n-1}|}{n}\,\Gamma\left(\frac{1}{p'} + \frac{\lambda}{n}\right)\Gamma\left(\frac{1}{p} - \frac{\lambda}{n}\right).$$

$\lhd$ In the case of the operator H^α we have

$$\mathscr{K}(1,t) = \begin{cases} t^\alpha, & t < 1, \\ 0, & t > 1, \end{cases}$$

so that $\varkappa^*(n) = \varkappa_*(n) = \frac{|S^{n-1}|}{\frac{n}{p}+\lambda-\alpha}$. Arguments for $\mathscr{H}^\beta$ are similar.

For the operator $\mathbb{H}$ we have

$$\varkappa^*(n) = \varkappa_*(n) = \left|S^{n-1}\right| \int\limits_0^\infty \frac{t^{\frac{n}{p'}+\lambda-1}}{1+t^n}\,dt,$$

where it remains to pass the Beta function via the change $\frac{1}{1+t^n} \to t$. $\rhd$

4.2. Operator K in grand Morrey type spaces.

Lemma 4.2. *Let $f_0(x) = \frac{1}{|x|^{n/p} w(|x|)}$ and $K(|x|,|y|) \geqslant 0$. Then*

$$Kf_0(x) \geqslant \varkappa_*(n) f_0(x).$$

$\lhd$ We have

$$Kf_0(x) = \int\limits_{\mathbb{R}^n} \mathscr{K}(|x|,|y|) \frac{dy}{|y|^{\frac{n}{p}} w(|y|)} = \frac{|S^{n-1}|}{|x|^{\frac{n}{p}}} \int\limits_0^\infty \rho^{\frac{n}{p'}-1} \mathscr{K}(1,\rho) \frac{d\rho}{w(\rho|x|)}$$

$$\geqslant \frac{|S^{n-1}|}{|x|^{\frac{n}{p}} w(|x|)} \int\limits_0^\infty \rho^{\frac{n}{p'}-1} \mathscr{K}(1,\rho) \inf_{t>0} \frac{w(t)}{w(\rho t)}\,d\rho = \varkappa_*(n) f_0(x). \quad \rhd$$

Theorem 4.4. *Let $1 \leqslant p < \infty$, $1 < q < \infty$ and $w \in \Omega_q(\mathbb{R}_+)$. If*

$$\sup_{0<\delta<\delta_0} \varphi(\delta) \int\limits_0^\infty t^{\frac{n}{p'}-1} \left|\mathscr{K}(1,t)\right| w^*\left(\frac{1}{t}\right) \left[b^*\left(\frac{1}{t}\right)\right]^{\frac{\delta}{q(q-\delta)}} dt < \infty \tag{15}$$

for some $\delta_0 \in (0, q-1)$, then the operator K is bounded in the grand space $L_b^{p,q),w}(\mathbb{R}^n)$.

◁ By Lemma 3.1 we take

$$\|f\|_{L_b^{p,q),w}(\mathbb{R}^n)} = \sup_{0<\delta<\delta_0} \varphi(\delta)\, \|f\|_{L^{p,q-\delta,w_\delta}(\mathbb{R}^n)},$$

where $w_\delta(r) = w(r)[b(r)]^{\delta/q(q-\delta)}$. By using Theorem 4.2 and Remark 4.1, we get

$$\|Kf\|_{L_b^{p,q),w}(\mathbb{R}^n)} \leqslant \sup_{0<\delta<\delta_0} \varphi(\delta)C(\delta)\|f\|_{L^{p,q-\delta,w_\delta}(\mathbb{R}^n)},$$

where

$$C(\delta) = \left|S^{n-1}\right| \int\limits_0^\infty t^{\frac{n}{p'}-1} \left|\mathscr{K}(1,t)\right| w_\delta^*\left(\frac{1}{t}\right) dt. \quad \triangleright$$

Corollary 4.2. *Let assumptions of Theorem 4.4 be satisfied and,* $b(r) = r^\mu(1+r)^{-\nu}$, $0 < \mu < \nu$. *If there exists* $\varepsilon_0 > 0$ *such that*

$$\kappa_{\varepsilon_0} := \int\limits_0^1 t^{\frac{n}{p'}-\varepsilon_0-1} \left|\mathscr{K}(1,t)\right| w^*\left(\frac{1}{t}\right) dt + \int\limits_1^\infty t^{\frac{n}{p'}+\varepsilon_0-1} \left|\mathscr{K}(1,t)\right| w^*\left(\frac{1}{t}\right) dt < \infty, \tag{16}$$

then the operator K *is bounded in the grand space* $L_b^{p,q).w}(\mathbb{R}^n)$ *and*

$$\|Kf\|_{L_b^{p,q),w}(\mathbb{R}^n)} \leqslant \kappa_{\varepsilon_0}\|f\|_{L_b^{p,q).w}(\mathbb{R}^n)}.$$

◁ For $b(r) = r^\mu(1+r)^{-\nu}$, by (3) we have

$$b^*\left(\frac{1}{t}\right) = \begin{cases} t^{-\mu}, & t < 1, \\ t^{\nu-\mu}, & t > 1. \end{cases}$$

Then it is easy to see that (16) implies (15). ▷

Theorem 4.5. *The Hardy operators* H^α *and* $\mathscr{H}^\beta$ *are bounded in the grand space* $L_b^{p,q),\lambda}(\mathbb{R}^n)$, $1 \leqslant p < \infty$, $1 < q < \infty$, $\lambda > 0$, *with the grandizer* $b(r) = \frac{r^\mu}{(1+r)^\nu}$, $0 < \mu < \nu$, *if* $\alpha < \frac{n}{p'} + \lambda$ *and* $\beta > \lambda - \frac{n}{p}$, *respectively. If* $\varphi(\delta) \leqslant c\delta^{1/q}$, *then these conditions are also necessary for such a boundedness.*

◁ For the operator H^α we have

$$\kappa_{\varepsilon_0} = \int\limits_0^1 t^{\frac{n}{p'}+\lambda-\alpha-\varepsilon_0-1}\, dt,$$

which is finite under the choice $\varepsilon_0 \in (0, \frac{n}{p'} + \lambda - \alpha)$. This ensures the boundedness of H^α when $\alpha < \frac{n}{p'} + \lambda$.

Similarly, the sufficiency of the condition $\beta > \lambda - \frac{n}{p}$ for the boundedness of $\mathscr{H}^\beta$ is checked.

To prove the necessary, we choose $f = f_0(x) =: \frac{1}{|x|^{n/p-\lambda}}$, so that

$$\int\limits_{|y|<r} |f_0(y)|^p\, dy = c_1 r^{\lambda p}$$

and then

$$\|f_0\|_{L_b^{p,q),\lambda}(\mathbb{R}^n)} = \sup_{0<\delta<q-1} \varphi(\delta) \left(\int_0^\infty t^{\frac{\mu\delta}{q}} (1+t)^{-\frac{\nu\delta}{q}} \frac{dt}{t} \right)^{\frac{1}{q-\delta}}$$

$$= \sup_{0<\delta<q-1} \varphi(\delta) \left[B\left(\frac{\mu\delta}{q}, \frac{\nu\delta}{q} - \frac{\mu\delta}{q} \right) \right]^{\frac{1}{q-\delta}} \leqslant c \sup_{0<\delta<q-1} \frac{\varphi(\delta)}{\delta^{\frac{1}{q}}} < \infty.$$

Thus $f_0 \in L_b^{p,q),\lambda}(\mathbb{R}^n)$. On the other hand, direct colculation shows that

$$H^\alpha f_0(x) = c f_0(x), \quad c = |S^{n-1}| \int_0^\infty t^{\frac{n}{p'}+\lambda-\alpha-1} \, dt,$$

which implies that $\int_0^\infty t^{n/p'+\lambda-\alpha-1} dt$ must be finite.

The case of the operator $\mathscr{H}^\beta$ is analogously treated. $\rhd$

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Stefan G. Samko
University of Algarve,
Faro 8005-139, Portugal,
Professor Jubilado;
Kh. Ibragimov Complex Institute of the Russian Academy of Sciences,
21 a Staropromyslovskoe Hwy, Grozny 364051, Russia,
Principal Researcher
E-mail: ssamko@ualg.pt
https://orcid.org/0000-0002-8022-2863

Salaudin M. Umarkhadzhiev
Kh. Ibragimov Complex Institute of the Russian Academy of Sciences,
21 a Staropromyslovskoe Hwy, Grozny 364051, Russia,
Principal Researcher;
Academy of Sciences of Chechen Republic,
13 Esambaev Av., Grosny 364024, Russia,
Head of the Department of Applied Semiotics
E-mail: umsalaudin@gmail.com
https://orcid.org/0000-0002-8283-1515

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ГРАНД-ПРОСТРАНСТВА ТИПА МОРРИ

Самко С. Г.[1,2], Умархаджиев С. М.[2,3]

[1] Университет Алгарво, Португалия, 8005-139, Фаро;
[2] Комплексный научно-исследовательский институт им. Х. Ибрагимова РАН, Россия, 364051, Грозный, Старопромысловское шоссе, 21 а;
[3] Академия наук Чеченской Республики, Россия, 364024, Грозный, пр. им. М. Эсамбаева, 13

E-mail: ssamko@ualg.pt, umsalaudin@gmail.com

Аннотация. Так называемые гранд-пространства в настоящее время являются одним из основных объектов в теории функциональных пространств. Гранд-пространства Лебега были введены в работах T. Iwaniec и C. Sbordone в случае множеств Ω конечной меры $|\Omega| < \infty$, и авторами в случае $|\Omega| = \infty$.

Последнее основано на введении понятия грандизатора. Идея «грандизации» была также применена в контексте пространств Морри. В этой статье мы развиваем идею грандизации до более общих пространств Морри $L^{p,q,w}(\mathbb{R}^n)$, известных как пространства типа Морри. Мы вводим гранд-пространства типа Морри, что включает смешанные и частные гранд версии таких пространств. Смешанное гранд-пространство определяется нормой

$$\sup_{\varepsilon,\delta}\varphi(\varepsilon,\delta)\sup_{x\in E}\left(\int\limits_0^\infty w(r)^{q-\delta}b(r)^{\frac{\delta}{q}}\left(\int\limits_{|x-y|<r}\big|f(y)\big|^{p-\varepsilon}a(y)^{\frac{\varepsilon}{p}}\,dy\right)^{\frac{q-\delta}{p-\varepsilon}}\frac{dr}{r}\right)^{\frac{1}{q-\varepsilon}}$$

с использованием двух грандизаторов a и b. В случае гранд-пространств, частных относительно показателя q, мы изучаем ограниченность некоторых интегральных операторов. Класс этих операторов содержит, в частности, многомерные версии операторов типа Харди и операторов Гильберта.

Ключевые слова: пространство типа Морри, гранд-пространство, гранд-пространство типа Морри, грандизатор, частная грандизация, смешанная грандизация, однородное ядро, оператор типа Харди, оператор Гильберта.

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