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LOCAL GRAND LEBESGUE SPACES#

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Abstract. We introduce "local grand" Lebesgue spaces $L^{p),\theta}_{x_0,a}(\Omega)$, $0 < p < \infty$, $\Omega \subseteq \mathbb{R}^n$, where the process of "grandization" relates to a single point $x_0 \in \Omega$, contrast to the case of usual known grand spaces $L^{p),\theta}(\Omega)$, where "grandization" relates to all the points of Ω. We define the space $L^{p),\theta}_{x_0,a}(\Omega)$ by means of the weight $a(|x-x_0|)^{\varepsilon p}$ with small exponent, $a(0) = 0$. Under some rather wide assumptions on the choice of the local "grandizer" $a(t)$, we prove some properties of these spaces including their equivalence under different choices of the grandizers $a(t)$ and show that the maximal, singular and Hardy operators preserve such a "single-point grandization" of Lebesgue spaces $L^p(\Omega)$, $1 < p < \infty$, provided that the lower Matuszewska–Orlicz index of the function a is positive. A Sobolev-type theorem is also proved in local grand spaces under the same condition on the grandizer.

Key words: grand space, Lebesgue space, Muckenhoupt weight, maximal operator, singular operator, Hardy operator, Stein–Weiss interpolation theorem, Matuszewska–Orlicz indices.

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1. Introduction

We introduce the so called "local grand" Lebesgue spaces $L^{p),\theta}_{x_0,a}(\Omega)$, where the process of "grandization" relates to a single point $x_0 \in \overline{\Omega}$, contrast to the case of usual known grand spaces $L^{p),\theta}(\Omega)$, where "grandization" relates to all the points of Ω.

The grand spaces $L^{p),\theta}(\Omega)$, defined by the norm

$$\|f\|_{L^{p),\theta}(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int\limits_\Omega |f(x)|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}},$$

were introduced in [1, 2] in the case of a set Ω with finite measure. They were widely investigated during the last decades. We refer e.g. to [3–8]. An approach to grandize Lebesgue

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spaces on sets of infinite measure was suggested and developed in [9–13]. We refer also to [14] and references therein.

Let Ω be an open set in $\mathbb{R}^n$, bounded or unbounded and $x_0 \in \overline{\Omega}$ be fixed. We introduce the spaces $L^{p),\theta}_{x_0,a}(\Omega)$ via the (quasi)-norm

$$\|f\|_{L^{p),\theta}_{x_0,a}(\Omega)} := \sup_{0<\varepsilon<\ell} \varepsilon^\theta \left(\int_\Omega |f(x)|^p a(|x-x_0|)^{p\varepsilon}\, dx \right)^{\frac{1}{p}}, \quad 0<p<\infty,$$

where $a(t)$, $0 \leqslant t < \operatorname{diam}\Omega$, is a non-negative continuous bounded function vanishing only at $t=0$.

Under some rather wide assumptions on the choice of the local "grandizer" $a(t)$, we prove some properties of the spaces $L^{p),\theta}_{x_0,a}(\Omega)$ and we show that the maximal, singular and Hardy operators preserve such a "single-point grandization" of Lebesgue spaces $L^p(\Omega)$, $1<p<\infty$.

As a motivation for the introduction of such local grand spaces, we mention the following. When we study in Lebesgue spaces such operators as Hardy and Hilbert operators, or more generally integral operators with homogeneous kernel with fixed singularity, of principal importance is the study of mapping properties near the single point $x=0$, because beyond this point such operators essentially improve properties of functions.

In Section 2 we give precise definitions and prove some properties of the spaces $L^{p),\theta}_{x_0,a}(\Omega)$, including their equivalence under different choices of the grandizers $a(t)$. In Section 3 we prove the main statements on the boundedness of operators in the spaces $L^{p),\theta}_{x_0,a}(\Omega)$.

2. Definitions and Properties of Local Grand Lebesgue Spaces

2.1. Definitions. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $x_0 \in \overline{\Omega}$, $|x_0| < \infty$ and $d = \operatorname{diam}\Omega$, $0 < d \leqslant \infty$. By $G(0,d)$ we denote the set of functions continuous and bounded on $[0,d)$, satisfying the conditions:

$$a(0)=0 \quad \text{and} \quad \inf_{\delta\leqslant t<d} a(t) > 0 \quad \text{for every} \quad \delta \in (0,d). \tag{1}$$

Definition 2.1. Let $a \in G(0,d)$. We define the local grand Lebesgue space $L^{p),\theta}_{x_0,a}(\Omega)$, where $0<p\leqslant\infty$, $\theta>0$, by the (quasi)-norm

$$\|f\|_{L^{p),\theta}_{x_0,a}(\Omega)} := \sup_{0<\varepsilon<\ell} \varepsilon^\theta \left(\int_\Omega |f(x)|^p a(|x-x_0|)^{p\varepsilon}\, dx \right)^{\frac{1}{p}}, \tag{2}$$

when $p<\infty$ and

$$\|f\|_{L^{\infty)}_{x_0,a}(\Omega)} := \sup_{0<\varepsilon<\ell} \varepsilon^\theta \sup_{x\in\Omega} |f(x)| a(|x-x_0|)^\varepsilon, \tag{3}$$

where $\ell \in (0,\infty)$ is any fixed number.

By (1), the norm (2) is equivalent to

$$\|f\|_{L^{p),\theta}_{x_0,a}(\Omega)} := \sup_{0<\varepsilon<\ell} \varepsilon^\theta \left(\int_{\Omega\cap B(x_0,\delta)} |f(x)|^p a(|x-x_0|)^{p\varepsilon}\, dx \right)^{\frac{1}{p}} + \|f\|_{L^p(\Omega\setminus B(x_0,\delta))} \tag{4}$$

for every $\delta \in (0,d)$. Everywhere in Section 2 we take $0<p\leqslant\infty$.

The function $a \in G(0,d)$ will be referred to as *grandizer*.

The norm will be sometimes written as $\|f\|_{L^{p),\theta}_{x_0,a;\ell}(\Omega)}$ to underline dependence on the range for ε.

Lemma 2.1. *The space $L^{p),\theta}_{x_0,a}(\Omega)$ does not depend on the choice of ℓ, up to equivalence of norms:*

$$\|f\|_{L^{p),\theta}_{x_0,a;\ell_1}(\Omega)} \leqslant \|f\|_{L^{p),\theta}_{x_0,a;\ell_2}(\Omega)} \leqslant C\|f\|_{L^{p),\theta}_{x_0,a;\ell_1}(\Omega)}, \quad 0<\ell_1<\ell_2<\infty, \tag{5}$$

where

$$C = \max\left\{1, \frac{1}{\ell_1^{\theta}\|a\|^{\ell_1}_{L^\infty}} \sup_{\ell_1\leqslant\varepsilon<\ell_2} \varepsilon^{\theta}\|a\|^{\varepsilon}_{L^\infty}\right\}.$$

$\lhd$ In the case $p<\infty$ we have

$$\|f\|_{L^{p),\theta}_{x_0,a;\ell_2}(\Omega)} = \max\left\{\|f\|_{L^{p),\theta}_{x_0,a;\ell_1}(\Omega)}, E\right\},$$

where we denoted

$$E := \sup_{\ell_1\leqslant\varepsilon<\ell_2} \varepsilon^{\theta}\Bigg(\int\limits_{\Omega} |f(x)|^p a(|x-x_0|)^{p\varepsilon}\,dx\Bigg)^{\frac{1}{p}}.$$

Let $A := \|a\|_{L^\infty}$. We have

$$\begin{aligned} E = \sup_{\ell_1\leqslant\varepsilon<\ell_2} \varepsilon^{\theta}A^{\varepsilon}\Bigg(\int\limits_{\Omega} |f(x)|^p \left(\frac{a(|x-x_0|)}{A}\right)^{p\varepsilon} dx\Bigg)^{\frac{1}{p}} \\ \leqslant \sup_{\ell_1\leqslant\varepsilon<\ell_2} \varepsilon^{\theta}A^{\varepsilon}\Bigg(\int\limits_{\Omega} |f(x)|^p \left(\frac{a(|x-x_0|)}{A}\right)^{p\ell_1} dx\Bigg)^{\frac{1}{p}} \\ \leqslant \ell_1^{-\theta}A^{-\ell_1} \sup_{\ell_1\leqslant\varepsilon<\ell_2} \varepsilon^{\theta}A^{\varepsilon}\|f\|_{L^{p),\theta}_{x_0,a;\ell_1}(\Omega)}, \end{aligned}$$

q.e.d.

Arguments for $p=\infty$ are similar. $\rhd$

The embedding

$$L^p(\Omega) \hookrightarrow L^{p),\theta}_{x_0,a}(\Omega), \quad 0<p\leqslant\infty,\ \theta>0,$$

holds, whenever $a \in L^\infty(0,d)$.

A natural choice of grandizers a in the case of bounded sets Ω, may be:

$$a_0(t) = t\Big(\ln\frac{d\cdot e}{t}\Big)^{\nu}, \quad a_1(t)=t, \quad a_2(t) = \frac{1}{\ln\frac{d\cdot e}{t}}, \quad a_3(t) = \frac{1}{\ln\ln\frac{d\cdot e^2}{t}}, \tag{6}$$

where $\nu \in \mathbb{R}$, though this list may be continued.

If Ω is unbounded, the above functions may be modified e.g. as follows:

$$a_0(t) = t\Big(\ln\frac{e}{t}\Big)^{\nu}, \quad a_1(t)=t, \quad a_2(t) = \frac{1}{\ln\frac{e}{t}}, \quad a_3(t) = \frac{1}{\ln\ln\frac{e^2}{t}} \tag{7}$$

for $0<t\leqslant 1$ and identically equal to 1 for $1\leqslant t<\infty$.

Definition 2.2. We define the vanishing local grand Lebesgue space $VL^{p),\theta}_{x_0,a}(\Omega)$, $0<p<\infty$, as the subspace of functions $f\in L^{p),\theta}_{x_0,a}(\Omega)$ such that

$$\lim_{\varepsilon\to 0}\varepsilon^{p\theta}\int\limits_{\Omega}|f(x)|^p a(|x-x_0|)^{p\varepsilon}\,dx=0. \tag{8}$$

Clearly, the space $L^{p),\theta}_{x_0,a}(\Omega)$ contains non-integrable functions when $0<p<1$. The same holds for $p=1$, since $a(0)=0$. This may happen also for $p>1$, if $a(t)$ rapidly vanishs at $t=0$, e.g. $a(t)=e^{-\frac{\lambda}{t}}$, $\lambda>0$. It is easy to check that the condition

$$\sup_{0<\varepsilon<\varepsilon_0}\varepsilon^{-\theta p'}\int\limits_0^d t^{n-1}a(t)^{-\varepsilon p'}<\infty,\quad 1<p<\infty,$$

guarantees the embedding $L^{p),\theta}_{x_0,a}(\Omega)\subset L^1(\Omega)$.

Similar local "grandization" may be made not only with respect to a single point $x_0\in\overline{\Omega}$, but a finite number of points $x^{(1)},\dots,x^{(N)}\in\overline{\Omega}$ via the grandizer $a(x)=\prod_{k=1}^N a_k(|x-x^{(k)}|)$, $a_k\in G(0,d)$, $k=1,\dots,N$. Such a space coincides with the algebraic sum of the "single-point" local grand spaces $L^{p),\theta}_{x^{(k)},a_k}(\Omega)$, $k=1,\dots,N$.

2.2. Basic properties.

Lemma 2.2. *Let $a,b\in G(0,d)$. If there exists a number $\alpha>0$ such that*

$$a(t)\leqslant Cb(t)^{\alpha},\quad t\in(0,d),$$

then $L^{p),\theta}_{x_0,b}(\Omega)\hookrightarrow L^{p),\theta}_{x_0,a}(\Omega)$.

$\lhd$ The proof is straightforward, with Lemma 2.1 taken into account. $\rhd$

From Lemma 2.2 it follows that

$$L^{p),\theta}_{x_0,a}(\Omega)\big|_{a=t^{\lambda}}=L^{p),\theta}_{x_0,b}(\Omega)\big|_{b=t^{\mu}},\quad d<\infty \tag{9}$$

for all $\lambda>0$, $\mu>0$.

By Lemma 2.2 we have

$$L^{p),\theta}_{x_0,a_3}(\Omega)\hookrightarrow L^{p),\theta}_{x_0,a_2}(\Omega)\hookrightarrow L^{p),\theta}_{x_0,a_1}(\Omega)=L^{p),\theta}_{x_0,a_0}(\Omega), \tag{10}$$

where the grandizers a_0, a_1, a_2 and a_3 are from (6) or (7) and coincidence of spaces holds up to equivalence of norms. The embeddings (10) are strict, see Lemma 2.4.

The coincidence of spaces in (10) and (9) may be observed in a more general situation, as given in Theorem 2.1, where we use the notion of Matuszewska–Orlicz indices $m(a)$ and $M(a)$ of a non-negative function a ([15], see also [16]), where properties of these indices are given in a from convenient for us. The lower index $m(a)$ is defined by

$$m(a):=\sup_{0<x<1}\frac{\ln\left(\limsup\limits_{h\to 0}\frac{a(hx)}{a(h)}\right)}{\ln x}.$$

Note also that

$$m(t^{\alpha})=\alpha,\ m\left[\left(\ln\frac{d\cdot e}{t}\right)^{\pm 1}\right]=0,\quad m\left(t^{\alpha}a(t)\right)=\alpha+m(a),\quad m\big[a(t)^{\beta}\big]=\beta m(a),$$

where $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{R}_+$.

A non-negative function $a(t)$ on $(0,d)$, $0 < d \leqslant \infty$ is called quasi-monotone, if there exist $\alpha, \beta \in \mathbb{R}$, such that $\frac{a(t)}{t^\alpha}$ is almost increasing (a.i.) and $\frac{a(t)}{t^\beta}$ is almost decreasing (a.d.). A quasi-monotone function has finite indices and $m(a) = \sup\left\{\alpha : \frac{a(t)}{t^\alpha} \text{ is a.i.}\right\}$ and $M(a) = \inf\left\{\beta : \frac{a(t)}{t^\beta} \text{ is a.d.}\right\}$.

Theorem 2.1. *Let a and b be quasi-monotone on $(0,\delta)$ for some $\delta \in (0,d)$. If $m(a) > 0$ and $m(b) > 0$, then*

$$L^{p),\theta}_{x_0,a}(\Omega) = L^{p),\theta}_{x_0,b}(\Omega)$$

up to equivalence of norms.

$\lhd$ It suffices to refer to (4), use the fact that for an arbitrarily small $\varepsilon > 0$ there exist constants $c(\varepsilon)$ and $C(\varepsilon)$ such that

$$c(\varepsilon)t^{M(a)+\varepsilon} \leqslant a(t) \leqslant C(\varepsilon)t^{m(a)-\varepsilon}, \quad t \in (0,\delta),$$

where $M(a)$ is the upper Matuszewska–Orlicz index of a, $M(a) \geqslant m(a)$ (see [16, Section 6] and apply Lemma 2.2). $\rhd$

Keeping in mind that the function $\frac{1}{|x-x_0|^{\frac{n}{p}}}$ belongs to the usual grand Lebesgue space $L^{p),\theta}(\Omega)$, $\theta \geqslant 1$, below we consider similar inclusion of functions $u = u(|x-x_0|)$ into the space $L^{p),\theta}_{x_0,a}(\Omega)$.

For the *cone condition* used in the lemma below we refer e.g. to [17].

Lemma 2.3. *Let $x_0 \in \overline{\Omega}$ and assume that Ω satisfies the cone condition at the point x_0, when x_0 lies on the boundary of Ω. Let $u(t)$ be a non-negative function on $(0,d)$ such that $\int_\delta^d t^{n-1}u(t)^p\,dt < \infty$ for every $\delta \in (0,d)$. Then the condition*

$$\sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{p\theta} \int\limits_0^d t^{n-1}u(t)^p a(t)^{p\varepsilon}\,dt < \infty \tag{11}$$

for some $\varepsilon_0 > 0$ is necessary and sufficient for the inclusion

$$u(|x-x_0|) \in L^{p),\theta}_{x_0,a}(\Omega).$$

$\lhd$ The proof is straightforward. $\rhd$

When Ω is bounded, we put

$$u_1(t) = \frac{1}{t^{\frac{n}{p}}}, \quad u_2(t) = \frac{1}{\left(t^n \ln\frac{d\cdot e}{t}\right)^{\frac{1}{p}}}, \quad u_3(t) = \frac{1}{\left[t^n \left(\ln\frac{d\cdot e}{t}\right)\left(\ln\ln\frac{d\cdot e^2}{t}\right)\right]^{\frac{1}{p}}} \tag{12}$$

correspondingly to the grandizers $a_1(t)$, $a_2(t)$ and $a_3(t)$.

When Ω is unbounded, we define the functions $u_i(t)$ for $0 < t < 1$ by (12) with $d = 1$ and continue them for $t > 1$ so that $\int_1^\infty t^{n-1}u_i(t)^p\,dt < \infty$ (e.g. $u_i(t) \equiv 0$, $t > 1$, $i = 1,2,3$).

Lemma 2.4. *Let a_i, $i = 1,2,3$, be the grandizers defined in* (6) *and u_k, $k = 1,2,3$, be the functions* (12). *Then*

$$u_k \in L^{p),\theta}_{x_0,a_k}(\Omega), \text{ if } \theta \geqslant \frac{1}{p}, \quad \text{and} \quad u_k \notin L^{p),\theta}_{x_0,a_k}(\Omega), \text{ if } 0 < \theta < \frac{1}{p}, \; k = 1,2,3, \tag{13}$$

and

$$u_k \notin L^{p),\theta}_{x_0,a_i}(\Omega), \text{ if } i > k, \; \theta > 0. \tag{14}$$

◁ Let $d < \infty$. For u_1 and a_1 we have

$$\varepsilon^{p\theta} \int_0^d t^{n-1} u_1(t)^p a_1(t)^{p\varepsilon}\, dt = \varepsilon^{\theta p} \int_0^d t^{-1+\varepsilon p}\, dt,$$

so that the statement for u_1 and a_1 becoms evident by Lemma 2.2. For u_1 and a_2 we have

$$\varepsilon^{p\theta} \int_0^d t^{n-1} u_1(t)^p a_2(t)^{p\varepsilon}\, dt = \varepsilon^{p\theta} \int_0^d t^{-1} \left(\ln \frac{d \cdot e}{t}\right)^{-p\varepsilon} dt = \infty,$$

so that $u_1 \notin L^{p),\theta}_{x_0,a_2}(\Omega)$ by Lemma 2.2 and then $u_1 \notin L^{p),\theta}_{x_0,a_3}(\Omega)$.

Similarly other cases are verified. ▷

3. Interpolation of Sublinear Operators in Local Grand Lebesgue Spaces

Everywhere in Section 3 we take $1 \leqslant p < \infty$.

3.1. On interpolation. The proof of Theorem 3.2 in this section is based on the following theorem known as Stein–Weiss interpolation theorem with change of measure (see [18]; [19, p. 17]). We formulate it in weight terms.

We use the notation

$$L^p(\Omega, w) := \left\{ f : \int_\Omega |f(x)|^p w(x)\, dx < \infty \right\}$$

for weighted Lebesgue spaces.

Theorem 3.1. *Let $p_k, q_k \in [1, \infty)$ and v_k, w_k be weights on Ω, $k = 1, 2$, and T — a sublinear operator defined on $L^{p_1}(\Omega, w_1) \cup L^{p_2}(\Omega, w_2)$. If $T : L^{p_1}(\Omega, w_1) \hookrightarrow L^{q_1}(\Omega, v_1)$ with the norm K_1 and $T : L^{p_2}(\Omega, w_2) \hookrightarrow L^{q_2}(\Omega, v_2)$ with the norm K_2, then*

$$T : \ L^{p_t}(\Omega, w_t) \hookrightarrow L^{q_t}(\Omega, v_t)$$

with the norm $K \leqslant K_1^{1-t} K_2^t$, where

$$\frac{1}{p_t} = \frac{1-t}{p_1} + \frac{t}{p_2}, \quad \frac{1}{q_t} = \frac{1-t}{q_1} + \frac{t}{q_2}, \tag{15}$$

$$w_t = w_1^{(1-t)\frac{p_t}{p_1}} w_2^{t\frac{p_t}{p_2}}, \quad v_t = v_1^{(1-t)\frac{q_t}{q_1}} v_2^{t\frac{q_t}{q_2}}, \quad 0 < t < 1. \tag{16}$$

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^n$, $1 \leqslant p < \infty$, $\theta > 0$ and a and b be grandizers. Assume that a sublinear operator T is bounded from the space $L^p(\Omega)$ to the space $L^q(\Omega)$ and there exists an $\varepsilon_0 > 0$ such that it is bounded from the space $L^p(\Omega, a(|\cdot - x_0|)^{p\varepsilon_0})$ to the space $L^q(\Omega, b(|\cdot - x_0|)^{q\varepsilon_0})$. Then the operator T is bounded from $L^{p),\theta}_{x_0,a}(\Omega)$ to $L^{q),\theta}_{x_0,b}(\Omega)$ and from $VL^{p),\theta}_{x_0,a}(\Omega)$ to $VL^{q),\theta}_{x_0,b}(\Omega)$.*

◁ By Theorem 3.1 we obtain

$$\|Tf\|_{L^q(\Omega, b(|\cdot - x_0|)^{\varepsilon q})} \leqslant C\|f\|_{L^p(\Omega, a(|\cdot - x_0|)^{\varepsilon p})}, \quad 0 < \varepsilon < \varepsilon_0,$$

where C does not depend on f and ε. Hence the statements of the theorem follow for both the spaces $L^{p),\theta}_{x_0,a}(\Omega)$ and $VL^{p),\theta}_{x_0,a}(\Omega)$, with Lemma 2.1 taken into account. $\rhd$

3.2. Boundeddness of some classical operators of harmonic analysis in local grand Lebesgue spaces. In this section we take $\Omega = \mathbb{R}^n$ and stady the action, in the frameworks of the spaces $L^{p),\theta}_{x_0,a}(\mathbb{R}^n)$, of the following operators:

1) *the maximal operator*

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|\,dy, \tag{17}$$

2) *singular Calderón–Zygmund operators*

$$Tf(x) = \int\limits_{\mathbb{R}^n} K(x,y)f(y)\,dy,$$

with *standard kernel* (see [20, p. 144]),

3) *the Riesz potential operator*

$$I^\alpha f(x) := \int\limits_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\,dy, \quad 0<\alpha<n,$$

4) *the Hardy operators*

$$H^\alpha f(x) = |x|^{\alpha-n} \int\limits_{|y|<|x|} f(y)\,dy, \quad \mathscr{H}^\alpha f(x) = |x|^\alpha \int\limits_{|y|>|x|} \frac{f(y)}{|y|^n}\,dy. \tag{18}$$

We show that these operators act in the Lebesgue spaces, preserving their grandization at a single point $x_0 \in \mathbb{R}^n$, under a wide choice of the grandizers $a(t)$.

Maximal and singular operators. By A_p we denote the Muckenhoupt class of weights.

Theorem 3.3. *Let $1<p<\infty$, $\theta>0$ and $a \in G(\mathbb{R}_+)$. If there exists an $\varepsilon_0>0$ such that*

$$a^{\varepsilon_0} \in A_p, \tag{19}$$

then the maximal operator M and singular Calderón–Zygmund operators T with standard kernel, bounded in $L^2(\mathbb{R}^n)$, are bounded in the space $L^{p),\theta}_{x_0,a}(\mathbb{R}^n)$.

$\lhd$ It suffices to apply Theorem 3.2 and use the known fact that both M and T are bounded in Lebesgue spaces with A_p-weights (see e.g. [20, pp. 137, 144]). $\rhd$

Corollary 3.1. *Let $1<p<\infty$, $\theta>0$, and let $a(t)$ be quasi-monotone with $m(a)>0$. Then the maximal operator M and singular Caldrón–Zygmund operators T with standard kernel, bounded in $L^2(\mathbb{R}^n)$, are bounded in the space $L^{p),\theta}_{x_0,a}(\mathbb{R}^n)$.*

$\lhd$ By Theorem 2.1 we have

$$\|f\|_{L^{p),\theta}_{x_0,a}(\mathbb{R}^n)} \simeq \|f\|_{L^{p),\theta}_{x_0,a_0}(\mathbb{R}^n)},$$

where

$$a_0(t) = \begin{cases} t, & 0<t<1, \\ 1, & t \geqslant 1. \end{cases}$$

It remains to note that $a_0(|x|)^{\varepsilon_0} \in A_p$ under the choice $\varepsilon_0 \in (0, n(p-1))$. This is well known, if a_0 was $a_0(t)^{\varepsilon_0} = t^{\varepsilon_0}$, $t \in \mathbb{R}_+$. For the truncated power function it is easily obtained from the fact that for radial weights the Muckenhoupt condition is equivalent to (see [21])

$$\sup_{r>0} \int_0^r t^{n-1} a_0(t)^{\varepsilon_0}\, dt \left(\int_0^r t^{n-1} a_0(t)^{-\frac{\varepsilon_0}{p-1}}\, dt \right)^{p-1} < \infty. \ \rhd$$

Potential operators. In the proof of Theorem 3.4 we use the known (see [21, 22]) Muckenhoupt–Wheeden class $A_{p,q}$ defined by the condition

$$\sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x)^q\, dx \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q w(x)^{-p'}\, dx \right)^{\frac{1}{p'}} < \infty, \tag{20}$$

which goes back to [23].

Theorem 3.4. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If there exists an $\varepsilon_0 > 0$ such that $a^{\varepsilon_0} \in A_{1+\frac{q}{p'}}$, then the operator I^α is bounded from $L^{p),\theta}_{x_0,a}(\mathbb{R}^n)$ to $L^{q),\theta}_{x_0,a}(\mathbb{R}^n)$.*

$\lhd$ We apply Theorem 3.2. The $L^p \to L^q$-boundedness holds by the well known Sobolev theorem. The weighted $L^p(\mathbb{R}^n, a(|\cdot - x_0|)^{p\varepsilon_0}) \to L^q(\mathbb{R}^n, a(|\cdot - x_0|)^{q\varepsilon_0})$ holds, if $a^{\varepsilon_0} \in A_{p,q}$ (see [22, 23]). It remains to note that, as is known, $w \in A_{p,q} \Leftrightarrow w^q \in A_{1+\frac{q}{p'}}$ (see e.g. [21]). $\rhd$

Corollary 3.2. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and let $a \in G(\mathbb{R}_+)$ be quasi-monotone with $m(a) > 0$. Then the operator I^α is bounded from $L^{p),\theta}_{x_0,a}(\mathbb{R}^n)$ to $L^{q),\theta}_{x_0,a}(\mathbb{R}^n)$.*

$\lhd$ The arguments are similar to those in the proof of Corollary 3.1. $\rhd$

Hardy operators. In this case we take $x_0 = 0$.

Weighted boundedness of Hardy operators in Lebesgue spaces was thoroughly studied in the one-dimensional case (see [21, 24, 25]). The multidimensional versions (18) of Hardy operators were in particular studied in the case of power weights in [26], where the sharp constants were also found.

Though the weighted $L^p \to L^q$-boundedness of Hardy operators is well studied for all $p, q \in (1, \infty)$, we consider, for simplicity, only the case $p \leqslant q$.

By $B_{p,q}$ and $\mathscr{B}_{p,q}$ we denote the classes of pairs (u, v) of weights on $\mathbb{R}_+$, satisfying the conditions

$$B_{p,q}: \qquad \sup_{x\in\mathbb{R}_+} \left(\int_x^\infty u(t)\, dt \right)^{\frac{1}{q}} \left(\int_0^x v(t)^{1-p'}\, dt \right)^{\frac{1}{p'}} < \infty,$$

$$\mathscr{B}_{p,q}: \qquad \sup_{x\in\mathbb{R}_+} \left(\int_0^x u(t)\, dt \right)^{\frac{1}{q}} \left(\int_x^\infty v(t)^{1-p'}\, dt \right)^{\frac{1}{p'}} < \infty,$$

respectively.

Denote

$$u_\gamma(t) = \begin{cases} t^{\gamma_0}, & 0 < t < 1, \\ t^{\gamma_\infty}, & t \geqslant 1 \end{cases} \quad \text{and} \quad v_\lambda(t) = \begin{cases} t^{\lambda_0}, & 0 < t < 1, \\ t^{\lambda_\infty}, & t \geqslant 1. \end{cases}$$

It is easy to check that

$$(u_\gamma, v_\lambda) \in B_{p,q} \ \Leftrightarrow \ \gamma_\infty < -1,\ \lambda_0 < p-1,\ \frac{\gamma_0 + 1}{q} + \frac{1}{p'} = \frac{\lambda_0}{p} \ \text{and} \ \frac{\gamma_\infty + 1}{q} + \frac{1}{p'} = \frac{\lambda_\infty}{p}, \tag{21}$$

and

$$(u_\gamma, v_\lambda) \in \mathscr{B}_{p,q} \;\Leftrightarrow\; \gamma_\infty > -1,\; \lambda_0 > p-1,\; \frac{\gamma_0+1}{q} + \frac{1}{p'} = \frac{\lambda_0}{p} \;\text{ and }\; \frac{\gamma_\infty+1}{q} + \frac{1}{p'} = \frac{\lambda_\infty}{p}. \tag{22}$$

The known results for the one-dimensional Hardy operators

$$Hf(x) = \int_0^x f(t)\,dt \quad \text{and} \quad \mathscr{H}f(x) = \int_x^\infty f(t)\,dt, \quad x \in \mathbb{R}_+,$$

in the case $1 < p \leqslant q < \infty$ state that (see [24, p. 6–7]; [25, p. 12–13])

$$\left(\int_0^\infty |Hf(x)|^q u(x)\,dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty |f(x)|^p v(x)\,dx\right)^{\frac{1}{p}} \;\Leftrightarrow\; (u,v) \in B_{p,q}, \tag{23}$$

$$\left(\int_0^\infty |\mathscr{H}f(x)|^q u(x)\,dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty |f(x)|^p v(x)\,dx\right)^{\frac{1}{p}} \;\Leftrightarrow\; (u,v) \in \mathscr{B}_{p,q}. \tag{24}$$

Note that norm estimates of multi-dimensional integral operators with kernel $k(|x|,|y|)$ and radial weights reduce in a sense to similar one-dimensional estimates of spherical mens, see [26] in the case of Hardy operators and [27] in the case of operators with homogeneous kernel. In the lemma below we show this in the case of Hardy operators and arbitrary radial weights.

Lemma 3.1. *Let $1 < p \leqslant q < \infty$ and $\alpha \in \mathbb{R}$. The multi-dimensional inequality*

$$\left(\int_{\mathbb{R}^n} |H^\alpha f(x)|^q U(|x|)\,dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n} |f(x)|^p V(|x|)\,dx\right)^{\frac{1}{p}} \tag{25}$$

with radial weights holds, if there holds the one-dimensional inequality

$$\left(\int_0^\infty |Hg(t)|^q u(t)\,dt\right)^{\frac{1}{q}} \leqslant \frac{C}{|S^{n-1}|^{\frac{1}{p'}}}\left(\int_0^\infty |g(t)|^p v(t)\,dt\right)^{\frac{1}{p}}, \tag{26}$$

where

$$u(t) = t^{n-1+(\alpha-n)q}U(t), \quad v(t) = t^{(n-1)(1-p)}V(t).$$

Similarly

$$\left(\int_0^\infty |\mathscr{H}g(t)|^q u(t)\,dt\right)^{\frac{1}{q}} \leqslant \frac{C}{|S^{n-1}|^{\frac{1}{p'}}}\left(\int_0^\infty |g(t)|^p v(t)\,dt\right)^{\frac{1}{p}} \tag{27}$$

implies

$$\left(\int_{\mathbb{R}^n} |\mathscr{H}^\alpha f(x)|^q U(|x|)\,dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n} |f(x)|^p V(|x|)\,dx\right)^{\frac{1}{p}}, \tag{28}$$

were

$$u(t) = t^{n-1+\alpha q}U(t), \quad v(t) = t^{-p(n-1)-1}V(t).$$

◁ Passing to polar coordinates, we rewrite (25) as

$$\left\{\int_0^\infty r^{n-1+(\alpha-n)q}\left|\int_0^r t^{n-1}\Phi(t)\,dt\right|^q U(r)\,dr\right\}^{\frac{1}{q}} \leqslant C\left\{\int_0^\infty t^{n-1}\Phi_p(t)V(t)\,dt\right\}^{\frac{1}{p}}, \tag{29}$$

where

$$\Phi(t) = \int_{S^{n-1}} f(t\sigma)\,d\sigma, \quad \Phi_p(t) = \int_{S^{n-1}} |f(t\sigma)|^p\,d\sigma.$$

By Jensen inequality, $|\Phi(t)|^p \leqslant |S^{n-1}|^{p-1}\Phi_p(t)$. Therefore, (29) will be moreover satisfied, if

$$\left\{\int_0^\infty r^{n-1+(\alpha-n)q}U(r)\left|\int_0^r t^{n-1}\Phi(t)\,dt\right|^q dr\right\}^{\frac{1}{q}} \leqslant \frac{C}{|S^{n-1}|}\left\{\int_0^\infty t^{n-1}V(t)|\Phi(t)|^p\,dt\right\}^{\frac{1}{p}}, \tag{30}$$

which is nothing else bat (26) with $g(t) = t^{n-1}\Phi(t)$.

The case of the operator $\mathscr{H}^\alpha$ is similarly treated. ▷

Corollary 3.3. *The conditions*

$$(t^{n-1+(\alpha-n)q}U(t), t^{(n-1)(1-p)}V(t)) \in B_{p,q} \tag{31}$$

and

$$(t^{n-1+\alpha q}U(t), t^{-p(n-1)-1}V(t)) \in \mathscr{B}_{p,q} \tag{32}$$

are sufficient for the validity of the inequalities (25) *and* (28), *respectively.*

Theorem 3.5. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $a, b \in G(R_+)$. *The Hardy operators* H^α *and* $\mathscr{H}^\alpha$ *are bounded from* $L^{p),\theta}_{0,a}(\mathbb{R}^n)$ *to* $L^{q),\theta}_{0,b}(\mathbb{R}^n)$, $\theta > 0$, *if there exists an* $\varepsilon_0 > 0$ *such that*

$$\left(t^{n-1+(\alpha-n)q}b(t)^{\varepsilon_0 q}, t^{(n-1)(1-p)}a(t)^{\varepsilon_0 p}\right) \in B_{p,q} \tag{33}$$

and

$$\left(t^{n-1+\alpha q}b(t)^{\varepsilon_0 q}, t^{-p(n-1)-1}a(t)^{\varepsilon_0 p}\right) \in \mathscr{B}_{p,q}, \tag{34}$$

respectively.

◁ We apply Theorem 3.2. The $L^p \to L^q$ boundedness of H^α and $\mathscr{H}^\alpha$ is known (see [26, Section 4]). By Corollary 3.3, the weighted $L^p(\mathbb{R}^n, a^{\varepsilon_0 p}) \to L^q(\mathbb{R}^n, b^{\varepsilon_0 p})$-boundedness for the operators H^α and $\mathscr{H}^\alpha$ is guaranteed by the conditions (31) with $U(t) = b(t)^{\varepsilon_0 q}$ and $V(t) = a(t)^{\varepsilon_0 p}$, which proves the theorem. ▷

Theorem 3.6. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *The operators* H^α *and* $\mathscr{H}^\alpha$ *are bounded from* $L^{p),\theta}_{0,a}(\mathbb{R}^n)$ *to* $L^{q),\theta}_{0,b}(\mathbb{R}^n)$, $\theta > 0$, *for all grandizers* $a, b \in G(R_+)$, *quasi-monotone in a neighbourhood of the origin, having positive indices* $m(a) > 0$ *and* $m(b) > 0$.

◁ By Theorem 2.1, it suffices to prove the theorem in the case

$$a(t) = b(t) = \begin{cases} t, & 0 < t < 1, \\ 1, & t > 1. \end{cases}$$

Under this choice we have to verify the conditions (33) and (34) for sufficiently small ε_0.

This verification is easily done by means of the relations (21) and (22). ▷

4. On a Weight Generalisation

In a similar way we can consider *local grandization of weighted Lebesgue spaces*, defined by the norm

$$\|f\|_{L^{p),\theta}_{x_0,a}(\Omega,w)} = \sup_{0<\varepsilon<\ell} \varepsilon^{\theta} \left(\int\limits_{\Omega} |f(x)|^p w(x) a(|x-x_0|)^{\varepsilon p}\, dx \right)^{\frac{1}{p}}.$$

It is easy to see that statements of Lemmas 2.1, 2.2, 2.3 and Theorem 2.1 hold also in the weighted case in the corresponding reformulation. In the case of radial weights $w = w(|x-x_0|)$, an extension of Lemma 2.4 may be also obtained.

As regards the boundedness of operators in the weighted local grand space $L^{p),\theta}_{x_0,a}(\Omega, w)$, Theorem 3.1 allows to extend all the results of Section 3 to this case. We leave this to the reader.

References

1. *Iwaniec, T. and Sbordone, C.* On the Integrability of the Jacobian under Minimal Hypotheses, *Arch. Rational Mech. Anal.*, 1992 , vol. 119, no. 2, pp. 129–143. DOI: 10.1007/BF00375119.
2. *Greco, L., Iwaniec, T. and Sbordone, C.* Inverting the p-Harmonic Operator, *Manuscripta Math.*, 1997, vol. 92, no. 1, pp. 249–258. DOI: 10.1007/BF02678192.
3. *Fiorenza, A., Gupta, B. and Jain, P.* The Maximal Theorem in Weighted Grand Lebesgue Spaces, *Studia Math.*, 2008, vol. 188, no. 2, pp. 123–133. DOI: 10.4064/sm188-2-2.
4. *Jain, P., Singh, A. P., Singh, M. and Stepanov, V. D.* Sawyer's Duality Principle for Grand Lebesgue Spaces, *Math. Nachr.*, 2019, vol. 292, no. 4, pp. 841–849. DOI: 10.1002/mana.201700312.
5. *Kokilashvili, V. and Meskhi, A.* A Note on the Boundedness of the Hilbert Transform in Weighted Grand Lebesgue Spaces, *Georgian Math. J.*, 2009, vol. 16, no. 3, pp. 547–551. DOI: 10.1515/GMJ.2009.547.
6. *Fiorenza, A., Formica, M. R., Gogatishvili, A., Kopaliani, T. and Rakotoson, J. M.* Characterization of Interpolation Between Grand, Small or Classical Lebesgue Spaces, *Nonlinear Analysis*, 2018, vol. 177, pp. 422–453. DOI: 10.1016/j.na.2017.09.005.
7. *Kokilashvili, V. and Meskhi, A.* Fractional Integrals with Measure in Grand Lebesgue and Morrey Spaces, *Integral Transforms and Special Functions*, 2020, pp. 1–15. DOI: 10.1080/10652469.2020.1833003.
8. *Edmunds, D. E., Kokilashvili, V. and Meskhi, A.* Sobolev-Type Inequalities for Potentials in Grand Variable Exponent Lebesgue Spaces, *Mathematische Nachrichten*, 2019, vol. 292, no. 10, pp. 2174–2188. DOI: 10.1002/mana.201800239.
9. *Samko, S. G. and Umarkhadzhiev, S. M.* On Iwaniec–Sbordone Spaces on Sets which May Have Infinite Measure, *Azerb. J. Math.*, 2011, vol. 1, no. 1, pp. 67–84. URL: https://www.azjm.org/volumes/1-1.html.
10. *Samko, S. G. and Umarkhadzhiev, S. M.* Grand Morrey Type Spaces, *Vladikavkaz Math. J.*, 2020, vol. 22, no. 4, pp. 104–118. DOI: 10.46698/c3825-5071-7579-i.
11. *Samko, S. and Umarkhadzhiev, S.* Riesz Fractional Integrals in Grand Lebesgue Spaces, *Fract. Calc. Appl. Anal.*, 2016, vol. 19, no. 3, pp. 608–624. DOI: 10.1515/fca-2016-0033.
12. *Samko, S. and Umarkhadzhiev, S.* On Grand Lebesgue Spaces on Sets of Infinite Measure, *Mathematische Nachrichten*, 2017, vol. 290, no. 5–6, pp. 913–919. DOI: 10.1002/mana.201600136.
13. *Umarkhadzhiev, S.* Generalization of the Notion of Grand Lebesgue Space, *Russian Mathematics*, 2014, vol. 58, no. 4, pp. 35–43, DOI: 10.3103/S1066369X14040057.
14. *Kokilashvili, V., Meskhi, A., Rafeiro, H. and Samko, S. Integral Operators in Non-Standard Function Spaces. Vol. 2: Variable exponent Hölder, Morrey–Campanato and Grand Spaces*, Birkhäuser, 2016.
15. *Matuszewska, W. and Orlicz, W.* On Some Classes of Functions with Regard to their Orders of Growth, *Studia Math.*, 1965, vol. 26, pp. 11–24.
16. *Samko, N. G.* Weighted Hardy Operators in the Local Generalized Vanishing Morrey Spaces, *Positivity*, 2013, vol. 17, no. 3, pp. 683–706. DOI: 10.1007/s11117-012-0199-z.
17. *Besov, O. V., Il'in, V. P. and Nikol'skii, S. M. Integral Representations of Functions and Embedding of Functions*, Moscow, Nauka, 1975, 480 p. (in Russian).
18. *Stein, E. M. and Weiss, G.* Interpolation of Operators with Change of Measures, *Trans. Amer. Math. Soc.*, 1958, vol. 87, pp. 159–172.

19. *Bergh, J. and Löfström, J. Interpolation Spaces. An Introduction*, Berlin, Springer, 1976.
20. *Duoandikoetxea, J. Fourier Analysis*, Graduate Studies in Math., Vol. 29, Providence, Amer. Math. Soc., 2001.
21. *Dyn'kin, E. M. and Osilenker, B. B.* Weighted Norm Estimates for Singular Integrals and their Applications, *J. Sov. Math.*, 1985, vol. 30, pp. 2094–2154.
22. *Adams, D. R. and Hedberg, L. I. Function Spaces and Potential Theory*, Berlin, Springer, 1996.
23. *Muckenhoupt, B. and Wheeden, R. L.* Weighted Norm Inequalities for Fractional Integrals, *Trans. Amer. Math. Soc.*, 1974, vol. 192, pp. 261–274.
24. *Kufner, A., Persson, L. E. and Samko, N. Weighted Inequalities of Hardy Type*, Second Edition, New Jersey, World Scientific, 2017.
25. *Opic, B. and Kufner, A. Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, Vol. 219, Harlow, Longman Scientific & Technical, 1990.
26. *Persson, E.-L. and Samko, S. G.* A Note on the Best Constants in Some Hardy Inequalities, *J. Math. Inequal.*, 2015, vol. 9, no. 2, pp. 437–447. DOI: 10.7153/jmi-09-37.
27. *Umarkhadzhiev, S. M.* Integral Operators with Homogeneous Kernels in Grand Lebesgue Spaces, *Mathematical Notes*, 2017, vol. 102, no. 5, pp. 710–721. DOI: 10.1134/S0001434617110104.

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ЛОКАЛЬНЫЕ ГРАНД ПРОСТРАНСТВА ЛЕБЕГА

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Аннотация. Мы вводим «локальные гранд» пространства Лебега $L^{p),\theta}_{x_0,a}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, где процесс «грандизации» относится к единственной точке $x_0 \in \Omega$, в отличие от случая обычных известных гранд пространств $L^{p),\theta}(\Omega)$, где «грандизация» относится ко всем точкам Ω. Мы определяем пространство $L^{p),\theta}_{x_0,a}(\Omega)$ с помощью веса $a(|x-x_0|)^{\varepsilon p}$ с малым показателем степени, $a(0)=0$. При некоторых довольно широких предположениях о выборе локального «грандизатора» $a(t)$ мы доказываем некоторые свойства этих пространств, включая их эквивалентность при различном выборе грандизаторов $a(t)$, и показываем, что максимальный, сингулярный операторы и операторы Харди сохраняют такую «одноточечную грандизацию» пространств Лебега $L^p(\Omega)$, $1 < p < \infty$, при условии, что нижний индекс Матушевской — Орлича функции a положительный. Доказана также теорема типа Соболева в локальных гранд пространствах при том же условии на грандизатор.

Ключевые слова: гранд-пространство, пространство Лебега, вес Макенхаупта, максимальный оператор, сингулярный оператор, оператор Харди, интерполяционная теорема Стейна — Вейса, индексы Матушевской — Орлича.

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