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EVERY LATERAL BAND IS THE KERNEL OF AN ORTHOGONALLY ADDITIVE OPERATOR#

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Abstract. In this paper we continue a study of relationships between the lateral partial order $\sqsubseteq$ in a vector lattice (the relation $x \sqsubseteq y$ means that x is a fragment of y) and the theory of orthogonally additive operators on vector lattices. It was shown in [1] that the concepts of lateral ideal and lateral band play the same important role in the theory of orthogonally additive operators as ideals and bands play in the theory for linear operators in vector lattices. We show that, for a vector lattice E and a lateral band G of E, there exists a vector lattice F and a positive, disjointness preserving orthogonally additive operator $T\colon E \to F$ such that $\ker T = G$. As a consequence, we partially resolve the following open problem suggested in [1]: Are there a vector lattice E and a lateral ideal in E which is not equal to the kernel of any positive orthogonally additive operator $T\colon E \to F$ for any vector lattice F?

Key words: orthogonally operator, lateral ideal, lateral band, lateral disjointness, orthogonally additive projection, vector lattice.

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1. Introduction and Preliminaries

The lateral order introduced in [1] shows its importance for the theory of orthogonally additive operators (OAOs) [1–3]. In this paper we continue investigation of order properties of OAOs connected with the lateral order. For all unexplained notions concerning vector lattices and orthogonally additive operators we refer the reader to [1, 4, 5].

We shall write $x = y \sqcup z$ if $x = y + z$ and $y \perp z$. We say that y is a *fragment* (a *component*) of $x \in E$, and use the notation $y \sqsubseteq x$, if $y \perp (x - y)$.

Definition 1.1. Let E be a vector lattice. A subset $\mathscr{I}$ of E is said to be a *lateral ideal* if the following hold:

(1) $x \sqcup y \in \mathscr{I}$ for every disjoint $x, y \in \mathscr{I}$;

(2) if $x \in \mathscr{I}$ then $y \in \mathscr{I}$ for all $y \in \mathfrak{F}_x$.

It is clear that every order ideal of E is a lateral ideal of E. The set $\mathfrak{F}_x$ of all fragments of an element $x \in E$ provides the example of a lateral ideal that is not a linear subspace of E.

The importance of lateral ideals for orthogonally additive operators is demonstrated in the following statement.

Proposition 1.1 [1, Proposition 6.4]**.** *Let E, F be vector lattices and $T\colon E \to F$ a positive orthogonally additive operator. Then $\ker T := \{x \in E : Tx = 0\}$ is a lateral ideal in E.*

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The following problem was stated in [1].

Problem 1.2 [1, Problem 6.7]: Are there a vector lattice E and a lateral ideal in E which is not equal to the kernel of any positive orthogonally additive operator $T\colon E \to F$ for any vector lattice F?

Below we resolve this problem for lateral bands. That is the special subclass of lateral ideals.

Definition 1.2. We recall that a net $(x_\lambda)_{\lambda\in\Lambda}$ in a vector lattice E is called *order fundamental* if the net $(x_\lambda - x_{\lambda'})_{(\lambda,\lambda')\in\Lambda\times\Lambda}$ order converges to zero. An order fundamental net $(x_\lambda)_{\lambda\in\Lambda}$ in E is called *laterally fundamental* if $x_\lambda \sqsubseteq x_{\lambda'}$ for all $\lambda, \lambda' \in \Lambda$ with $\lambda \leqslant \lambda'$. A subset D of the vector lattice E is called *laterally closed*, if every laterally fundamental net $(x_\lambda)_{\lambda\in\Lambda}$ in D order converges to some $x \in D$. A laterally closed lateral ideal $\mathscr{B}$ in E is called a *lateral band*.

Example 1.1. Every band $\mathscr{B}$ of a Dedekind complete vector lattice E is a lateral band of E.

2. Main Result

The next theorem is the main result of these notes.

Theorem 2.1. *Let E be a vector lattice and G a lateral band in E. Then there exist a vector lattice F and a positive, disjointness preserving orthogonally additive operator $T\colon E \to F$ with* $\ker T = G$.

We need some auxiliary constructions to prove Theorem 2.1.

Definition 2.1. Let E be a vector lattice and $x, y \in E$. We say that x and y are *laterally disjoint* and write $x\dagger y$ if $\{z \in \mathscr{F}_x \cap \mathscr{F}_y\} = 0$. We say that two subsets H and D of E are *laterally disjoint* and use the notation $H\dagger D$ if $x\dagger y$ for every $x \in H$ and $y \in D$.

Let E be a vector lattice and G be a lateral ideal in E. Put

(1) $G^\dagger := \{x \in E : x\dagger y \ \text{ for all } \ y \in G\}$;

(2) $G^{\dagger\dagger} := (G^\dagger)^\dagger$.

Lemma 2.1. *$G^\dagger$ is a lateral band in E.*

$\lhd$ Take $x \in G^\dagger$ and $y \in \mathfrak{F}_x$. Since $\mathfrak{F}_y \subset \mathfrak{F}_x$ we have that $y \in G^\dagger$. Fix $x, y \in G^\dagger$ with $x \perp y$. We note that

$$\mathfrak{F}_{x\sqcup y} = \{u \sqcup v : u \in \mathfrak{F}_x,\, v \in \mathfrak{F}_y\}.$$

Suppose that $x \sqcup y \notin G^\dagger$. Then there exists $0 \neq w \in \mathfrak{F}_{x\sqcup y} \cap G$ and consequently $w = u \sqcup v$ for some $u \in \mathfrak{F}_x$ and $v \in \mathfrak{F}_y$. It follows that $u, v \in G$ and we come to the contradiction. Thus $G^\dagger$ is a lateral ideal in E. Now we show that $G^\dagger$ is laterally closed. Suppose that $(x_\lambda)_{\lambda\in\Lambda}$ is a laterally convergent net in $G^\dagger$ that converges to $x \in E$. We claim that $x \in G^\dagger$. Indeed, assume that $x \notin G^\dagger$. Then there exists $0 \neq y \in \mathfrak{F}_x \cap G$ and since all fragments of y belong to G we have that $(x_\lambda)_{\lambda\in\Lambda}$ does not laterally converges to x. $\rhd$

Lemma 2.2 [2, Proposition 5.6]**.** *Let E be a vector lattice, $x, y \in E$ and $x \perp y$. Then $x\dagger y$.*

The following example shows that the converse assertion, in general, is not true.

Example 2.1. Let (Ω, Σ, ν) be a finite measure spaces, $E = L_0(\nu)$. We denote the characteristic function of a set $D \in \Sigma$ by 1_D. Suppose that $x = 1_\Omega$ and $y = k1_\Omega$, for some $0 < k < 1$. Then $|x| \wedge |y| = x \wedge y = y > 0$. On the other hand we have that

$$\mathfrak{F}_x = \{1_D : D \in \Sigma\} \ \text{ and } \ \mathfrak{F}_x = \{k1_D : D \in \Sigma\}.$$

Hence, $\mathfrak{F}_x \cap \mathfrak{F}_y = 0$ and therefore $x\dagger y$.

The next lemma clarifies the relation between the lateral and the traditional disjointness.

Lemma 2.3 [2, Proposition 5.7]. *Let E be a vector lattice, $x, y \in E$, $x \dagger y$ and $x, y \in \mathfrak{F}_v$ for some $v \in E$. Then $x \perp y$.*

Lemma 2.4. *Let E be a vector lattice and G be a lateral ideal in E. Then $G \subset G^{\dagger\dagger}$ and $G = G^{\dagger\dagger}$ if G is a lateral band.*

$\lhd$ Take $x \in G$. Then $\mathfrak{F}_x \cap \mathfrak{F}_y = 0$ for all $y \in G^\dagger$ and therefore $x \in G^{\dagger\dagger}$. Suppose that G is a lateral band and assume that the inclusion $G \subset G^{\dagger\dagger}$ is strict. Then there is $x \in G^{\dagger\dagger}$ such that there exists $0 \neq y \in \mathfrak{F}_x$ with $\mathfrak{F}_y \cap G = 0$. It follows that $y \in G^\dagger$ and we come to the contradiction. $\rhd$

Lemma 2.5 [6, Lemma 3]. *Let E be a Dedekind complete vector lattice and G be a lateral band in E. Then, for every $x \in E$, the set $G(x) = \mathfrak{F}_x \cap G$ contains a maximal element x_G with respect to the lateral order.*

Lemma 2.6. *Let E be a Dedekind complete vector lattice and G be a lateral band in E. Then for every $x \in E$ there exists the unique decomposition $x = x_G \sqcup x_{G^\dagger}$, with $x_G \in G$ and $x_{G^\dagger} \in G^\dagger$.*

$\lhd$ Fix $x \in E$. By Lemma 2.5 there is the unique decomposition $x = x_G \sqcup (x - x_G)$. We claim that $(x - x_G) \in G^\dagger$. Indeed, assuming the contrary we can find $y \in \mathfrak{F}_{x - x_G}$ such that $y \in G$. Then $x_G \sqsubseteq x_G \sqcup y$ and it contradicts to the maximality of x_G. $\rhd$

Remark 2.1. We note that orthogonally additive projections $\mathfrak{p}_G$ and $\mathfrak{p}_{G^\dagger}$ onto lateral bands G and $G^\dagger$ are defined by the setting $\mathfrak{p}_G x := x_G$, $\mathfrak{p}_{G^\dagger} x := x_{G^\dagger}$, $x \in E$ and the identity operator Id_E on E has the representation $Id_E = \mathfrak{p}_G \sqcup \mathfrak{p}_{G^\dagger}$. We observe that different properties of projections onto lateral bands were investigated in [1]. In particular, an orthogonally additive projection $\mathfrak{p}_G$ onto a lateral band G in E preserves disjointness [1, Theorem 6.9].

Lemma 2.7. *Suppose that E is a Dedekind complete vector lattice and G is a band in E. Then the disjoint complement $G^\perp$ coincides with $G^\dagger$ and orthogonally additive projections $\mathfrak{p}_G$ and $\mathfrak{p}_{G^\dagger}$ onto G and $G^\dagger$ coincide with linear order projections π_G and $\pi_{G^\perp}$ onto projection bands G and $G^\perp$ respectively.*

$\lhd$ It is enough to prove that $G^\dagger = G^\perp$. The relation $G^\perp \subset G^\dagger$ is obvious. Assume that there exists $x \in E$ with $x \in G^\dagger$ and $x \notin G^\perp$. Then $|x| \wedge y = u > 0$ for some $0 < y \in G$. Let π_u is a band projection onto the projection band $\{u\}^{\perp\perp}$. Then $v := \pi_u x \in G$ and $\mathfrak{F}_x \cap \mathfrak{F}_v = \mathfrak{F}_v$. Thus $x \notin G^\dagger$ and $G^\dagger = G^\perp$. $\rhd$

We provide an example of a decomposition of a vector lattice into the disjoint sum of "nonlinear" bands.

Example 2.2. Let (Ω, Σ, ν) be a finite measure spaces, $E = L_0(\nu)$, $x \in E$ and $G = \mathfrak{F}_x$. Take $y \in E$ and define ν-measurable sets:

$$D_G^y := \{t \in \Omega : \, y(t) = x(t)\}, \quad D_{G^\dagger}^y := \{t \in \Omega : \, y(t) \neq x(t)\}.$$

Then for every and $y \in E$ there is the disjoint decomposition $y = y_G \sqcup y_{G^\dagger}$, where $y_G := y 1_{D_G^y}$ and $y_{G^\dagger} := y 1_{D_{G^\dagger}^y}$.

Now we are ready to present a positive orthogonally additive operator that is vanished on a lateral band G.

$\lhd$ Proof of Theorem 2.1. Suppose that F is the Dedekind completion of E. Then by Lemma 2.6 $\mathfrak{p}_{G^\dagger} : F \to F$ is a well defined orthogonally additive operator. By $\mathfrak{p}_{G^\dagger}|_E$ we denote the restriction of $\mathfrak{p}_{G^\dagger}$ on E. Clearly, $\ker \mathfrak{p}_{G^\dagger}|_E = G$. Put

$$Tx := |\mathfrak{p}_{G^\dagger}|_E x|, \quad x \in E.$$

Since $\mathfrak{p}_{G^\dagger}|_E$ preserves disjointness we have that $T \colon E \to F$ is a well defined positive, disjointness preserving orthogonally additive operator and $\ker T = G$. $\rhd$

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КАЖДАЯ ЛАТЕРАЛЬНАЯ ПОЛОСА ЯВЛЯЕТСЯ ЯДРОМ ПОЛОЖИТЕЛЬНОГО ОРТОГОНАЛЬНО АДДИТИВНОГО ОПЕРАТОРА

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Аннотация. В данной статье мы продолжим изучение приложений латерального порядка $\sqsubseteq$ в векторных решетках (запись $x \sqsubseteq y$ означает, что x — это осколок y) к теории ортогонально аддитивных операторов. В работе [1] было установлено, что понятия латерального идеала и латеральной полосы играют такую же важную роль в теории ортогонально аддитивных операторов, как и понятия порядкового идеала и полосы — в теории линейных операторов в векторных решетках. В заметке установлено, что для произвольной векторной решетки E и латеральной полосы G в E найдется векторная решетка F и положительный ортогонально аддитивный оператор $T\colon E \to F$, сохраняющий дизъюнктность, такой, что $\ker T = G$. Данный результат частично решает следующую открытую проблему, указанную в работе [1]. Верно ли, что для любой векторной решетки E и латерального идеала G в E существуют векторная решетка F и положительный ортогонально аддитивный оператор $T\colon E \to F$ такие, что $\ker T = G$?

Ключевые слова: ортогонально аддитивный оператор, латеральный идеал, латеральная полоса, латеральная дизъюнктность, ортогонально аддитивный проектор, векторная решетка.

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