Vladikavkaz Mathematical Journal
2022, Volume 24, Issue 2, P. 5–13

УДК 517.954
DOI 10.46698/s8393-0239-0126-b

EXISTENCE RESULTS FOR A DIRICHLET BOUNDARY VALUE PROBLEM INVOLVING THE $P(X)$-LAPLACIAN OPERATOR

M. Ait Hammou

[1] Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar el Mahraz,
Laboratory LAMA, Department of Mathematics, Fez, P. O. Box 1796, Morocco

E-mail: mustapha.aithammou@usmba.ac.ma

Abstract. The aim of this paper is to establish the existence of weak solutions, in $W_0^{1,p(x)}(\Omega)$, for a Dirichlet boundary value problem involving the $p(x)$-Laplacian operator. Our technical approach is based on the Berkovits topological degree theory for a class of demicontinuous operators of generalized (S_+) type. We also use as a necessary tool the properties of variable Lebesgue and Sobolev spaces, and specially properties of $p(x)$-Laplacian operator. In order to use this theory, we will transform our problem into an abstract Hammerstein equation of the form $v+S\circ Tv=0$ in the reflexive Banach space $W^{-1,p'(x)}(\Omega)$ which is the dual space of $W_0^{1,p(x)}(\Omega)$. Note also that the problem can be seen as a nonlinear eigenvalue problem of the form$Au=\lambda u$, where $Au:=-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)-f(x,u)$. When this problem admits a non-zero weak solution u, λ is an eigenvalue of it and u is an associated eigenfunction.

Key words: Dirichlet problem, topological degree, $p(x)$-Laplacian operator.

AMS Subject Classification: 35J60, 47J05, 47H11.

For citation: *Ait Hammou, M.* Existence Results for a Dirichlet Boundary Value Problem Involving the $p(x)$-Laplacian Operator, *Vladikavkaz Math. J.*, 2022, vol. 24, no. 2, pp. 5–13. DOI: 10.46698/s8393-0239-0126-b.

1. Introduction

Topological degree is an effective tool in the study of nonlinear equations. Brouwer had published a degree theory in 1912 for continuous maps defined in finite dimensional Euclidean space [1]. Leray and Schauder generalized in 1934 the degree theory in infinite-dimensional Banach spaces [2]. Since 1934 various extension and generalizations of degree theory have been defined. The theory was constructed later by Berkovits and Mustonen [3–6].

In this paper, we prove the existence of weak solutions for the Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)=\lambda u+f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1}$$

where $\lambda\in\mathbb{R}$, $\Omega\subset\mathbb{R}^N$ is a bounded domain, $2\leqslant p(x)$ and $p(x)\in C(\bar{\Omega})$ by using the topological degree theory for a class of bounded and demicontinuous operators of generalized (S_+) type.

For $\lambda = 0$, Fan and Zhang (in [7]) presents several sufficient conditions for the existence of solutions for the problem (1), at first when f is independent of u [7, Theorem 4.2], then when f satisfies a growth condition of the form

$$|f(x,t)| \leqslant C_1 + C_2|t|^{\beta-1}$$

(here the exponent $\beta - 1$ is constante) [7, Theorem 4.3] and finally, in [7, Theorem 4.7], they also come from the existences of solution when f satisfies Carathéodory condition and a growth condition with a variable exponent but with other additional conditions. The same problem is studied after by P. S. Iliaş (in [8]) who gives sufficient conditions which allow to use variational and topological methods to prove the existence of weak solutions.

With another approach (theory of topological degree), we prove in this paper the existence of a weak solution for (1) when f is a Carathéodory function satisfying only a growth condition and with an additional term λu. Note that with this term, the problem (1) can be seen as a nonlinear eigenvalue problem of the form

$$Au = \lambda u, \tag{2}$$

where $Au := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - f(x,u)$. When (2) admits a non-zero weak solution u, λ is an eigenvalue of (2) and u is an associated eigenfunction. So, proving that (2) admits a weak solution, we prove at the same time that each real λ can be chosen as a eigenvalue of the problem (2).

This paper is divided into four sections. In the second section, we introduce some classes of mappings of generalized (S_+) type and the recent Berkovits degree. In the third section, some basic properties of variable Lebesgue and Sobolev spaces and several important properties of $p(x)$-Laplacian operator are presented. Finally, in the fourth section, we give the assumptions and our main results concerning the weak solutions of problem (1).

2. Classes of Mapping and Topological Degree

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle\,\cdot\,,\cdot\,\rangle$ and let Ω be a nonempty subset of X. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

Let Y be a real Banach space. We recall that a mapping $F : \Omega \subset X \to Y$ is *bounded*, if it takes any bounded set into a bounded set. F is said to be *demicontinuous*, if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$. F is said to be *compact* if it is continuous and the image of any bounded set is relatively compact.

A mapping $F : \Omega \subset X \to X^*$ is said to be *of class* (S_+), if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup\langle Fu_n, u_n - u\rangle \leqslant 0$, it follows that $u_n \to u$. F is said to be *quasimonotone*, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $\limsup\langle Fu_n, u_n - u\rangle \geqslant 0$.

For any operator $F : \Omega \subset X \to X$ and any bounded operator $T : \Omega_1 \subset X \to X^*$ such that $\Omega \subset \Omega_1$, we say that F satisfies condition $(S_+)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup\langle Fu_n, y_n - y\rangle \leqslant 0$, we have $u_n \to u$. We say that F has the property $(QM)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $\limsup\langle Fu_n, y - y_n\rangle \geqslant 0$.

Let $\mathscr{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$\mathscr{F}_1(\Omega) := \big\{F : \Omega \to X^* \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)\big\},$$
$$\mathscr{F}_{T,B}(\Omega) := \big\{F : \Omega \to X \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)_T\big\},$$
$$\mathscr{F}_T(\Omega) := \big\{F : \Omega \to X \mid F \text{ is demicontinuous and satisfies condition } (S_+)_T\big\},$$
$$\mathscr{F}_B(X) := \big\{F \in \mathscr{F}_{T,B}(\bar{G}) \mid G \in \mathscr{O}, \mathrm{T} \in \mathscr{F}_1(\bar{\mathrm{G}})\big\}.$$

Here, $T \in \mathscr{F}_1(\bar{G})$ is called an *essential inner map* to F.

Lemma 2.1 [4, Lemma 2.2 and Lemma 2.4]**.** *Suppose that $T \in \mathscr{F}_1(\bar{G})$ is continuous and $S : D_S \subset X^* \to X$ is demicontinuous such that $T(\bar{G}) \subset D_s$, where G is a bounded open set in a real reflexive Banach space X. Then the following statement are true:*

(i) *If S is quasimonotone, then $I + S \circ T \in \mathscr{F}_T(\bar{G})$, where I denotes the identity operator.*

(ii) *If S is of class (S_+), then $S \circ T \in \mathscr{F}_T(\bar{G})$.*

Definition 2.1. Let G be a bounded open subset of a real reflexive Banach space X, $T \in \mathscr{F}_1(\bar{G})$ be continuous and let $F, S \in \mathscr{F}_T(\bar{G})$. The affine homotopy $H : [0,1] \times \bar{G} \to X$ defined by

$$H(t,u) := (1-t)Fu + tSu \quad \text{for} \quad (t,u) \in [0,1] \times \bar{G}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.1 [4]. The above affine homotopy satisfies condition $(S_+)_T$.

We introduce the topological degree for the class $\mathscr{F}_B(X)$ due to Berkovits [4].

Theorem 2.1. *There exists a unique degree function*

$$d : \ \big\{(F,G,h) \mid G \in \mathscr{O},\ T \in \mathscr{F}_1(\bar{G}),\ F \in \mathscr{F}_{T,B}(\bar{G}),\ h \notin F(\partial G)\big\} \to \mathbb{Z}$$

that satisfies the following properties:

1. (*Existence*) *If $d(F,G,h) \neq 0$, then the equation $Fu = h$ has a solution in G.*

2. (*Additivity*) *Let $F \in \mathscr{F}_{T,B}(\bar{G})$. If G_1 and G_2 are two disjoint open subset of G such that $h \notin F(\bar{G} \setminus (G_1 \cup G_2))$, then we have*

$$d(F,G,h) = d(F,G_1,h) + d(F,G_2,h).$$

3. *(Homotopy invariance) If $H : [0,1] \times \bar{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h : [0,1] \to X$ is a continuous path in X such that $h(t) \notin H(t, \partial G)$ for all $t \in [0,1]$, then the value of $d(H(t,\cdot), G, h(t))$ is constant for all $t \in [0,1]$.*

4. (*Normalization*) *For any $h \in G$, we have $d(I,G,h) = 1$.*

3. Variable Lebesgue and Sobolev Spaces and Prpoperty of $p(x)$-Laplacian Operator

In the sequel, we consider a naturel number N and a bounded domain $\Omega \subset \mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$.

We introduce the setting of our problem with some auxiliary results of the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. For convenience, we only recall some basic facts with will be used later, we refer to [9–11] for more details.

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geqslant 2$, with a Lipschitz boundary denoted by $\partial\Omega$. Denote

$$C_+(\bar{\Omega}) = \big\{h \in C(\bar{\Omega}) \,\big|\, \inf_{x \in \bar{\Omega}} h(x) > 1\big\}.$$

For any $h \in C_+(\bar{\Omega})$ we define

$$h^+ := \max\big\{h(x), x \in \bar{\Omega}\big\}, \quad h^- := \min\big\{h(x), x \in \bar{\Omega}\big\}.$$

For any $p \in C_+(\bar{\Omega})$ we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u;\, u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\right\}$$

endowed with *Luxemburg norm*

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 / \rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \leqslant 1\Big\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx \quad (\forall\, u \in L^{p(x)}(\Omega)),$$

$(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a Banach space [10, Theorem 2.5], separable and reflexive [10, Corollary 2.7]. Its conjugate space is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the Hölder inequality holds [10, Theorem 2.1]

$$\Big|\int_\Omega uv\,dx\Big| \leqslant \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \leqslant 2|u|_{p(x)}|v|_{p'(x)}. \tag{3}$$

Notice that if (u_n) and $u \in L^{p(\cdot)}(\Omega)$ then the following relations hold true (see [9])

$$|u|_{p(x)} < 1\ (= 1;\ > 1) \iff \rho_{p(x)}(u) < 1\ (= 1;\ > 1),$$

$$|u|_{p(x)} > 1 \implies |u|^{p^-}_{p(x)} \leqslant \rho_{p(x)}(u) \leqslant |u|^{p^+}_{p(x)}, \tag{4}$$

$$|u|_{p(x)} < 1 \implies |u|^{p^+}_{p(x)} \leqslant \rho_{p(x)}(u) \leqslant |u|^{p^-}_{p(x)}, \tag{5}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \iff \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0. \tag{6}$$

From (4) and (5), we can deduce the inequalities

$$|u|_{p(x)} \leqslant \rho_{p(x)}(u) + 1, \tag{7}$$

$$\rho_{p(x)}(u) \leqslant |u|^{p^-}_{p(x)} + |u|^{p^+}_{p(x)}. \tag{8}$$

If $p_1, p_2 \in C_+(\bar\Omega)$, $p_1(x) \leqslant p_2(x)$ for any $x \in \bar\Omega$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Next, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) / |\nabla u| \in L^{p(x)}(\Omega)\big\}.$$

It is a Banach space under the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We also define $W^{1,p(\cdot)}_0(\Omega)$ as the subspace of $W^{1,p(\cdot)}(\Omega)$ which is the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|$. If the exponent $p(\cdot)$ satisfies the log-Hölder continuity condition, i. e., there is a constant $\alpha > 0$ such that for every $x, y \in \Omega$, $x \neq y$ with $|x - y| \leqslant \frac{1}{2}$ one has

$$|p(x) - p(y)| \leqslant \frac{\alpha}{-\log|x - y|}\,, \tag{9}$$

then we have the Poincaré inequality (see [12]), i. e., the exists a constant $C > 0$ depending only on Ω and the function p such that

$$|u|_{p(x)} \leqslant C\,|\nabla u|_{p(x)} \quad (\forall\, u \in W^{1,p(\cdot)}_0(\Omega)). \tag{10}$$

In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has a norm $|\cdot|$ given by

$$|u|_{1,p(x)} = |\nabla u|_{p(\cdot)} \quad (\forall\, u \in W_0^{1,p(x)}(\Omega)),$$

which is equivalent to $\|\cdot\|$. In addition, we have the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ (see [10]). The space $(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)})$ is a Banach space, separable and reflexive (see [9, 10]). The dual space of $W_0^{1,p(x)}(\Omega)$, denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$|v|_{-1,p'(x)} = \inf\left\{|v_0|_{p'(x)} + \sum_{i=1}^{N} |v_i|_{p'(x)}\right\},$$

where the infinimum is taken on all possible decompositions $v = v_0 - \operatorname{div} F$ with $v_0 \in L^{p'(x)}(\Omega)$ and $F = (v_1, \ldots, v_N) \in (L^{p'(x)}(\Omega))^N$.

Next, we discuss the $p(x)$-Laplacian operator

$$-\Delta_{p(x)} u := -\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big).$$

Consider the following functional:

$$J(u) = \int\limits_{\Omega} \frac{1}{p(x)}\, |\nabla u|^{p(x)}\, dx, \quad u \in W_0^{1,p(x)}(\Omega).$$

We know that (see [13]), $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$, and the $p(x)$-Laplacian operator is the derivative operator of J in the weak sense.

We denote $L = J' : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, then

$$\langle Lu, v\rangle = \int\limits_{\Omega} |\nabla u|^{p(x)-2}\, \nabla u \nabla v dx \quad (\forall\, u, v \in W_0^{1,p(x)}(\Omega)).$$

Theorem 3.1 [13, Theorem 3.1]**.** (i) $L : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *is a continuous, bounded and strictly monotone operator;*
(ii) L *is a mapping of class* (S_+)*;*
(iii) L *is a homeomorphism.*

4. Assumption and Main Results

In this section, we study the Dirichlet boundary value problem (1) based on the degree theory in Section 2, where $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\bar{\Omega})$ satisfy the log-Hölder continuity condition (9), $2 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a real-valued function such that:

(f_1) f satisfies the Carathéodory condition, that is, $f(\cdot, \eta)$ is measurable on Ω for all $\eta \in \mathbb{R}$ and $f(x, \cdot)$ is continuous on $\mathbb{R}$ for a. e. $x \in \Omega$.

(f_2) f has the growth condition

$$|f(x, \eta)| \leqslant c\big(k(x) + |\eta|^{q(x)-1}\big)$$

for a. e. $x \in \Omega$ and all $\eta \in \mathbb{R}$, where c is a positive constant, $k \in L^{p'(x)}(\Omega)$ and $q \in C_+(\bar{\Omega})$ with $q^+ < p^-$.

Definition 4.1. We call that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1) if

$$\int\limits_{\Omega} |\nabla u|^{p(x)-2}\, \nabla u \nabla v\, dx = \int\limits_{\Omega} (\lambda u + f(x,u))v\, dx \quad (\forall\, v \in W_0^{1,p(x)}(\Omega)).$$

Lemma 4.1. *Under assumptions* (f_1) *and* (f_2), *the operator* $S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *setting by*

$$\langle Su, v\rangle = -\int\limits_{\Omega} (\lambda u + f(x,u))v\, dx \quad (\forall\, u, v \in W_0^{1,p(x)}(\Omega))$$

is compact.

$\lhd$ Let $\phi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\phi u(x) := -f(x,u) \;\text{ for }\; u \in W_0^{1,p(x)}(\Omega) \;\text{ and }\; x \in \Omega.$$

We first show that ϕ is bounded and continuous.

For each $u \in W_0^{1,p(x)}(\Omega)$, we have by the growth condition (f_2), the inequalities (7) and (8) that

$$\begin{aligned}|\phi u|_{p'(x)} &\leqslant \rho_{p'(x)}(\phi u) + 1 = \int\limits_{\Omega} \big|f(x,u(x))\big|^{p'(x)} + 1 \\ &\leqslant \text{const}\,\big(\rho_{p'(x)}(k) + \rho_{r(x)}(u)\big) + 1 \leqslant \text{const}\,\big(|k|_{p'(x)}^{p'^{+}} + |u|_{r(x)}^{r^{+}} + |u|_{r(x)}^{r^{-}}\big) + 1,\end{aligned}$$

where $r(x) = (q(x)-1)p'(x) \in C_{+}(\bar\Omega)$ with $r(x) < p(x)$. Then, by the continuous embedding $L^{p(x)} \hookrightarrow L^{r(x)}$ and the Poincaré inequality (10), we have

$$|\phi u|_{p'(x)} \leqslant \text{const}\,\big(|k|_{p'(x)}^{p'^{+}} + |u|_{1,p(x)}^{r^{+}} + |u|_{1,p(x)}^{r^{-}}\big) + 1.$$

This implies that ϕ is bounded on $W_0^{1,p(x)}(\Omega)$.

To show that ϕ is continuous, let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and measurable functions h in $L^{p(x)}(\Omega)$ such that

$$u_k(x) \to u(x) \;\text{ and }\; |u_k(x)| \leqslant h(x)$$

for a. e. $x \in \Omega$ and all $k \in \mathbb{N}$. Since f satisfies the Carathodory condition, we obtain that

$$f(x,u_k(x)) \to f(x,u(x)) \;\text{ a. e. }\; x \in \Omega,$$

it follows from (f_2) that

$$\big|f(x,u_k(x))\big| \leqslant c\big(k(x) + |h(x)|^{q(x)-1}\big)$$

for a. e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Since

$$k + |h|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\phi u_k - \phi u) = \int\limits_{\Omega} \big|f(x,u_k(x)) - f(x,u(x))\big|^{p'(x)}\, dx,$$

the dominated convergence theorem and the equivalence (6) implies that

$$\phi u_k \to \phi u \quad \text{in} \quad L^{p'(x)}(\Omega).$$

Thus the entire sequence (ϕu_n) converges to ϕu in $L^{p'(x)}(\Omega)$.

Since the embedding $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is also compact. Therefore, the composition $I^* \circ \phi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is compact. Moreover, considering the operator $K : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ given by

$$\langle Ku, v\rangle = -\int_\Omega \lambda uv \, dx \quad \text{for} \quad u, v \in W_0^{1,p(x)}(\Omega),$$

it can be seen that K is compact, by nothing that the embedding $i : L^{p(x)} \hookrightarrow L^{p'(x)}$ is continuous and $K = -\lambda I^* \circ i \circ I$. We conclude that $S = K + I^* \circ \phi$ is compact. This completes the proof. $\rhd$

Theorem 4.1. *Under assumptions* (f_1) *and* (f_2), *problem* (1) *has a weak solution* u *in* $W_0^{1,p(x)}(\Omega)$.

$\lhd$ Let $S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be as in Lemma 4.1 and $L : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, as in subsection 3.2, setting by

$$\langle Lu, v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx, \quad \forall\, u, v \in W_0^{1,p(x)}(\Omega).$$

Then $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1) if and only if

$$Lu = -Su. \tag{11}$$

Thanks to the properties of the operator L seen in Theorem 3.1 and in view of Minty–Browder Theorem (see [14, Theorem 26A]), the inverse operator $T := L^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ is bounded, continuous and satisfies condition (S_+). Moreover, note by Lemma 4.1 that the operator S is bounded, continuous and quasimonotone.

Consequently, equation (2) is equivalent to

$$u = Tv \quad \text{and} \quad v + S \circ Tv = 0. \tag{12}$$

Following the terminology of [14], the equation $v + S \circ Tv = 0$ is an *abstract Hammerstein equation* in the reflexive Banach space $W^{-1,p'(x)}(\Omega)$.

To solve equation (3), we will apply the degree theory introducing in Section 2. To do this, we first claim that the set

$$B := \big\{v \in W^{-1,p'(x)}(\Omega) \mid v + tS \circ Tv = 0 \ \text{for some} \ t \in [0, 1]\big\}$$

is bounded. Indeed, let $v \in B$. Set $u := Tv$, then $|Tv|_{1,p(x)} = |\nabla u|_{p(x)}$.

If $|\nabla u|_{p(x)} \leqslant 1$, then $|Tv|_{1,p(x)}$ is bounded.

If $|\nabla u|_{p(x)} > 1$, then we get by the implication (4), the growth condition (f_2), the Hölder inequality (3) and the inequality (8) the estimate

$$|Tv|^{p^-}_{1,p(x)} = |\nabla u|^{p^-}_{p(x)} \leqslant \rho_{p(x)}(\nabla u) = \langle Lu, u\rangle = \langle v, Tv\rangle = -t\langle S \circ Tv, Tv\rangle$$
$$= t\int_\Omega (\lambda u + f(x, u))u \, dx \leqslant \text{const} \left(\int_\Omega \lambda |u(x)|^2 \, dx + \int_\Omega |k(x)u(x)| \, dx + \rho_{q(x)}(u)\right)$$
$$\leqslant \text{const}\,\big(\lambda|u|^2_{L^2} + |k|_{p'(x)}|u|_{p(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)}\big) \leqslant \text{const}\,\big(|u|^2_{L^2} + |u|_{p(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)}\big).$$

From the Poincaré inequality (10) and the continuous embedding $L^{p(x)} \hookrightarrow L^2$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we can deduct the estimate

$$|Tv|_{1,p(x)}^{p^-} \leqslant \text{const}\left(|Tv|_{1,p(x)}^2 + |Tv|_{1,p(x)} + |Tv|_{1,p(x)}^{q^+}\right).$$

It follows that $\{Tv|v \in B\}$ is bounded.

Since the operator S is bounded, it is obvious from (3) that the set B is bounded in $W^{-1,p'(x)}(\Omega)$. Consequently, there exists $R > 0$ such that

$$|v|_{-1,p'(x)} < R \quad \text{for all} \quad v \in B.$$

This says that

$$v + tS \circ Tv \neq 0 \quad \text{for all} \quad v \in \partial B_R(0) \quad \text{and all} \quad t \in [0,1].$$

From Lemma 2.1 it follows that

$$I + S \circ T \in \mathscr{F}_T\big(\overline{B_R(0)}\big) \quad \text{and} \quad I = L \circ T \in \mathscr{F}_T\big(\overline{B_R(0)}\big).$$

Since the operators I, S and T are bounded, $I + S \circ T$ is also bounded. We conclude that

$$I + S \circ T \in \mathscr{F}_{T,B}\big(\overline{B_R(0)}\big) \quad \text{and} \quad I \in F_{T,B}\big(\overline{B_R(0)}\big).$$

Consider a homotopy $H : [0,1] \times \overline{B_R(0)} \to W^{-1,p'(x)}(\Omega)$ given by

$$H(t,v) := v + tS \circ Tv \quad \text{for} \quad (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree d stated in Theorem 2.1, we get

$$d(I + S \circ T, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + S \circ Tv = 0.$$

We conclude that $u = Tv$ is a weak solution of (1). This completes the proof. $\rhd$

References

1. *Brouwer, L. E. J.* Uber Abbildung von Mannigfaltigkeiten, *Mathematische Annalen*, 1912, vol. 71, no. 1, pp. 97–115. DOI: 10.1007/BF01456931.
2. *Leray, J. and Schauder, J.* Topologie et Équationes Fonctionnelles, *Annales Scientifiques de l'École Normale Supérieure*, 1934, vol. 51, pp. 45–78. DOI: 10.24033/asens.836.
3. *Berkovits, J.* On the Degree Theory for Nonlinear Mappings of Monotone Type, *Annales Academia Scientiarum Fennica. Series A I. Mathematica Dissertationes*, 1986, vol. 58, 58 p.
4. *Berkovits, J.* Extension of the Leray-Schauder Degree for abstract Hammerstein Type Mappings, *Journal of Differential Equations*, 2007, vol. 234, no. 1, pp. 289–310. DOI: 10.1016/j.jde.2006.11.012.
5. *Berkovits, J. and Mustonen, V.* On Topological Degree for Mappings of Monotone Type, *Nonlinear Analysis: Theory, Methods and Applications*, 1986, vol. 10, no. 12, pp. 1373–1383. DOI: 10.1016/0362-546X(86)90108-2.
6. *Berkovits, J. and Mustonen, V.*, *Nonlinear Mappings of Monotone Type I. Classification and Degree Theory*, Preprint no. 2/88, Mathematics, University of Oulu.

7. *Fan, X. L. and Zhang, Q. H.* Existence of Solutions for $p(x)$-Laplacian Dirichlet Problem, *Nonlinear Analysis: Theory, Methods and Applications*, 2003, vol. 52, no. 8, pp. 1843–1852. DOI: 10.1016/S0362-546X(02)00150-5.
8. *Iliaş, P. S.* Existence and Multiplicity of Solutions of a $p(x)$-Laplacian Equations in a Bounded Domain, *Revue Roumaine des Mathematiques Pures et Appliquees*, 2007, vol. 52, no. 6, pp. 639–653.
9. *Fan, X. L. and Zhao, D.* On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *Journal of Mathematical Analysis and Applications*, 2001, vol. 263, no. 2, pp. 424–446. DOI: 10.1006/jmaa.2000.7617.
10. *Kováčik, O. and Rákosník, J.* On Spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Mathematical Journal*, 1991, vol. 41, pp. 592–618.
11. *Zhao, D., Qiang, W. J. and Fan, X. L.* On Generalizerd Orlicz Spaces $L^{p(x)}(\Omega)$, *Journal of Gansu Sciences*, 1996, vol. 9, no. 2, pp. 1–7.
12. *Samko, S. G.* Density of $\mathscr{C}_0^\infty(\mathbb{R}^N)$ in the Generalized Sobolev Spaces $W^{m,p(x)}(\mathbb{R}^N)$, *Doklady Akademii Nauk*, 1999, vol. 369, no. 4, pp. 451–454.
13. *Chang, K. C. Critical Point Theory and Applications*, Shanghai, Shanghai Scientific and Technology Press, 1986.
14. *Zeidler, E. Nonlinear Functional Analysis and its Applications, II/B: Nonlinear Monotone Operators*, New York, Springer-Verlag, 1985.

Received March 26, 2021

Mustapha Ait Hammou
Sidi Mohamed Ben Abdellah University,
Faculty of Sciences Dhar el Mahraz,
Laboratory LAMA, Department of Mathematics,
Fez, P. O. Box 1796, Morocco,
Professor
E-mail: mustapha.aithammou@usmba.ac.ma
https://orcid.org/0000-0002-3930-3469

Владикавказский математический журнал
2022, Том 24, Выпуск 2, С. 5–13

СУЩЕСТВОВАНИЕ РЕШЕНИЯ КРАЕВОЙ ЗАДАЧИ ДИРИХЛЕ ДЛЯ $p(x)$-ЛАПЛАСИАНА

Мустафа Айт Хамму[1]

[1] Университет Сиди Мохамеда бен Абделла, Фес, Марокко
E-mail: mustapha.aithammou@usmba.ac.ma

Аннотация. Цель настоящей статьи — установить существование слабого решения в пространстве $W_0^{1,p(x)}(\Omega)$ краевой задачи Дирихле для $p(x)$-лапласиана. Наш подход основан на теории топологической степени Берковича для класса деминепрерывных операторов обобщенного (S_+) типа. Используются также свойства лебеговых и соболевских пространство с переменными показателями и специальные свойства $p(x)$-лапласиана. Для того, чтобы использовать упомянутую теорию, задача преобразуется в абстрактное уравнение Гаммерштейна вида $v + S \circ Tv = 0$ в рефлексивном банаховом пространстве $W^{-1,p'(x)}(\Omega)$, которое является двойственным к $W_0^{1,p(x)}(\Omega)$ пространством. Заметим также, что изучаемую проблему можно рассматривать как нелинейную задачу на собственные значения вида $Au = \lambda u$, где $Au := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - f(x,u)$. Если исходная задача имеет слабое решение u, то u является собственной функцией, ассоциированной с собственным значением λ.

Ключевые слова: задача Дирихле, топологическая степень, $p(x)$-лапласиан.

AMS Subject Classification: 35J60,47J05,47H11.

Образец цитирования: *Ait Hammou M.* Existence Results for a Dirichlet Boundary Value Problem Involving the $p(x)$-Laplacian Operator // Владикавк. мат. журн.—2022.—Т. 24, № 2.—С. 5–13 (in English). DOI: 10.46698/s8393-0239-0126-b.