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A COUNTER-EXAMPLE TO THE ANDREOTTI–GRAUERT CONJECTURE

Y. Alaoui[1]

[1] Department of Fundamental Sciences,
Hassan II Institute of Agronomy and Veterinary Sciences,
Madinat Al Irfane, BP 6202, Rabat, 10101, Morocco
E-mail: `y.alaoui@iav.ac.ma`, `comp5123ster@gmail.com`

Abstract. In 1962, Andreotti and Grauert showed that every q-complete complex space X is cohomologically q-complete, that is for every coherent analytic sheaf $\mathscr{F}$ on X, the cohomology group $H^p(X, \mathscr{F})$ vanishes if $p \geqslant q$. Since then the question whether the reciprocal statements of these theorems are true have been subject to extensive studies, where more specific assumptions have been added. Until now it is not known if these two conditions are equivalent. Using test cohomology classes, it was shown however that if X is a Stein manifold and, if $D \subset X$ is an open subset which has C^2 boundary such that $H^p(D, \mathscr{O}_D) = 0$ for all $p \geqslant q$, then D is q-complete. The aim of the present article is to give a counterexample to the conjecture posed in 1962 by Andreotti and Grauert [1] to show that a cohomologically q-complete space is not necessarily q-complete. More precisely, we show that there exist for each $n \geqslant 3$ open subsets $\Omega \subset \mathbb{C}^n$ such that for every $\mathscr{F} \in coh(\Omega)$, the cohomology groups $H^p(\Omega, \mathscr{F})$ vanish for all $p \geqslant n-1$ but Ω is not $(n-1)$-complete.

Key words: q-convex functions, q-convex with corners functions, q-complete spaces, cohomologically q-complete spaces, q-Runge spaces.

AMS Subject Classification: 32E10, 32E40.

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1. Introduction

In 1962, A. Andreotti and H. Grauert [1] showed finiteness and vanishing theorems for cohomology groups of analytic spaces under geometric conditions of q-convexity. Since then the question whether the reciprocal statements of these theorems are true have been subject to extensive studies, where for $q > 1$ more specific assumptions have been added. For example, it is known from the theory of Andreotti–Grauert [1] that a q-complete complex space is always cohomologically q-complete, but it is not known if these two conditions are equivalent except when X is a Stein manifold, $\Omega \subset X$ is cohomologically q-complete with respect to O_Ω and Ω has a smooth boundary [2].

The aim of the present article is to give a counterexample to the conjecture posed by Andreotti and Grauert [1] to show that a cohomologically q-complete space is not necessarily q-complete.

More precisely we will show

Theorem 1. *For each integer $n \geqslant 3$, there is a domain $\Omega \subset \mathbb{C}^n$ which is cohomologically $(n-1)$-complete but Ω is not $(n-1)$-complete.*

2. Preliminaries

Let ϕ be a real valued function in $C^\infty(\Omega)$, where Ω is an open set in $\mathbb{C}^n$ with complex coordinates $z_1, \dots, z_n$. Then we say that ϕ is q-convex if its complex Hessian $\left(\frac{\partial^2\phi(z)}{\partial z_i \partial \bar{z}_j}\right)_{1\leqslant i,j\leqslant n}$ has at most $q-1$ negative or zero eingenvalues for every $z \in \Omega$.

A function $\rho \in C^o(\Omega, \mathbb{R})$ is said to be q-convex with corners, if every point of Ω admits a neighborhood U on which there exist finitely many q-convex functions $\phi_1, \dots, \phi_l$ such that $\rho|_U = \operatorname{Max}(\phi_1, \dots, \phi_l)$.

The open set Ω is called q-complete if there exists a smooth q-convex exhaustion function on Ω.

We say that Ω is cohomologically q-complete, if for every coherent analytic sheaf $\mathscr{F}$ on Ω, the cohomology group $H^p(\Omega, \mathscr{F}) = 0$ for all $p \geqslant q$.

Finally, an open subset D of Ω is called q-Runge, if for each compact set $K \subset \Omega$, there is a q-convex exhaustion function $\phi \in C^\infty(\Omega)$ such that

$$K \subset \{z \in \Omega : \phi(z) < c\} \subset\subset D.$$

It is known from [1] that if D is q-Runge in Ω, then for every coherent analytic sheaf $\mathscr{F} \in \operatorname{coh}(\Omega)$, the restriction map $H^p(\Omega, \mathscr{F}) \longrightarrow H^p(D, \mathscr{F})$ has dense image for all $p \geqslant q-1$, or equivalently, for every open covering $\mathscr{U} = (U_i)_{i\in I}$ of Ω with a fundamental system of Stein neighborhoods of Ω, the restriction map between spaces of cocycles

$$Z^p(\mathscr{U}, \mathscr{F}) \longrightarrow Z^p(\mathscr{U}|_D, \mathscr{F})$$

has dense range for $p \geqslant q-1$.

3. Proof of the Theorem

We consider for $n \geqslant 3$ the functions $\phi_1, \phi_2 : \mathbb{C}^n \to \mathbb{R}$ defined by

$$\phi_1(z) = \sigma_1(z) + \sigma_1(z)^2 + N\|z\|^4 - \frac{1}{4}\|z\|^2,$$

$$\phi_2(z) = -\sigma_1(z) + \sigma_1(z)^2 + N\|z\|^4 - \frac{1}{4}\|z\|^2,$$

where $\sigma_1(z) = \operatorname{Im}(z_1) + \sum_{i=3}^n |z_i|^2 - |z_2|^2$, $z = (z_1, z_2, \dots, z_n)$, and $N > 0$ a positive constant. Then, if N is large enough, the functions ϕ_1 and ϕ_2 are $(n-1)$-convex on $\mathbb{C}^n$ and, if $\rho = \operatorname{Max}(\phi_1, \phi_2)$, then, for $\varepsilon_o > 0$ small enough, the set $D_{\varepsilon_o} = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon_o\}$ is relatively compact in the unit ball $B = B(0,1)$, if N is sufficiently large. This is a special case of an example given and utilized by Diederich and Fornaess in a different context (see [3]).

Proposition 1. *In the situation described above for every coherent analytic sheaf $\mathscr{F}$ on D_{ε_o}, the cohomology groups $H^p(D_{\varepsilon_o}, \mathscr{F})$ vanish for all $p \geqslant n-1$.*

$\lhd$ We consider the set A of all real numbers $\varepsilon \geqslant \varepsilon_o$ such that $H^{n-1}(D_\varepsilon, \mathscr{F}) = 0$, where $D_\varepsilon = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon\}$. To prove Proposition 1, it will be sufficient to show that:

(a) $A \neq \varnothing$ and, if $\varepsilon \in A$ and $\varepsilon' > \varepsilon$, then $\varepsilon' \in A$;

(b) if $\varepsilon_j \searrow \varepsilon$ and $\varepsilon_j \in A$ for all j, then $\varepsilon \in A$;

(c) if $\varepsilon \in A$, $\varepsilon > \varepsilon_o$, there exists $\varepsilon_o \leqslant \varepsilon' < \varepsilon$ such that $\varepsilon' \in A$.

We first prove (a). Choose $\varepsilon_1 > \varepsilon_o$ such that

$$-\varepsilon_1 < \operatorname{Inf}_{z\in\partial D_{\varepsilon_o}}\{\phi_i(z),\ i = 1,2\},$$

and set $D_i = \{z \in D_{\varepsilon_o} : \phi_i(z) < -\varepsilon\}$ for $\varepsilon \geqslant \varepsilon_1$. Then, if D_ε is not empty, the sets $D_i \subset\subset D_{\varepsilon_o}$ are clearly $(n-1)$-complete, since $\frac{-1}{\phi_i+\varepsilon}$ is a $(n-1)$-convex exhaustion function on D_i. Therefore, using the exact sequence of cohomology associated to the Mayer–Vietoris sequence

$$\to H^{n-1}(D_1, \mathscr{F}) \oplus H^{n-1}(D_2, \mathscr{F}) \to H^{n-1}(D_\varepsilon, \mathscr{F}) \to H^n(D_1 \cup D_2, \mathscr{F}) \to$$

one obtains $H^{n-1}(D_\varepsilon, \mathscr{F}) = 0$ and, obviously $[\varepsilon_1, +\infty[\subset A$.

To complete the proof of assertion (a), we remark at first that if $\varepsilon > \varepsilon_o$, then $\dim_{\mathbb{C}} H^{n-1}(D_\varepsilon, \mathscr{F}) < \infty$. In fact, choose finitely many Stein open sets $U_i \subset\subset D_{\varepsilon_o}$, $i = 1, \ldots, k$, such that $\partial D_\varepsilon \subset \bigcup_{i=1}^k U_i$. Let $\theta_j \in C_o^\infty(U_j, \mathbb{R}^+)$ such that $\sum_{j=1}^k \theta_j(x) > 0$ at any point $x \in \partial D_\varepsilon$. Let also $c_i > 0$ be sufficiently small constants such that the functions $\phi_i - \sum_{i=1}^j c_i\theta_i$ are $(n-1)$-convex for $i = 1, 2$ and $1 \leqslant j \leqslant k$. We now define the continuous functions $\rho_j : \mathbb{C}^n \to \mathbb{R}$ by

$$\rho_j = \rho - \sum_{i=1}^j c_i\theta_i, \quad j = 1, \ldots, k.$$

Then ρ_j are $(n-1)$-convex with corners and, if $D_j = \{z \in D_{\varepsilon_o} : \rho_j(z) < -\varepsilon\}$, $j = 1, \ldots, k$, and $D_o = D_\varepsilon$, then $D_o \subset D_1 \subset \cdots \subset D_k$, $D_o \subset\subset D_k \subset\subset D_{\varepsilon_o}$ and $D_j \setminus D_{j-1} \subset\subset U_j$ for $j = 1, \ldots, k$. Moreover, we claim that $H^{n-1}(D_j \cap U_l, \mathscr{F}) = 0$ for $0 \leqslant j \leqslant k$, $1 \leqslant l \leqslant k$. In fact, since $D_j \cap U_l$ can be written in the form $D_j \cap U_l = B_{1,j} \cap B_{2,j}$, where $B_{i,j} = \{z \in U_l : \phi_i - \sum_{r=1}^j c_r\theta_r < -\varepsilon\}$, $i = 1, 2$, is $(n-1)$-complete, because U_l is Stein and $\phi_i - \sum_{r=1}^j c_r\theta_r$ is $(n-1)$-convex in U_l, then from the Mayer–Vietoris sequence

$$\to H^{n-1}(B_{1,j}, \mathscr{F}) \oplus H^{n-1}(B_{2,j}, \mathscr{F}) \to H^{n-1}(D_j \cap U_l, \mathscr{F}) \to H^n(B_{1,j} \cup B_{2,j}, \mathscr{F}) \to$$

it follows that $H^{n-1}(D_j \cap U_l, \mathscr{F}) = 0$. Now using the Mayer–Vietoris sequence

$$\to H^{n-1}(D_j, \mathscr{F}) \to H^{n-1}(D_{j-1}, \mathscr{F}) \oplus H^{n-1}(D_j \cap U_j, \mathscr{F}) \to H^{n-1}(D_{j-1} \cap U_j, \mathscr{F}) \to$$

and noting that $H^{n-1}(D_j \cap U_j, \mathscr{F}) = H^{n-1}(D_{j-1} \cap U_j, \mathscr{F}) = 0$ for all $1 \leqslant j \leqslant k$, we find that $H^{n-1}(D_k, \mathscr{F}) \to H^{n-1}(D_\varepsilon, \mathscr{F})$ is surjective. It follows from Theorem 11 of [1] that $\dim_{\mathbb{C}} H^{n-1}(D_\varepsilon, \mathscr{F}) < \infty$.

Let now $\varepsilon \in A$ and $\varepsilon' > \varepsilon$. Then $D_{\varepsilon'} \subset\subset D_\varepsilon$ is n-Runge in D_ε. Indeed, if $K \subset D_{\varepsilon'}$ is a compact set, there exists a $(n-1)$-convex exhaustion function $\psi_i \in C^\infty(B)$, such that $K \subset \{\psi_i < 0\} \subset\subset D_i = \{z \in B : \phi_i(z) < -\varepsilon'\}$, $i = 1, 2$ because D_i is obviously $(n-1)$-Runge in B, the function ϕ_i being $(n-1)$-convex and B is Stein. Then a suitable smooth n-convex approximation of $\operatorname{Max}(\psi_1, \psi_2)$ [4] shows that $D_{\varepsilon'}$ is n-Runge in D_ε. We deduce from [3] that $D_\varepsilon \setminus D_{\varepsilon'}$ has no compact connected components and, therefore the restriction map

$$H^{n-1}(D_\varepsilon, \mathscr{F}) \longrightarrow H^{n-1}(D_{\varepsilon'}, \mathscr{F})$$

has dense image. This proves that $H^{n-1}(D_{\varepsilon'}, \mathscr{F}) = 0$ and $\varepsilon' \in A$.

The proof of statement (c) will result from two lemmas.

We now put $\psi_{i,j} = \phi_i - \sum_{i=1}^j c_i\theta_i$, $\psi_{i,o} = \phi_i$, $i = 1, 2$, $1 \leqslant j \leqslant k$, and define the open sets $D_{j,i}$, as follows $D_{1,o} = \{\phi_1 < -\varepsilon, \phi_2 < -\varepsilon\}$, $D_{1,1} = \{\psi_{1,1} < -\varepsilon, \phi_2 < -\varepsilon\}$, $D_{1,2} = \{\psi_{1,1} < -\varepsilon, \psi_{2,1} < -\varepsilon\}$. And for $2 \leqslant j \leqslant k$ we set $D_{j,o} = \{\psi_{1,j-1} < -\varepsilon, \psi_{2,j-1} < -\varepsilon\}$, $D_{j,1} = \{\psi_{1,j} < -\varepsilon, \psi_{2,j-1} < -\varepsilon\}$, $D_{j,2} = \{\psi_{1,j} < -\varepsilon, \psi_{2,j} < -\varepsilon\}$. Obviously, $D_{1,o} = D_o$ and $D_{j,2} = D_{j+1,o} = D_j$ for $1 \leqslant j \leqslant k$. Therefore $D_o = D_{1,o} \subset D_{1,1} \subset D_{1,2} = D_{2,o} \subset \cdots \subset D_{k,o} \subset D_{k,1} \subset D_{k,2} = D_k$.

Lemma 1. *The restriction map $H^{n-2}(D_{j+1},\mathscr{F}) \to H^{n-2}(D_j,\mathscr{F})$ has a dens image for all $0 \leqslant j \leqslant k-1$.*

$\lhd$ It is clear that it is sufficient to show that the restriction map

$$H^{n-2}(D_{j+1,1},\mathscr{F}) \longrightarrow H^{n-2}(D_j,\mathscr{F})$$

has a dense image. We have $D_j \cap U_{j+1} = B_{1,j} \cap B_{2,j}$ and $D_{j+1,1} \cap U_{j+1} = B_{1,j+1} \cap B_{2,j}$, where $B_{i,j} = \{z \in U_{j+1} : \psi_{i,j} < -\varepsilon\}$, $i = 1,2$, and $B_{1,j+1} = \{z \in U_{j+1} : \psi_{1,j+1}(z) < -\varepsilon\}$. Note also that $B_{i,j}$ are $(n-1)$-complete and $(n-1)$-Runge in the Stein set U_{j+1}. Also it is easy to see that $B_{1,j}$ is $(n-1)$-Runge in $B_{1,j+1}$, which shows that the restriction map

$$H^{n-2}(B_{1,j+1},\mathscr{F}) \longrightarrow H^{n-2}(B_{1,j},\mathscr{F})$$

has a dense range. Moreover, we can choose the open sets U_i, $1 \leqslant i \leqslant k$, so that $U_{j+1} \setminus (B_{1,j} \cup B_{2,j})$ has no compact connected components, which implies according to [4] that for every coherent analytic sheaf $\mathscr{G}$ on U_{j+1}, the restriction map

$$H^{n-1}(B_{1,j+1} \cup B_{2,j},\mathscr{G}) \longrightarrow H^{n-1}(B_{1,j} \cup B_{2,j},\mathscr{G})$$

has also a dense image. Now consider the following commutative diagrams given by the Mayer–Vietoris sequence for cohomology

$$\begin{array}{ccccccc}
\to H^{n-2}(B_{1,j+1},\mathscr{F}) \oplus H^{n-2}(B_{2,j},\mathscr{F}) & \to & H^{n-2}(D_{j+1,1} \cap U_{j+1},\mathscr{F}) & \to & H^{n-1}(B_{1,j+1} \cup B_{2,j},\mathscr{F}) & \to & 0 \\
\downarrow \rho_1 \oplus id & & \rho_2 \downarrow & & \rho_3 \downarrow & & \\
\to H^{n-2}(B_{1,j},\mathscr{F}) \oplus H^{n-2}(B_{2,j},\mathscr{F}) & \to & H^{n-2}(D_j \cap U_{j+1},\mathscr{F}) & \xrightarrow{u} & H^{n-1}(B_{1,j} \cup B_{2,j},\mathscr{F}) & \to & 0.
\end{array}$$

Since u is surjective, then u is open by Lemma 3.2 of [2] and, since $\rho_1 \oplus id$ and ρ_3 have dense image, it follows that ρ_2 has also a dense image.

Now since $\operatorname{Supp} \theta_j \subset U_j$, $j = 1,\dots,k$, then $D_{j,i+1} \setminus D_{j,i} \subset\subset U_j$ and $D_{j,i+1} = D_{j,i} \cup (D_{j,i+1} \cap U_j)$. So the Mayer–Vietoris sequence for cohomology gives the exactness of the sequence $\dots \to H^{n-2}(D_{j+1,1},\mathscr{F}) \to H^{n-2}(D_j,\mathscr{F}) \oplus H^{n-2}(D_{j+1,1} \cap U_{j+1},\mathscr{F}) \to H^{n-2}(D_j \cap U_{j+1},\mathscr{F}) \to H^{n-1}(D_{j+1,1},\mathscr{F}) \to \dots$ Since $H^{n-2}(D_{j+1,1} \cap U_{j+1},\mathscr{F}) \to H^{n-2}(D_j \cap U_{j+1},\mathscr{F})$ has a dense image and $\dim_{\mathbb{C}} H^{n-1}(D_{j+1,1},\mathscr{F}) < \infty$, then in view of the proof of Theorem 11 of [1] the restriction map

$$H^{n-2}(D_{j+1,1},\mathscr{F}) \to H^{n-2}(D_j,\mathscr{F})$$

has also a dense image. $\rhd$

Lemma 2. *Suppose that $\varepsilon \in A$. Then there is $\varepsilon_o \leqslant \varepsilon' < \varepsilon$ such that $\varepsilon' \in A$.*

$\lhd$ Let $\mathscr{V} = (V_i)_{i\in\mathbb{N}}$ be an open covering of D_{ε_o} with a fundamental system of Stein neighborhoods of D_{ε_o} such that if $V_{i_o} \cap \dots \cap V_{i_r} \neq \varnothing$ and $V_{i_o} \cup \dots \cup V_{i_r} \subset D_{j+1}$, then $V_{i_o} \cup \dots \cup V_{i_r} \subset D_j$ or $V_{i_o} \cup \dots \cup V_{i_r} \subset U_{j+1} \cap D_{j+1}$.

We first show that $H^{n-1}(D_k,\mathscr{F}) = 0$. We shall prove it assuming that it has already been proved for $j < k$. For this, we consider the Mayer–Vietoris sequence for cohomology

$$\to H^{n-2}(D_j,\mathscr{F}) \oplus H^{n-2}(D_{j+1} \cap U_{j+1},\mathscr{F}) \xrightarrow{r^*} H^{n-2}(D_j \cap U_{j+1},\mathscr{F}) \xrightarrow{j^*} H^{n-1}(D_{j+1},\mathscr{F}) \xrightarrow{\rho^*}.$$

Let ξ be a cocycle in $Z^{n-1}(\mathscr{V}|_{D_{j+1}},\mathscr{F})$ and let $\rho(\xi)$ be its restriction to a cocycle in $Z^{n-1}(\mathscr{V}|_{D_j},\mathscr{F})$. Since $\rho(\xi)$ is a coboundary by induction and $H^{n-1}(D_{j+1} \cap U_{j+1},\mathscr{F}) = 0$, from the Mayer–Vietoris sequence, it follows that there exist

$$\eta \in Z^{n-2}(\mathscr{V}|_{D_j \cap U_{j+1}},\mathscr{F}) \quad \text{and} \quad \mu \in C^{n-2}(\mathscr{V}|_{D_{j+1}},\mathscr{F}),$$

such that $\xi = j(\eta) + \delta\mu$. There exists a sequence $\{\eta_n\} \subset Z^{n-2}(\mathscr{V}|_{D_{j+1}\cap U_{j+1}}, \mathscr{F})$ with $r(\eta_n) - \eta \to 0$, when $n \to \infty$. This is possible because $Z^{n-2}(\mathscr{V}|_{D_{j+1}\cap U_{j+1}}, \mathscr{F}) \to Z^{n-2}(\mathscr{V}|_{D_j\cap U_{j+1}}, \mathscr{F})$ has a dense range. Now choose a sequence $\{\gamma_n\} \subset C^{n-2}(\mathscr{V}|D_{j+1}, \mathscr{F})$ such that $j(r(\eta_n)) = \delta\gamma_n$. Then

$$\xi - \delta\mu - \delta\gamma_n = j(\eta - r(\eta_n)).$$

This proves that $\delta\mu + \delta\gamma_n$ converges to ξ when $n \to \infty$. Since $\dim_{\mathbb{C}} H^{n-1}(D_{j+1}, \mathscr{F}) < \infty$, then the coboundary space $B^{n-1}(\mathscr{V}|_{D_{j+1}}, \mathscr{F})$ is closed in $Z^{n-1}(\mathscr{V}|_{D_{j+1}}, \mathscr{F})$. Therefore $\xi \in B^{n-1}(\mathscr{V}|_{D_{j+1}}, \mathscr{F})$ and $H^{n-1}(D_j, \mathscr{F}) = 0$ for all $0 \leqslant j \leqslant k$. On the other hand, there exists $\varepsilon' > 0$ such that $\varepsilon - \varepsilon' > \varepsilon_o$ and $D_{\varepsilon-\varepsilon'} = \{z \in D_{\varepsilon_o} : \rho(z) < \varepsilon' - \varepsilon\} \subset\subset D_k$.

Since $H^{n-1}(D_k, \mathscr{F}) \to H^{n-1}(D_{\varepsilon-\varepsilon'}, \mathscr{F})$ has a dense image, $H^{n-1}(D_k, \mathscr{F}) = 0$ and $\dim_{\mathbb{C}} H^{n-1}(D_{\varepsilon-\varepsilon'}, \mathscr{F}) < \infty$, then $H^{n-1}(D_{\varepsilon'-\varepsilon'}, \mathscr{F}) = 0$. Whence $\varepsilon - \varepsilon' \in A$. $\triangleright$

In order to prove statement (b), it is sufficient to show that if $\varepsilon_j \searrow \varepsilon$ and $\varepsilon_j \in A$ for all j, then

$$H^{n-2}(D_{\varepsilon_{j+1}}, \mathscr{F}) \longrightarrow H^{n-2}(D_{\varepsilon_j}, \mathscr{F})$$

has dense image (Cf. [1, p. 250]). To complete the proof of Proposition 1, it is therefore enough to prove the following lemma.

Lemma 3. *The restriction map $H^{n-2}(D_{\varepsilon_o}, \mathscr{F}) \to H^{n-2}(D_\varepsilon, \mathscr{F})$ has dense image for every real number $\varepsilon \geqslant \varepsilon_o$.*

$\triangleleft$ We consider the set T of all $\varepsilon \geqslant \varepsilon_o$, such that $H^{n-2}(D_\varepsilon, \mathscr{F}) \to H^{n-2}(D_{\varepsilon_1}, \mathscr{F})$ has dense image for every real number $\varepsilon_1 > \varepsilon$.

To see that $T \neq \varnothing$, we choose $\varepsilon > \varepsilon_o$, such that $-\varepsilon < \mathrm{Min}_{\overline{D}_{\varepsilon_o}}\{\phi_i(z),\, i = 1, 2\}$, and let $\varepsilon_1 > \varepsilon$. If D_{ε_1} is not empty, $D_i = \{z \in B : \phi_i(z) < -\varepsilon_1\}$ and $D_i' = \{z \in B : \phi_i(z) < -\varepsilon\}$ are relatively compact in D_{ε_o}, $(n-1)$-complete and $(n-1)$-Runge in B. Moreover, $D_i \subset\subset D_i'$ is clearly $(n-1)$-Runge in D_i'. Therefore $H^{n-2}(D_i', \mathscr{F}) \to H^{n-2}(D_i, \mathscr{F})$ has dense image for $i = 1, 2$. Also it is easy to see that $D_{\varepsilon_0} \setminus D_1 \cup D_2$ has no compact connected component, which means that $D_1 \cup D_2$ is n-Runge in D_{ε_0}, or equivalently for every coherent analytic sheaf $\mathscr{G}$ on D_{ε_0}, the restriction homomorphisms

$$H^{n-1}(D_1' \cup D_2, \mathscr{F}) \to H^{n-1}(D_1 \cup D_2, \mathscr{F}) \quad \text{and} \quad H^{n-1}(D_1' \cup D_2', \mathscr{F}) \to H^{n-1}(D_1' \cup D_2, \mathscr{F})$$

have dense images. Consequently we can show exactly as in Lemma 1 that if $D_{\varepsilon,1} = D_1' \cap D_2$, then we have the density of the image of the restriction maps $H^{n-2}(D_{\varepsilon,1}, \mathscr{F}) \to H^{n-2}(D_{\varepsilon_1}, \mathscr{F})$, and $H^{n-2}(D_\varepsilon, \mathscr{F}) \to H^{n-2}(D_{\varepsilon,1}, \mathscr{F})$. This proves that $\varepsilon \in T$ and, clearly $[\varepsilon, +\infty[\subset T$.

Let now $\varepsilon_j \in T$, $j \geqslant 0$, such that $\varepsilon_j \searrow \varepsilon$, and let $\mathscr{U} = (U_i)_{i\in I}$ be a Stein open covering of D_{ε_o} with a countable base of open subsets of D_{ε_o}. Then the restriction map between spaces of cocycles $Z^{n-2}(\mathscr{U}|_{D_{\varepsilon_{j+1}}}, \mathscr{F}) \to Z^{n-2}(\mathscr{U}|_{D_{\varepsilon_j}}, \mathscr{F})$ has dense image for $j \geqslant 0$. Let $\varepsilon' > \varepsilon$ and $j \in \mathbb{N}$, such that $\varepsilon' > \varepsilon_j$. By [1, p. 246], the restriction map $Z^{n-2}(\mathscr{U}|_{D_\varepsilon}, \mathscr{F}) \to Z^{n-2}(\mathscr{U}|_{D_{\varepsilon_j}}, \mathscr{F})$ has dense image. Since $\varepsilon_j \in T$, then $Z^{n-2}(\mathscr{U}|_{D_{\varepsilon_j}}, \mathscr{F}) \to Z^{n-2}(\mathscr{U}|_{D_{\varepsilon'}}, \mathscr{F})$ has also dense image, and hence $\varepsilon \in T$.

To prove that T is open in $[\varepsilon_o, +\infty[$ it is sufficient to show that if $\varepsilon \in T$, $\varepsilon > \varepsilon_o$, then there is $\varepsilon_o < \varepsilon' < \varepsilon$, such that $\varepsilon' \in T$. But this can be done in the same way as in the proof of Lemma 1. We consider a finite covering $(U_i)_{1\leqslant i\leqslant k}$ of ∂D_ε by Stein open sets $U_i \subset\subset D_{\varepsilon_o}$ and compactly supported functions $\theta_i \in C_o^\infty(U_i)$, $\theta_j \geqslant 0$, $j = 1, \ldots, k$, such that $\sum_{i=1}^k \theta_i(x) > 0$ at any point of ∂D_ε. Define $D_j = \{z \in D_{\varepsilon_o} : \rho_j(z) < -\varepsilon\}$, where $\rho_j(z) = \mathrm{Max}(\phi_1 - \sum_1^j c_i\theta_i, \phi_2 - \sum_1^j c_i\theta_i)$

with $c_i > 0$ sufficiently small so that $\psi_{i,j} = \phi_i - \sum_1^j c_i\theta_i$ are still $(n-1)$-convex for $i = 1, 2$ and $1 \leqslant j \leqslant k$. By Lemma 1, the restriction map

$$H^{n-2}(D_k, \mathscr{F}) \longrightarrow H^{n-2}(D_\varepsilon, \mathscr{F})$$

has dense image. Since ρ is proper and $D_\varepsilon \subset\subset D_k$, there exists $\varepsilon_o < \varepsilon' < \varepsilon$, such that $D_\varepsilon \subset\subset D_{\varepsilon'} \subset\subset D_k$. Then obviously the restriction

$$H^{n-2}(D_{\varepsilon'}, \mathscr{F}) \longrightarrow H^{n-2}(D_\varepsilon, \mathscr{F})$$

has dense range, and hence $\varepsilon' \in T$, which completes the proof of Lemma 3. $\triangleright$

We have shown that D_{ε_0} is cohomologically $(n-1)$-complete. We are now going to prove that for a good choice of the contants ε_0 and N, introduced in Proposition 1, we can find a constant $\varepsilon > \varepsilon_0$ such that D_ε is cohomologically $(n-1)$-complete but not $(n-1)$-complete.

In fact, it was shown by Diederich–Fornaess [4] that if $\delta > 0$ is small enough, then the topological sphere

$$S_\delta = \left\{ z \in \mathbb{C}^n : x_1^2 + |z_2|^2 + \cdots + |z_n|^2 = \delta, \quad y_1 = -\sum_{i=3}^{n} |z_i|^2 + |z_2|^2 \right\}$$

is not homologous to 0 in D_{ε_0}. This follows from the fact that the set $E = \{z \in \mathbb{C}^n : x_1 = z_2 = \ldots = z_n = 0\}$ does not intersect D_{ε_0}, since on E

$$\rho(z) = \mathrm{Max}\left(y_1 + \frac{3}{4}y_1^2 + Ny_1^4, -y_1 + \frac{3}{4}y_1^2 + Ny_1^4 \right) \geqslant 0.$$

So the following real form of degree $2n-2$

$$\omega = \left(\sum_{i=1}^{n} x_i^2 + \sum_{i=2}^{n} y_i^2 \right)^{-2n+1} \left(\sum_{i=1}^{n} (-1)^i x_i dx_1 \wedge \ldots \widehat{dx_i} \wedge \cdots \wedge dx_n \wedge dy_2 \wedge \cdots \wedge dy_n \right.$$
$$\left. + \sum_{i=1}^{n-1} (-1)^{n+i} y_{1+i} dx_1 \wedge \cdots \wedge dx_n \wedge dy_2 \wedge \cdots \wedge \widehat{dy_{1+i}} \wedge \cdots \wedge dy_n \right)$$

is well-defined and d-closed on D_{ε_0}. Since ω does not depend on y_1, then by the standard argument $\int_{S_\delta} \omega \neq 0$. Therefore S_δ is not homologous to 0 in D_{ε_0}.

Let $\mathscr{E}_q$ be the sheaf of germs of C^∞ q-forms on $\mathbb{C}^n$ and $\mathscr{T}_q$ the sheaf of germs of C^∞ d-closed q-forms. Then we have an exact sequence of sheaf homomorphisms

$$0 \to \mathscr{T}_q \to \mathscr{E}_q \xrightarrow{d} \mathscr{T}_{q+1} \to 0.$$

Since by the de Rham theorem for every $p \geqslant 1$, the cohomology group $H^p(D_{\varepsilon_0}, \mathbb{C})$ is isomorphic to

$$\frac{\{\omega \in \Gamma(D_{\varepsilon_0}, \mathscr{E}_q) : df = 0\}}{\{d\omega : \omega \in \Gamma(D_{\varepsilon_0}, \mathscr{E}_{q-1})\}},$$

it follows from Stockes formula that $H^{2n-2}(D_{\varepsilon_0}, \mathbb{C})$ does not vanish.

We are going to show that $H^r(D_{\varepsilon_0}, \mathscr{O}_{D_{\varepsilon_0}}) = 0$ for all r with $1 \leqslant r \leqslant n-3$.

We first assert that we can choose N, ε_0 and $\varepsilon > \varepsilon_0$ such that, if with the notations of Proposition 1 we set

$$\sigma_1(z) = \mathrm{Im}\ z_1 + \sum_{i=3}^{n} |z_i|^2 - |z_2|^2, \quad \psi(z) = N\|z\|^4 - \frac{1}{4}\|z\|^2 + \varepsilon_0,$$
$$\rho(z) = |\sigma_1(z)| + \sigma_1(z)^2 + \psi(z) - \varepsilon_0,$$

then we obtain

$$D_\varepsilon = \big\{z \in D_\varepsilon : m + M < \phi(z) < \varepsilon_0 - \varepsilon\big\},$$

where $m = \mathrm{Min}_{z\in\overline{D}_{\varepsilon_0}} \psi(z)$, $M = \mathrm{Max}_{z\in\overline{D}_{\varepsilon_0}} \psi(z)$ and $\phi(z) = \sigma_1(z) + \sigma_1(z)^2 + m$.

In fact, we can choose $\varepsilon > \varepsilon_0$ sufficiently big and $\lambda > 0$ small enough so that $\varepsilon_0 - \varepsilon < m - M$, $m < (1+\lambda)$. $\mathrm{Min}_{z\in\overline{D}_\varepsilon} \psi(z)$ and $\lambda\varepsilon - (1+\lambda)\varepsilon_0 > 0$.

On the other hand, if $\delta = \mathrm{Min}_{z\in\overline{D}_{\varepsilon_0}} \|z\|^2$, then we have

$$0 < \delta \leqslant \|z\|^2 < \frac{1}{4N} - \frac{\varepsilon_0}{N} \quad \text{for every} \quad z \in \overline{D}_{\varepsilon_0}.$$

Therefore by suitable choice of ε_0, ε and N we can also achieve that

$$\left(N\|z\|^4 - \frac{1}{4}\|z\|^2\right) - \mathrm{Min}_{z\in\overline{D}_{\varepsilon_0}}\left(N\|z\|^4 - \frac{1}{4}\|z\|^2\right) < \mathrm{Min}\left(\frac{\varepsilon-\varepsilon_0}{2}, \lambda\varepsilon - (1+\lambda)\varepsilon_0\right),$$

and

$$\mathrm{Max}_{z\in\overline{D}_{\varepsilon_0}}\left(N\|z\|^4 - \frac{1}{4}\|z\|^2\right) - \left(N\|z\|^4 - \frac{1}{4}\|z\|^2\right) < \mathrm{Min}\left(\frac{\varepsilon-\varepsilon_0}{2}, \lambda\varepsilon - (1+\lambda)\varepsilon_0\right)$$

for every $z \in \overline{D}_\varepsilon$.

Because $\psi(z) < \varepsilon_0 - \varepsilon$ on $\overline{D}_\varepsilon$, then clearly we obtain

$$\phi(z) = \sigma_1(z) + \sigma_1(z)^2 + m < \sigma_1(z) + \sigma_1(z)^2 + (1+\lambda)\psi(z)) < (1+\lambda)(\varepsilon_0 - \varepsilon), \quad z \in \overline{D}_\varepsilon.$$

Moreover,

$$\phi(z) = m - (\psi(z) + \varepsilon - \varepsilon_0) > m + M, \quad \text{when} \quad z \in \partial D_\varepsilon \quad \text{and} \quad \sigma_1(z) \geqslant 0.$$

Furthermore, if $z \in \partial D_\varepsilon$ and $\sigma_1(z) < 0$, then

$$-\sigma_1(z) + \sigma_1(z)^2 + M = -\sigma_1(z) + \sigma_1(z)^2 + \left(N\|z\|^4 - \frac{1}{4}\|z\|^2\right) + M - \left(N\|z\|^4 - \frac{1}{4}\|z\|^2\right).$$

Hence

$$-\sigma_1(z) + \sigma_1(z)^2 + M = -\varepsilon + \varepsilon_0 + \mathrm{Max}_{z\in\overline{D}_{\varepsilon_0}}\left(N\|z\|^4 - \frac{1}{4}\|z\|^2\right) - \left(N\|z\|^4 - \frac{1}{4}\|z\|^2\right) < \frac{\varepsilon_0 - \varepsilon}{2}.$$

This implies that $\sigma_1(z) > \sigma_1(z)^2 + M + \frac{\varepsilon-\varepsilon_0}{2}$ and, therefore

$$\phi(z) > 2\sigma_1(z)^2 + m + M + \frac{\varepsilon - \varepsilon_0}{2} > m + M,$$

when $z \in \overline{D}_\varepsilon$ and $\sigma_1(z) < 0$, which shows that

$$D_\varepsilon = \big\{z \in D_\varepsilon : m + M < \phi(z) < \varepsilon_0 - \varepsilon\big\}.$$

We are now going to show that for every none-positive real numbers α, β with $m+M<\alpha<\beta<\varepsilon_0-\varepsilon$, the open sets

$$B_{\alpha,\beta}=\{z\in D_\varepsilon:\ \alpha<\phi(z)<\beta\}$$

are relatively compact in D_ε.

To see this, we consider a sequence $(z_j)_{j\geqslant 0}\subset B_{\alpha,\beta}$ which converges to a point $z\in\overline{D}_\varepsilon$. Suppose first that $\sigma_1(z)>0$. Then one has for every sufficiently large integer j

$$\rho(z_j)=\sigma_1(z_j)+\sigma_1(z_j)^2+N\|z_j\|^4-\frac{1}{4}\|z_j\|^2<-\varepsilon.$$

Since

$$\phi(z_j)<\varepsilon_0-\varepsilon+\lambda\psi(z_j)<(1+\lambda)(\varepsilon_0-\varepsilon)$$

and

$$N\|z_j\|^4-\frac{1}{4}\|z_j\|^2-\mathrm{Min}_{z\in\overline{D}_{\varepsilon_0}}\left(N\|z\|^4-\frac{1}{4}\|z\|^2\right)<\lambda\varepsilon-(1+\lambda)\varepsilon_0,$$

then

$$\rho(z_j)=\phi(z_j)+N\left(\|z_j\|^4-\frac{1}{4}\|z_j\|^2\right)-m<\varepsilon_0-\varepsilon+\lambda\psi(z_j)+\lambda\varepsilon-(1+\lambda)\varepsilon_0.$$

A passage to the limit shows that

$$\rho(z)\leqslant\varepsilon_0-\varepsilon+\lambda\psi(z)+\lambda\varepsilon-(1+\lambda)\varepsilon_0<(1+\lambda)(\varepsilon_0-\varepsilon)+\lambda\varepsilon-(1+\lambda)\varepsilon_0=-\varepsilon,$$

because $\psi(z)<\varepsilon_0-\varepsilon$, which implies that $z\in D_\varepsilon$.

If now $\sigma_1(z)<0$ and $z\in\partial D_\varepsilon$, then $\sigma_1(z)>\sigma_1(z)^2+M+\frac{\varepsilon-\varepsilon_0}{2}$. Therefore

$$\rho(z)<-M+\frac{\varepsilon_0-\varepsilon}{2}+N\|z\|^4-\frac{1}{4}\|z\|^2-\mathrm{Min}_{z\in\overline{D}_{\varepsilon_0}}\left(N\|z\|^4-\frac{1}{4}\|z\|^2\right)+m-\varepsilon_0.$$

In order to have the inequality

$$-M+\frac{\varepsilon_0-\varepsilon}{2}+N\|z\|^4-\frac{1}{4}\|z\|^2-\mathrm{Min}_{z\in\overline{D}_{\varepsilon_0}}\left(N\|z\|^4-\frac{1}{4}\|z\|^2\right)+m-\varepsilon_0<-\varepsilon$$

it suffices to have $m-M-\varepsilon_0<-\varepsilon$, since

$$N\|z\|^4-\frac{1}{4}\|z\|^2-\mathrm{Min}_{z\in\overline{D}_{\varepsilon_0}}\left(N\|z\|^4-\frac{1}{4}\|z\|^2\right)<\frac{\varepsilon-\varepsilon_0}{2}.$$

As this condition is satisfied, we conclude that with such a choice of ε_0, N and ε the limit $z\in D_\varepsilon$, and hence the open set

$$B_{\alpha,\beta}=\left\{z\in D_\varepsilon:\ \alpha<\phi(z)<\beta\right\}$$

is relatively compact in D_ε for all real numbers α, β, with $m<\alpha<\beta<\varepsilon_0-\varepsilon$.

Now since ϕ is in addition 3-convex, then a similar proof of theorem 15 of [1] shows that, if Ω^i is the sheaf of germs of holomorphic i-forms on $\mathbb{C}^n$, $i\geqslant 0$ ($\Omega^0=\mathscr{O}_{\mathbb{C}^n}$), and $B_c=\{z\in D_\varepsilon:\ \phi(z)<c\}$ for $c\leqslant\varepsilon_0-\varepsilon$, then the map

$$H^r(D_\varepsilon,\Omega^i)\longrightarrow H^r(D_\varepsilon\setminus B_c,\Omega^i)$$

is injective for every $r < n-2$ and $c < \varepsilon_0 - \varepsilon$. Then obviously $H^r(D_\varepsilon, \Omega^i) = 0$ for $1 \leqslant r \leqslant n-3$ and $i \geqslant 0$. In fact, let $c_0 = \mathrm{Max}_{z \in \overline{D}_\varepsilon} \phi(z)$. Then there exists $z_1 \in \partial D_\varepsilon$ such that $\phi(z_1) = c_0$. Since $c_0 = \phi(z_1) = \sigma_1(z_1) + \sigma_1(z_1)^2 + m < \rho(z_1) + \varepsilon_0 \leqslant \varepsilon_0 - \varepsilon$, then $B_{c_0} = D_\varepsilon$, and hence $H^r(D_\varepsilon, \Omega^i) = 0$ for $1 \leqslant r \leqslant n-3$.

We now consider the resolution of the constant sheaf $\mathbb{C}$ on D_ε

$$0 \to \mathbb{C} \to \mathscr{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \to \xrightarrow{d} \Omega^n \to 0.$$

If we set $Z^j = \mathrm{Im}\left(\Omega^{j-1} \xrightarrow{d} \Omega^j\right)$ for $1 \leqslant j \leqslant n-1$, then we get short exact sequences

$$0 \to \mathbb{C} \to \mathscr{O} \to Z^1 \to 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 \to Z^j \to \Omega^j \to Z^{j+1} \to 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 \to Z^{n-2} \to \Omega^{n-2} \to Z^{n-1} \to 0$$

$$0 \to Z^{n-1} \to \Omega^{n-1} \to \Omega^n \to 0.$$

Since, by Proposition 1, D_ε is cohomologically $(n-1)$-complete, then $H^r(D_\varepsilon, \Omega^i) = 0$ for all $r \geqslant n-1$ and $i \geqslant 0$. So we obtain the isomorphisms

$$H^{n-1}(D_\varepsilon, Z^{n-1}) \cong \ldots \cong H^{2n-3}(D_\varepsilon, Z^1) \cong H^{2n-2}(D_\varepsilon, \mathbb{C})$$

and the exact sequence

$$\cdots \to H^{n-2}(D_\varepsilon, \Omega^n) \to H^{n-1}(D_\varepsilon, Z^{n-1}) \to H^{n-1}(D_\varepsilon, \Omega^{n-1}) = 0.$$

We deduce that the map

$$H^{n-2}(D_\varepsilon, \Omega^n) \xrightarrow{\varphi} H^{2n-2}(D_\varepsilon, \mathbb{C})$$

is surjective. The map φ is defined as follows: If a differential form $\omega \in C^\infty_{n,n-2}(D_\varepsilon)$ satisfies the equation $\overline{\partial}\omega = 0$, then ω is also d-closed and therefore defines a cohomology class in $H^{2n-2}(D_\varepsilon, \mathbb{C})$.

Moreover, since, by [6], every d-closed differential form $\omega \in C^\infty_{n,n-2}(D_\varepsilon)$ is cohomologous to a $\overline{\partial}$-closed $(n, n-2)$ differential form $\omega' \in C^\infty_{n,n-2}(D_\varepsilon)$, it follows that the map

$$H^{n-2}(D_\varepsilon, \Omega^n) \xrightarrow{\varphi} H^{2n-2}(D_\varepsilon, \mathbb{C})$$

is bijective.

Now, if we suppose that D_ε is $(n-1)$-complete, then there exists a C^∞ strictly $(n-1)$-convex function $\psi : D_\varepsilon \to \mathbb{R}$, such that $D_{\varepsilon,c} = \{z \in D_\varepsilon : \psi(z) < c\}$ is relatively compact in D_ε for every $c \in \mathbb{R}$.

Notice that for the given ε, if $\delta > 0$ is small enough, the topological sphere

$$S_\delta = \left\{z \in \mathbb{C}^n : x_1^2 + |z_2|^2 + \cdots + |z_n|^2 = \delta,\ \sigma_1(z) = 0\right\} \subset D_\varepsilon.$$

Since ψ is exhaustive on D_ε, there exists $c' > 0$, such that S_δ is not homologuous to 0 in $D_{\varepsilon,c'}$. Let $c > c'$. Then $D_{\varepsilon,c}$ and $D_{\varepsilon,c'}$ are $(n-1)$-complete and, similarly $H^p(D_{\varepsilon,c}, \Omega^i) = H^p(D_{\varepsilon,c'}, \Omega^i) = 0$ for $1 \leqslant p \leqslant n-3$ and $i \geqslant 0$. Also the maps $H^{n-2}(D_{\varepsilon,c}, \Omega^n) \to H^{2n-2}(D_{\varepsilon,c}, \mathbb{C})$ and $H^{n-2}(D_{\varepsilon,c'}, \Omega^n) \to H^{2n-2}(D_{\varepsilon,c'}, \mathbb{C})$ are bijective. Moreover, since the levi form of ψ has at least 2 strictly positive eingenvalues, then by using Morse theory (see for instance [7]) we find that

$$H^{2n-2}(D_{\varepsilon,c}, \mathbb{C}) \cong H^{2n-2}(D_{\varepsilon,c'}, \mathbb{C}).$$

It follows from the commutative diagram of continuous maps

$$\begin{array}{ccc} H^{n-2}(D_{\varepsilon,c}, \Omega^n) & \to & H^{2n-2}(D_{\varepsilon,c}\mathbb{C}) \\ \downarrow & & \downarrow \\ H^{n-2}(D_{\varepsilon,c'}, \Omega^n) & \to & H^{2n-2}(D_{\varepsilon,c'}, \mathbb{C}) \end{array}$$

that the restriction homomorphism

$$H^{n-2}(D_{\varepsilon,c}, \Omega^n) \to H^{n-2}(D_{\varepsilon,c'}, \Omega^n)$$

is bijective. Since in addition $D_{\varepsilon,c'}$ is relatively compact in $D_{\varepsilon,c}$, the function ψ being exhaustive on D_ε, then, according to theorem 11 of [1], one obtains

$$\dim_{\mathbb{C}} H^{n-2}(D_{\varepsilon,c'}, \Omega^n) < \infty.$$

Since the sheaf Ω^n is isomorphic to $\mathscr{O}_{D_\varepsilon}$, then we have also $\dim_{\mathbb{C}} H^{n-2}(D_{\varepsilon,c'}, \mathscr{O}_{D_\varepsilon}) < \infty$. Furthermore, since $D_{\varepsilon,c'}$ is cohomologically $(n-1)$-complete and $H^r(D_\varepsilon, \mathscr{O}_{D_\varepsilon}) = 0$ for $1 \leqslant r \leqslant n-3$, it follows from Theorem 1 of [8] that $D_{\varepsilon,c'}$ is Stein, which is in contradiction with the fact that $H^{2n-2}(D_{\varepsilon,c'}, \mathbb{C}) \neq 0$, since $S_\delta \subset D_{\varepsilon,c'}$ is not homologous to 0 in $D_{\varepsilon,c'}$. We conclude that D_ε is cohomologically $(n-1)$-complete but not $(n-1)$-complete.

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Youssef Alaoui
Hassan II Institute of Agronomy and Veterinary Sciences,
Madinat Al Irfane, BP 6202, Rabat, 10101, Morocco,
Professor
E-mail: `y.alaoui@iav.ac.ma`, `comp5123ster@gmail.com`
https://orcid.org/0000-0002-7014-8671

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КОНТРПРИМЕР К ГИПОТЕЗЕ АНДРЕОТТИ — ГРАУЭРТА

Алауи Ю.[1]

[1] Институт агрономии и ветеринарии Хасана II,
Марокко, BP 6202, Рабат

E-mail: `ashaw2912@gmail.com`

Аннотация. В 1962 г. Андреотти и Грауэрт показали, что любое q-полное комплексное пространство X когомологически q-полно, т. е. для любого когерентного аналитического пучка $\mathscr{F}$ на X группа когомологии $H^p(X, \mathscr{F})$ исчезает при $p \geqslant q$. С тех пор вопрос о том, верно ли обратное утверждение, является предметом обширных исследований, в ходе которых появились и другие специальные предположения. До сих пор неизвестно, являются ли эти два утверждения эквивалентны. Используя тестовые классы когомологий было показано, что если X — многообразие Стейна, а $D \subset X$ — открытое множество с C^2 границей, причем $H^p(D, \mathscr{O}_D) = 0$ для всех $p \geqslant q$, то D является q-полным. Цель настоящей статьи — дать контрпример к гипотезе Андреотти и Грауэрта 1962 г., показывающий, что когомологически q-полное пространство не обязательно является q-полным. Точнее мы показали, что для любого $n \geqslant 3$ существует открытое множество $\Omega \subset \mathbb{C}^n$ такое, что для всех $\mathscr{F} \in coh(\Omega)$, группы когомологий $H^p(\Omega, \mathscr{F})$ исчезают для всех $p \geqslant n-1$, но Ω не является $(n-1)$-полным.

Ключевые слова: q-выпуклая функция, q-выпуклая функция с углами, q-полное пространство, когомологически q-полное пространство, пространство q-Рунге.

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