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MOLLIFICATIONS OF CONTACT MAPPINGS OF ENGEL GROUP

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Abstract. The contact mappings belonging to the metric Sobolev classes are studied on an Engel group with a left-invariant sub-Riemannian metric. In the Euclidean space one of the main methods to handle non-smooth mappings is the mollification, i. e., the convolution with a smooth kernel. An extra difficulty arising with contact mappings of Carnot groups is that the mollification of a contact mapping is usually not contact. Nevertheless, in the case considered it is possible to estimate the magnitude of deviation of contactness sufficiently to obtain useful results. We obtain estimates on convergence (or sometimes divergence) of the components of the differential of the mollified mapping to the corresponding components of the Pansu differential of the contact mapping. As an application to the quasiconformal analysis, we present alternative proofs of the convergence of mollified horizontal exterior forms and the commutativity of the pull-back of the exterior form by the Pansu differential with the exterior differential in the weak sense. These results in turn allow us to obtain such basic properties of mappings with bounded distortion as Hölder continuity, differentiability almost everywhere in the sense of Pansu, Luzin $\mathscr{N}$-property.

Key words: Carnot group, Engel group, quasiconformal mappings, bounded distortion.

AMS Subject Classification: 30C65, 58C25.

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1. Introduction

Mollification or convolution with a smooth kernel is one of the main tools of analysis that allow us to transfer results from spaces of smooth functions to more general function spaces. However, for contact mappings, that is mappings that preserve a fixed distribution in the tangent bundle, mollifications lead to an undesirable effect: the mollified mapping is not contact anymore. In [1, 2] it is shown that at least for mappings between two-step Carnot groups we can give estimates on the failure of the mollified mapping to be contact sufficient to prove convergence of horizontal differential forms

$$(f^\varepsilon)^*\omega \to f^\#\omega \tag{1.1}$$

as $\varepsilon \to 0$ in L^ν, here $f^\#$ is a pull-back of the form with the Pansu differential $\widehat{d}$ of the contact mapping f, ν is the Hausdorff dimension of Carnot group. The formula (1.1) in turn allows to obtain a crucial fact of quasiconformal analysis

$$df^\#\omega = f^\# d\omega \tag{1.2}$$

in distributional sense. In the theory of mappings with bounded distortion developed by Yu. G. Reshetnyak [3, 4] the commutativity of outer differential and superposition allows to prove that the superposition $u = v \circ f$ of n-harmonic function v on $\mathbb{R}^n$ with the bounded distortion mapping f is a weak solution of the quasilinear elliptic equation $\operatorname{div} \mathscr{A}(x, \nabla u) = 0$.

For 2-step Carnot groups in [1, 2] a weaker statement is proved: if $d\omega = 0$ in weak sense, then $df^{\#}\omega = 0$. In full the formula (1.2) is proved in [5] for mappings of 2-step Carnot groups and in the recent preprint [6] for mappings of Carnot groups of arbitrary step. The proof of (1.2) in [5] does not rely on the convergence of mollifiers (1.1) but relies essentially on 2-step structure of the group. In [6] the formula (1.2) is derived from (1.1).

In this paper we obtain for a contact mapping f on Engel group $\mathbb{E}$ (the simplest example of 3-step Carnot group) estimates on convergence (or sometimes divergence) of components of the "Jacobi matrix" of the mollified mapping f^ε. However, in order to obtain meaningful estimates we should compute this matrix not in the Euclidean basis but in the graded basis of left invariant vector fields. Precisely, if $X_1, \dots, X_4$ is a graded basis of the algebra of left-invariant vector fields of Engel group, we are interested in the behavior as $\varepsilon \to 0$ of the coefficients a_{ij}^ε in the decomposition

$$X_i f^\varepsilon = df^\varepsilon \langle X_i \rangle = \sum_{j=1}^{4} a_{ij}^\varepsilon X_j, \quad i = 1, \dots, 4.$$

If we introduce the dual basis of 1-forms $\xi_1, \dots, \xi_4$ such that $\xi_i \langle X_j \rangle = \delta_{ij}$, $i, j = 1, \dots, 4$, the coefficients are $a_{ij}^\varepsilon = \xi_j \langle X_i f^\varepsilon \rangle$. By a *form of the weight* l we mean 1-form dual to the vector field of the degree l (and thus having the homogeneous degree $-l$). The main result of the paper is the following

Theorem 1.1. *Let $\Omega, \Omega' \subseteq \mathbb{E}$, $f : \Omega \to \Omega'$ be weakly contact of class $W^{1,p}_{\mathrm{loc}}(\Omega)$. If X is a left-invariant vector field of the degree k, ξ is a left-invariant 1-form of the weight l, $p \geqslant l$, then on every compact $K \subset \Omega$ we have*

$$\xi \langle X f^\varepsilon \rangle = \begin{cases} O(\varepsilon^{l-k}), & l > k, \\ \xi \langle \widehat{d} f \langle X \rangle \rangle + o(1), & l = k, \\ o\bigl(\frac{1}{\varepsilon^{k-l}}\bigr), & l < k, \end{cases}$$

as $\varepsilon \to 0$ in $L^{p/l}(K)$.

Section 3 of this paper is dedicated to the proof of the theorem in Engel case. For Engel group the result is new and may have independent interest, it also serves as an alternative way to prove (1.1) and (1.2). The analogue of Theorem 1.1 for 2-step groups follows from Lemma 2.1 and Lemma 3.3 in [2]. To prove Theorem 1.1 on Engel group we modify the approach of [2] to obtain finer estimates on divergence of components of the mollified mapping.

2. Carnot Groups

Recall that a *stratified graded nilpotent group* or a *Carnot group* (see e. g. [7–9]) is a connected simply connected Lie group $\mathbb{G}$ such that its algebra of left-invariant vector fields $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{g} = V_1 \oplus V_2 \oplus \dots \oplus V_M$ of vector spaces V_k satisfying $[V_1, V_k] = V_{k+1}$, $k = 1, \dots, M-1$, and $[V_1, V_M] = \{0\}$. Left-invariant vector field $L \in \mathfrak{g}$ is the *field of the degree* k if $L \in V_k$. The subspace $V_1 = H\mathbb{G}$ is a *horizontal space* of $\mathfrak{g}$, its elements are the *horizontal vector fields*.

Elements $g \in \mathbb{G}$ in some "privileged" coordinate system may be identified with elements $(x_1, \dots, x_M) \in \mathbb{R}^{\dim V_1} \times \dots \times \mathbb{R}^{\dim V_M} = \mathbb{R}^N$ in a way that *dilations* $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$, $\lambda > 0$, defined by $\delta_\lambda(x_1, x_2, \dots, x_M) = (\lambda x_1, \lambda^2 x_2, \dots, \lambda^M x_M)$ are group automorphisms.

The *homogeneous norm* on $\mathbb{G}$ is a function $\|\cdot\| : \mathbb{G} \to \mathbb{R}$ such that
1) $\|0\| = 0$, $\|g\| > 0$, when $g \neq 0$;
2) $\|\delta_\lambda g\| = \lambda \|g\|$ for all $\lambda \geqslant 0$, $g \in \mathbb{G}$;
3) there is $Q \geqslant 1$ such that $\|gh\| \leqslant Q(\|g\| + \|h\|)$ for all $g, h \in \mathbb{G}$.

The homogeneous norm generates left-invariant *homogeneous quasimetric* $d(g, h) = \|h^{-1} g\|$. In constract to metric it satisfies only the generalized triangle inequality

$$d(g_1, g_3) \leqslant Q\big(d(g_1, g_2) + d(g_2, g_3)\big), \quad g_1, g_2, g_3 \in \mathbb{G},$$

where $Q \geqslant 1$ is a constant from the definition of the homogeneous norm.

The ball in this quasimetric with the centre $g \in \mathbb{G}$ and the radius r denote by $B_r(g)$. The topology given by d coincides with the Euclidean topology of $\mathbb{R}^N$. The Lebesgue measure dx on $\mathbb{R}^N$ is the bi-invariant Haar measure on $\mathbb{G}$ and $d(\delta_\lambda x) = \lambda^\nu \, dx$, where $\nu = \sum_{j=1}^M j \dim V_j$ is the homogeneous dimension of the group.

Let $\Omega \subseteq \mathbb{G}$ be open. The space $L^p(\Omega)$, $p \geqslant 1$, consists of measurable functions $u : \Omega \to \mathbb{R}$ integrable in the p-th power. The norm on $L^p(\Omega)$ is defined by

$$\|u\|_{p,\Omega} = \bigg(\int\limits_\Omega |u(g)|^p \, dg \bigg)^{\frac{1}{p}}.$$

When $\Omega = \mathbb{G}$, we write $\|u\|_p = \|u\|_{p,\mathbb{G}}$.

Let left invariant vector fields $X_1, \dots, X_n$ be the basis of the horizontal space $H\mathbb{G}$. The Sobolev space $W^{1,p}(\Omega)$, $p \geqslant 1$, is a space of functions $u \in L^p(\Omega)$ that have distributional derivatives $X_j u \in L^p(\Omega)$ along the vector fields X_j, $j = 1, \dots, n$, that is such functions g_j that

$$\int\limits_\Omega g_j(x) \varphi(x) \, dx = -\int\limits_\Omega u(x) X_j \varphi(x) \, dx, \quad j = 1, \dots, n,$$

for all $\varphi \in C_0^\infty(\Omega)$. The norm on $W^{1,p}$ is $\|u \mid W^{1,p}(\Omega)\| = \|u\|_{p,\Omega} + \big\| |\nabla_H u| \big\|_{p,\Omega}$, where $\nabla_H u = (X_1 u, \dots, X_n u)$. We say that $u \in L^p_{\mathrm{loc}}(\Omega)$ and $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$, when $u \in L^p(K)$ and $v \in W^{1,p}(K)$ for each compact $K \subset \Omega$. A mapping $f : \Omega \to \mathbb{G}$ $f = (f_1, \dots, f_N)$ is in class $W^{1,p}(\Omega)$ or $W^{1,p}_{\mathrm{loc}}(\Omega)$ if all its components f_j are in the corresponding class.

A mapping $f : \Omega \to \mathbb{G}$ of class $W^{1,1}_{\mathrm{loc}}(\Omega)$ is (*weakly*) *contact* if $X_j f(x) \in H_{f(x)}\mathbb{G}$, $j = 1, \dots, n$, for a. e. $x \in \Omega$. The *formal horizontal differential* $d_H f(x) : H_x\mathbb{G} \to H_{f(x)}\mathbb{G}$ of a contact mapping f is a linear mapping such that $d_H f(x)\langle X_j \rangle = X_j f(x)$. It is proved in [10, 11] that the horizontal differential extends to the contact homomorphism of Lie algebras $\widehat{d}f(x) : T_x\mathbb{G} \to T_{f(x)}\mathbb{G}$ that we call the *formal Pansu differential* ($\mathscr{P}$-differential) of f at x.

The *convolution* of measurable functions u, v on Carnot group $\mathbb{G}$ is defined as

$$u * v(x) = \int\limits_{\mathbb{G}} u(y) v(y^{-1} x) \, dy = \int\limits_{\mathbb{G}} u(x y^{-1}) v(y) \, dy, \quad x \in \mathbb{G},$$

if the integral converges.

Lemma 2.1 (Convolution Properties [8])**.** 1. *Let* $p, q \in [1, +\infty]$, $u \in L^p(\mathbb{G})$, $v \in L^q(\mathbb{G})$. *Then* $u * v \in L^r(\mathbb{G})$, *where* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, *and the following Young inequality holds*

$$\|u * v\|_r \leqslant \|u\|_p \|v\|_q.$$

2. *If L is a left invariant vector field on $\mathbb{G}$, u, v are smooth compactly supported functions, then*

$$L(u * v) = u * (Lv), \quad (Lu) * v = u * (\widetilde{L}v),$$

where $\widetilde{L}$ is the right-invariant vector field agreeing with L at the origin.

For $\varphi : \mathbb{G} \to \mathbb{R}$ and $\varepsilon > 0$ define

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^\nu} \varphi \circ \delta_{1/\varepsilon}(x), \quad x \in \mathbb{G},$$

where ν is the homogeneous dimension of $\mathbb{G}$. If $\varphi \in L^1(\mathbb{G})$, then $\int_{\mathbb{G}} \varphi_\varepsilon(x)\, dx = \int_{\mathbb{G}} \varphi(x)\, dx$.

Lemma 2.2. *For every $\varphi \in C_0^\infty(\mathbb{G})$, left-invariant field L of the degree k and right-invariant field $\widetilde{L}$ such that $\widetilde{L}(0) = L(0)$ we have*

$$L\varphi_\varepsilon = \frac{1}{\varepsilon^k}(L\varphi)_\varepsilon, \quad \widetilde{L}\varphi_\varepsilon = \frac{1}{\varepsilon^k}(\widetilde{L}\varphi)_\varepsilon.$$

$\lhd$ Since dilation δ_λ is an automorphism of $\mathbb{G}$, that is $\delta_\lambda(x \cdot y) = \delta_\lambda x \cdot \delta_\lambda y$, for the left translation $\ell_x(y) = x \cdot y$ we have $\delta_\lambda \circ \ell_x = \ell_{\delta_\lambda x} \circ \delta_\lambda$. If $L(0) \in V_k(0)$, then $D\delta_\lambda L(0) = \lambda^k L(0)$. Therefore, for left-invariant $L \in V_k$

$$D\delta_\lambda \langle L(x) \rangle = D\delta_\lambda \circ D\ell_x \langle L(0) \rangle = D\ell_{\delta_\lambda x} \circ D\delta_\lambda \langle L(0) \rangle = \lambda^k L(\delta_\lambda x).$$

Thus,

$$L\varphi_\varepsilon(x) = \frac{1}{\varepsilon^\nu} L(\varphi \circ \delta_{1/\varepsilon})(x) = \frac{1}{\varepsilon^\nu} \cdot \frac{1}{\varepsilon^k}(L\varphi)(\delta_{1/\varepsilon} x) = \frac{1}{\varepsilon^k}(L\varphi)_\varepsilon(x).$$

Obviously, the same argument holds for the right translation $r_x(y) = y \cdot x$. $\rhd$

Lemma 2.3 (Properties of Mollifications [8])**.** *Let $\varphi \in C_0^\infty(\mathbb{G})$ and $\int_{\mathbb{G}} \varphi(x)\, dx = a$. Then*

1. *If $u \in L^1_{\mathrm{loc}}(\mathbb{G})$ then $u * \varphi_\varepsilon \to au$ a. e. as $\varepsilon \to 0$.*

2. *If $u \in L^p(\mathbb{G})$, $p \in [1, +\infty)$, then $\|u * \varphi_\varepsilon - au\|_p \to 0$ as $\varepsilon \to 0$.*

3. *If u is bounded on $\mathbb{G}$ and is continuous on open set $\Omega \subseteq \mathbb{G}$ then $u * \varphi_\varepsilon \to au$ uniformly on compact subsets of Ω as $\varepsilon \to 0$.*

3. Mollifications on Engel Group

Engel group $\mathbb{E}$ is a 4-dimensional 3-step Carnot group i. e. $\mathbb{E} = (\mathbb{R}^4, \cdot)$, and $\mathfrak{g}(\mathbb{E}) = V_1 \oplus V_2 \oplus V_3$, $\dim V_1 = 2$, $\dim V_2 = \dim V_3 = 1$. Such a group is unique up to an isomorphism. For convenience we use coordinate system (x, y, z, t) such that the group operation has the form

$$(x, y, z, t)(x', y', z', t') = \left(x + x', y + y', z + z' + xy', t + t' + xz' + \frac{x^2 y'}{2} \right).$$

Thus, the dilation is

$$\delta_\lambda(x, y, z, t) = (\lambda x, \lambda y, \lambda^2 z, \lambda^3 t),$$

algebra of left-invariant vector fields is spanned by the graded basis

$$X = \partial_x, \quad Y = \partial_y + x\,\partial_z + \frac{1}{2}x^2\,\partial_t, \quad Z = [X, Y] = \partial_z + x\,\partial_t, \quad T = [X, Z] = \partial_t,$$

and algebra of right-invariant vector fields — by the basis

$$\widetilde{X} = \partial_x + y\,\partial_z + z\,\partial_t, \quad \widetilde{Y} = \partial_y, \quad \widetilde{Z} = -[\widetilde{X}, \widetilde{Y}] = \partial_z, \quad \widetilde{T} = -[\widetilde{X}, \widetilde{Z}] = \partial_t. \tag{3.1}$$

Further, we fix on Engel group infinitely smooth nonnegative function $\psi : \mathbb{E} \to \mathbb{R}$ supported in the unit ball such that $\int_{\mathbb{E}} \psi(g)\,dg = 1$.

For $\Omega \subseteq \mathbb{E}$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ define *mollification* $u^\varepsilon = \bar{u} * \psi_\varepsilon$, where $\bar{u}$ is u extended on $\mathbb{E} \setminus \Omega$ by zero.

Lemma 3.1. *There are functions $\chi_{ij} \in C_0^\infty(B_1(0))$, $i, j = 1, 2$, such that*

$$\int\limits_{\mathbb{E}} \chi_{ij}(g)\,dg = \delta_{ij}, \quad i, j = 1, 2, \tag{3.2}$$

where δ_{ij} is the Kronecker delta, and such that for every $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, compact $K \subset \Omega$, $g \in K$ and $0 < \varepsilon < \operatorname{dist}(K, \partial\Omega)$

$$Xu^\varepsilon = (Xu) * \chi_{11,\varepsilon} + (Yu) * \chi_{12,\varepsilon}, \quad Yu^\varepsilon = (Xu) * \chi_{21,\varepsilon} + (Yu) * \chi_{22,\varepsilon}. \tag{3.3}$$

$\lhd$ We find functions χ_{ij}, $i, j = 1, 2$, from the equations

$$X\psi = \widetilde{X}\chi_{11} + \widetilde{Y}\chi_{12}, \quad Y\psi = \widetilde{X}\chi_{21} + \widetilde{Y}\chi_{22}.$$

Using the expressions of the right-invariant vector fields (3.1) we obtain

$$\begin{aligned} X\psi &= \widetilde{X}\psi - y\widetilde{Z}\psi - z\widetilde{T}\psi = \widetilde{X}\psi - \widetilde{Z}(y\psi) - \widetilde{T}(z\psi) \\ &= \widetilde{X}\psi - \widetilde{Y}\widetilde{X}(y\psi) + \widetilde{X}\widetilde{Y}(y\psi) + \widetilde{X}\widetilde{Y}\widetilde{X}(z\psi) - \widetilde{X}\widetilde{X}\widetilde{Y}(z\psi) - \widetilde{Y}\widetilde{X}\widetilde{X}(z\psi) + \widetilde{X}\widetilde{Y}\widetilde{X}(z\psi) \\ &= \widetilde{X}\Big[\psi + \widetilde{Y}(y\psi) - \widetilde{Y}\widetilde{X}(z\psi) + 2\widetilde{Y}\widetilde{X}(z\psi)\Big] - \widetilde{Y}\Big[\widetilde{X}(y\psi) + \widetilde{X}\widetilde{X}(z\psi)\Big], \\ Y\psi &= \widetilde{Y}\psi + x\widetilde{Z}\psi + \frac{x^2}{2}\widetilde{T}\psi = \widetilde{Y}\psi + \widetilde{Z}(x\psi) + \widetilde{T}\Big(\frac{x^2}{2}\psi\Big) \\ &= \widetilde{Y}\psi + \widetilde{Y}\widetilde{X}(x\psi) - \widetilde{X}\widetilde{Y}(x\psi) + \widetilde{Y}\widetilde{X}\widetilde{X}\Big(\frac{x^2}{2}\psi\Big) - \widetilde{X}\widetilde{Y}\widetilde{X}\Big(\frac{x^2}{2}\psi\Big) - \widetilde{X}\widetilde{Y}\widetilde{X}\Big(\frac{x^2}{2}\psi\Big) + \widetilde{X}\widetilde{X}\widetilde{Y}\Big(\frac{x^2}{2}\psi\Big) \\ &= \widetilde{Y}\Big[\psi + \widetilde{X}(x\psi) + \widetilde{X}\widetilde{X}\Big(\frac{x^2}{2}\psi\Big)\Big] - \widetilde{X}\Big[\widetilde{Y}(x\psi) + 2\widetilde{Y}\widetilde{X}\Big(\frac{x^2}{2}\psi\Big) - \widetilde{X}\widetilde{Y}\Big(\frac{x^2}{2}\psi\Big)\Big]. \end{aligned}$$

The expressions in square brackets are the desired functions χ_{ij}, $i, j = 1, 2$, for instance, $\chi_{12} = -\widetilde{X}(y\psi) - \widetilde{X}\widetilde{X}(z\psi)$. On the one hand, the functions are of the form

$$\chi_{ij} = \delta_{ij}\psi + \widetilde{X}a_{ij} + \widetilde{Y}b_{ij}, \quad i, j = 1, 2,$$

where δ_{ij} is the Kronecker delta, $a_{ij}, b_{ij} \in C_0^\infty(B_1(0))$. Since integrating by parts yields $\int_{\mathbb{E}} \widetilde{X}a_{ij}(g)\,dg = \int_{\mathbb{E}} \widetilde{Y}b_{ij}(g)\,dg = 0$, the statement (3.2) of Lemma follows. On the other hand, from the expressions of χ_{ij} and Lemma 2.2 it follows

$$X\psi_\varepsilon = \frac{1}{\varepsilon}\big(X\psi\big)_\varepsilon = \frac{1}{\varepsilon}\big(\widetilde{X}\chi_{11}\big)_\varepsilon + \frac{1}{\varepsilon}\big(\widetilde{Y}\chi_{12}\big)_\varepsilon = \widetilde{X}\chi_{11,\varepsilon} + \widetilde{Y}\chi_{12,\varepsilon},$$

and in the same way $Y\psi_\varepsilon = \widetilde{X}\chi_{21,\varepsilon} + \widetilde{Y}\chi_{22,\varepsilon}$. This together with Lemma 2.1 leads to the statement (3.3) of Lemma. $\rhd$

Lemma 3.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p \geqslant 1$, $K \subset \Omega$ be a compact, X_1, X_2, X_3 be horizontal left-invariant vector fields. Then $X_1u^\varepsilon \to X_1u$ a. e. on K and*

$$\big\|X_1u^\varepsilon - X_1u\big\|_{p,K} = o(1), \quad \big\|X_2X_1u^\varepsilon\big\|_{p,K} = o\Big(\frac{1}{\varepsilon}\Big), \quad \big\|X_3X_2X_1u^\varepsilon\big\|_{p,K} = o\Big(\frac{1}{\varepsilon^2}\Big),$$

as $\varepsilon \to 0$.

◁ Any horizontal left-invariant vector field is a linear combination of basic vector fields, e. g. $X_1 = aX + bY$. For simplicity assume $X_1 = X$, the argument for $X_1 = Y$ is the same and the general case is a linear combination of the two. By Lemma 3.1 we have $Xu^\varepsilon = (Xu)*\chi_{11,\varepsilon} + (Yu)*\chi_{12,\varepsilon}$ and $\int_{\mathbb{E}} \chi_{11} = 1$, $\int_{\mathbb{E}} \chi_{12} = 0$. By properties of mollifications $Xu^\varepsilon \to Xu$ as $\varepsilon \to 0$ a. e. and in $L^p(K)$. Next,

$$X_2(Xu^\varepsilon) = (Xu) * X_2(\chi_{11,\varepsilon}) + (Yu) * X_2(\chi_{12,\varepsilon}) = \frac{1}{\varepsilon}(Xu) * (X_2\chi_{11})_\varepsilon + \frac{1}{\varepsilon}(Yu) * (X_2\chi_{12})_\varepsilon.$$

By properties of mollifications, e. g., $(Xu) * (X_2\chi_{11})_\varepsilon \to 0$ as $\varepsilon \to 0$ in $L^p(K)$. Analogously,

$$X_3X_2(Xu^\varepsilon) = \frac{1}{\varepsilon^2}(Xu) * (X_3X_2\chi_{11})_\varepsilon + \frac{1}{\varepsilon^2}(Yu) * (X_3X_2\chi_{12})_\varepsilon,$$

where each expression after the $\frac{1}{\varepsilon^2}$ term vanishes in $L^p(K)$ as $\varepsilon \to 0$. ▷

In the next proof we use the following pointwise estimate for Sobolev functions:

Proposition 3.1 (see [12–14]). *Let $K \subset \Omega$ be a compact. For every $u \in W^{1,p}(\Omega)$ there is $0 \leqslant g \in L^p(K)$, such that*

$$|u(y) - u(z)| \leqslant d(y, z)\big(g(y) + g(z)\big)$$

for a. e. $y, z \in V$ and $\|g\|_{p,K} \leqslant C\|\nabla_H u\|_{p,\Omega}$. Moreover, the constant C is independent of u.

Lemma 3.3. *Let $u \in W^{1,p}(\Omega)$, $v \in L^p(\Omega)$, X_1, X_2 be horizontal left invariant vector fields, $\varphi \in C_0^\infty(B(0,1))$, and $K \subset \Omega$ be a compact.*

1. *If $p \geqslant 2$, then for $F_\varepsilon(x) = (uv) * \varphi_\varepsilon(x) - u^\varepsilon(x)(v * \varphi_\varepsilon)(x)$ we have*

$$\|F_\varepsilon\|_{p/2,K} = O(\varepsilon), \quad \|X_1F_\varepsilon\|_{p/2,K} = o(1), \quad \|X_2X_1F_\varepsilon\|_{p/2,K} = o\left(\frac{1}{\varepsilon}\right), \tag{3.4}$$

as $\varepsilon \to 0$.

2. *If $p \geqslant 3$, then for $G_\varepsilon(x) = (u^2v) * \varphi_\varepsilon(x) - 2u^\varepsilon(x)(uv) * \varphi_\varepsilon(x) + (u^\varepsilon)^2(x)(v * \varphi_\varepsilon)(x)$ we have*

$$\|G_\varepsilon\|_{p/3,K} = O(\varepsilon^2), \quad \|X_1G_\varepsilon\|_{p/3,K} = O(\varepsilon), \quad \|X_2X_1G_\varepsilon\|_{p/3,K} = o(1), \tag{3.5}$$

as $\varepsilon \to 0$.

◁ Let $x \in K$, $0 < \varepsilon < \varepsilon_0 = \frac{1}{3}\operatorname{dist}(x, \partial\Omega)$, and $\widetilde{K}$ be an ε_0-neighborhood of K. For the summands in the expression of $F_\varepsilon(x)$ we have

$$(uv) * \varphi_\varepsilon(x) = \int\limits_{\widetilde{K}} u(z)v(z)\varphi_\varepsilon(z^{-1}x)\,dz = \iint\limits_{\widetilde{K}\times\widetilde{K}} u(z)v(z)\psi_\varepsilon(y^{-1}x)\varphi_\varepsilon(z^{-1}x)\,dy\,dz,$$

$$u^\varepsilon(x)(v * \varphi_\varepsilon)(x) = \iint\limits_{\widetilde{K}\times\widetilde{K}} u(y)v(z)\psi_\varepsilon(y^{-1}x)\varphi_\varepsilon(z^{-1}x)\,dy\,dz.$$

Thus,

$$(uv) * \varphi_\varepsilon(x) - u^\varepsilon(x)(v * \varphi_\varepsilon)(x) = \iint\limits_{\widetilde{K}\times\widetilde{K}} \big(u(y) - u(z)\big)v(z)\psi_\varepsilon(y^{-1}x)\varphi_\varepsilon(z^{-1}x)\,dy\,dz.$$

By Proposition 3.1 there is $g \in L^p(\widetilde{K})$, such that

$$u(y) - u(z) \leqslant d(y, z)\big(g(y) + g(z)\big)$$

and $\|g\|_{p,\widetilde{K}} \leqslant C\|\nabla_H u\|_{p,\Omega}$. Since the term under the integral is nonzero only when $d(x,y) \leqslant \varepsilon$ and $d(x,z) \leqslant \varepsilon$ we get

$$
\begin{aligned}
&|(uv) * \varphi_\varepsilon(x) - u^\varepsilon(x)(v * \varphi_\varepsilon)(x)| \\
&\qquad \leqslant 2Q\varepsilon \iint\limits_{\widetilde{K}\times\widetilde{K}} \big(g(y) + g(z)\big)\, |v(z)|\, \psi_\varepsilon(y^{-1}x)\, |\varphi_\varepsilon(z^{-1}x)|\, dy\, dz \\
&\qquad\qquad = 2Q\varepsilon\big(g^\varepsilon(x)(|v| * |\varphi_\varepsilon|)(x) + (g|v|) * |\varphi_\varepsilon|(x)\big),
\end{aligned}
$$

where Q is the constant from the generalized triangle inequality. Next, using Hölder and Young inequalities we derive

$$
\big\|(uv) * \varphi_\varepsilon - u^\varepsilon(v * \varphi_\varepsilon)\big\|_{p/2,K} \leqslant 4Q\varepsilon\|g\|_{p,\widetilde{K}}\|v\|_{p,\widetilde{K}}\|\varphi_\varepsilon\|_1 \leqslant \varepsilon(4QC\|\varphi\|_1)\|\nabla_H u\|_{p,\Omega}\|v\|_{p,\Omega}.
$$

Since for the horizontal vector field X_1 we have $X_1\varphi_\varepsilon = \frac{1}{\varepsilon}(X_1\varphi)_\varepsilon$, it follows

$$
\begin{aligned}
&X_1\big((uv) * \varphi_\varepsilon - u^\varepsilon(v * \varphi_\varepsilon)\big)(x) \\
&\qquad = \frac{1}{\varepsilon} \iint\limits_{\widetilde{K}\times\widetilde{K}} \big(u(y) - u(z)\big)v(z)\Big((X_1\psi)_\varepsilon(y^{-1}x)\varphi_\varepsilon(z^{-1}x) \\
&\qquad\qquad + \psi_\varepsilon(y^{-1}x)(X_1\varphi)_\varepsilon(z^{-1}x)\Big)\, dy\, dz,
\end{aligned}
$$

and from that

$$
\begin{aligned}
&\big|X_1\big((uv) * \varphi_\varepsilon - u^\varepsilon(v * \varphi_\varepsilon)\big)(x)\big| \\
&\qquad \leqslant 2Q \iint\limits_{\widetilde{K}\times\widetilde{K}} \big(g(y) + g(z)\big)|v(z)|\Big(\big|(X_1\psi)_\varepsilon(y^{-1}x)\big|\, \big|\varphi_\varepsilon(z^{-1}x)\big| \\
&\qquad\qquad + \psi_\varepsilon(y^{-1}x)\, \big|(X_1\varphi)_\varepsilon(z^{-1}x)\big|\Big)\, dy\, dz \\
&\qquad \leqslant 2Q\Big(g{*}|(X_1\psi)_\varepsilon| \cdot |v|{*}|\varphi_\varepsilon| + (g|v|){*}|\varphi_\varepsilon| \cdot \|(X_1\psi)_\varepsilon\|_1 \\
&\qquad\qquad + g^\varepsilon \cdot |v|{*}|(X_1\varphi)_\varepsilon| + g|v|{*}|(X_1\varphi)_\varepsilon|\Big)(x).
\end{aligned}
$$

Again, by the Hölder and the Young inequalities we derive

$$
\begin{aligned}
&\big\|X_1\big((uv) * \varphi_\varepsilon - u^\varepsilon(v * \varphi_\varepsilon)\big)\big\|_{p/2,K} \\
&\qquad \leqslant 4Q\|g\|_{p,\widetilde{K}}\|v\|_{p,\widetilde{K}}\|(X_1\psi)_\varepsilon\|_1\|\varphi_\varepsilon\|_1 + 4Q\|g\|_{p,\widetilde{K}}\|v\|_{p,\widetilde{K}}\|\psi_\varepsilon\|_1\|(X_1\varphi)_\varepsilon\|_1 \\
&\qquad\qquad \leqslant 4Q\Big(\|X_1\psi\|_1\|\varphi\|_1 + \|\psi\|_1\|X_1\varphi\|_1\Big)\|\nabla_H u\|_{p,\Omega}\|v\|_{p,\Omega}.
\end{aligned}
$$

The argument can be repeated for second derivatives giving

$$
\begin{aligned}
&\big\|X_2X_1\big((uv) * \varphi_\varepsilon - u^\varepsilon(v * \varphi_\varepsilon)\big)\big\|_{p/2,K} \\
&\qquad \leqslant \frac{4Q}{\varepsilon}\Big(\|X_2X_1\psi\|_1\|\varphi\|_1 + \|X_1\psi\|_1\|X_2\varphi\|_1 + \|X_2\psi\|_1\|X_1\varphi\|_1 \\
&\qquad\qquad + \|\psi\|_1\|X_2X_1\varphi\|_1\Big)\|\nabla_H u\|_{p,\Omega}\|v\|_{p,\Omega}.
\end{aligned}
$$

Thus, we obtain estimates uniform in $\varepsilon \in (0, \varepsilon_0)$

$$\|X_1 F_\varepsilon\|_{p/2,K} = \big\|X_1\big((uv) * \varphi_\varepsilon - u^\varepsilon(v * \varphi_\varepsilon)\big)\big\|_{p/2,K} \leqslant C\|\nabla_H u\|_{p,\Omega}\|v\|_{p,\Omega},$$
$$\big\|\varepsilon X_2 X_1 F_\varepsilon\big\|_{p/2,K} = \big\|\varepsilon X_2 X_1\big((uv) * \varphi_\varepsilon - u^\varepsilon(v * \varphi_\varepsilon)\big)\big\|_{p/2,K} \leqslant C\|\nabla_H u\|_{p,\Omega}\|v\|_{p,\Omega}.$$

Note that for smooth functions u, v we have

$$X_1\big((uv) * \varphi_\varepsilon - u^\varepsilon(v * \varphi_\varepsilon)\big) \to 0, \quad \varepsilon X_2 X_1\big((uv) * \varphi_\varepsilon - u^\varepsilon(v * \varphi_\varepsilon)\big) \to 0$$

in $L^{p/2}(K)$ as $\varepsilon \to 0$. Since smooth functions are dense in $L^p(K)$ and $W^{1,p}(K)$ we can choose sequences of smooth $u_n \to u$ in $W^{1,p}(K)$ and $v_n \to v$ in $L^p(K)$ as $n \to \infty$. The operator $X_1 F_\varepsilon = X_1 F_\varepsilon(u, v)$ is linear in both u and v, which implies

$$\|X_1 F_\varepsilon(u,v)\|_{p/2,K} \leqslant \|X_1 F_\varepsilon(u - u_n, v)\|_{p/2,K} + \|X_1 F_\varepsilon(u_n, v - v_n)\|_{p/2,K} + \|X_1 F_\varepsilon(u_n, v_n)\|_{p/2,K}$$
$$\leqslant C'\|\nabla_H(u - u_n)\|_{p,\Omega} + C'\|v - v_n\|_{p,\Omega} + \|X_1 F_\varepsilon(u_n, v_n)\|_{p/2,K}.$$

Therefore,

$$\varlimsup_{\varepsilon \to 0} \|X_1 F_\varepsilon(u,v)\|_{p/2,K} \leqslant C'\|\nabla_H(u - u_n)\|_{p,\Omega} + C'\|v - v_n\|_{p,\Omega} \to 0,$$

as $n \to \infty$. Analogously, $\varepsilon X_2 X_1 F_\varepsilon \to 0$ in $L^{p/2}(K)$ as $\varepsilon \to 0$. The estimates (3.4) follow.

The estimates (3.5) are obtained in a similar way. For the terms in the expression of G_ε we have

$$(u^2 v) * \varphi_\varepsilon(x) = \iiint\limits_{\widetilde{K} \times \widetilde{K} \times \widetilde{K}} u^2(z) v(z) \psi_\varepsilon(y_1^{-1}x)\psi_\varepsilon(y_2^{-1}x)\varphi_\varepsilon(z^{-1}x)\, dy_1\, dy_2\, dz,$$
$$u^\varepsilon(x) \cdot (uv) * \varphi_\varepsilon(x) = \iiint\limits_{\widetilde{K} \times \widetilde{K} \times \widetilde{K}} u(y_1) u(z) v(z) \psi_\varepsilon(y_1^{-1}x)\psi_\varepsilon(y_2^{-1}x)\varphi_\varepsilon(z^{-1}x)\, dy_1\, dy_2\, dz,$$
$$(u^\varepsilon)^2(x) \cdot (v * \varphi_\varepsilon)(x) = \iiint\limits_{\widetilde{K} \times \widetilde{K} \times \widetilde{K}} u(y_1) u(y_2) v(z) \psi_\varepsilon(y_1^{-1}x)\psi_\varepsilon(y_2^{-1}x)\varphi_\varepsilon(z^{-1}x)\, dy_1\, dy_2\, dz,$$

thus

$$G_\varepsilon(x) = \iiint\limits_{\widetilde{K} \times \widetilde{K} \times \widetilde{K}} \big(u(z) - u(y_1)\big)\big(u(z) - u(y_2)\big) v(z) \psi_\varepsilon(y_1^{-1}x)\psi_\varepsilon(y_2^{-1}x)\varphi_\varepsilon(z^{-1}x)\, dy_1\, dy_2\, dz.$$

Similar to what is already proved one can obtain the bounds

$$\|G_\varepsilon\|_{p/3,K} \leqslant C\varepsilon^2 \|\nabla_H u\|_{p,\Omega}^2 \|v\|_{p,\Omega},$$
$$\|X_1 G_\varepsilon\|_{p/3,K} \leqslant C\varepsilon \|\nabla_H u\|_{p,\Omega}^2 \|v\|_{p,\Omega},$$
$$\|X_2 X_1 G_\varepsilon\|_{p/3,K} \leqslant C \|\nabla_H u\|_{p,\Omega}^2 \|v\|_{p,\Omega}.$$

Moreover, since for smooth functions u, v we have $X_2 X_1 G_\varepsilon \to 0$ as $\varepsilon \to 0$ in $L^{p/3}(K)$, using for threelinear operator $G_\varepsilon(u, u, v)$ arguments analogous to the ones given for $F_\varepsilon(u, v)$ one can prove that $\|X_2 X_1 G_\varepsilon\|_{p/3,K} \to 0$ as $\varepsilon \to 0$. This concludes the proof of the estimates (3.5). $\rhd$

The dual basis of 1-forms to the basis of left-invariant vector fields X, Y, Z, T is

$$dx, \quad dy, \quad \zeta = dz - x\, dy, \quad \tau = dt - x\, dz + \frac{1}{2} x^2\, dy,$$

and satisfies the dual relations

$$d\zeta = -dx\wedge dy,\quad d\tau = -dx\wedge\zeta.$$

Lemma 3.4. *Let $f\in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p\geqslant 2$, be a contact mapping. Then on any compact $K\subset\Omega$*

$$\|\zeta\langle Xf^\varepsilon\rangle\|_{p/2,K} = O(\varepsilon),\quad \zeta\langle Zf^\varepsilon\rangle \to \zeta\langle\widehat{d}f\langle Z\rangle\rangle,$$
$$\|\zeta\langle Yf^\varepsilon\rangle\|_{p/2,K} = O(\varepsilon),\quad \|\zeta\langle Tf^\varepsilon\rangle\|_{p/2,K} = o\Big(\frac{1}{\varepsilon}\Big)$$

as $\varepsilon\to 0$ in $L^{p/2}(K)$.

◁ Since f is contact, we have

$$0 = \zeta\langle Xf\rangle = Xf_3 - f_1\,Xf_2,\quad 0 = \zeta\langle Yf\rangle = Yf_3 - f_1\,Yf_2,$$

It follows $Xf_3 = f_1\,Xf_2$, $Yf_3 = f_1\,Yf_2$. Next, for the mollification f^ε by Lemma 3.1

$$\begin{aligned}\zeta\langle Xf^\varepsilon\rangle &= Xf_3^\varepsilon - f_1^\varepsilon\,Xf_2^\varepsilon\\ &= (Xf_3)*\chi_{11,\varepsilon} + (Yf_3)*\chi_{12,\varepsilon} - f_1^\varepsilon(Xf_2)*\chi_{11,\varepsilon} - f_1^\varepsilon(Yf_2)*\chi_{12,\varepsilon}\\ &= \big[(f_1Xf_2)*\chi_{11,\varepsilon} - f_1^\varepsilon(Xf_2)*\chi_{11,\varepsilon}\big] + \big[(f_1Yf_2)*\chi_{12,\varepsilon} - f_1^\varepsilon(Yf_2)*\chi_{12,\varepsilon}\big].\end{aligned}\tag{3.6}$$

From Lemma 3.3 we have $\|\zeta\langle Xf^\varepsilon\rangle\|_{p/2,K} = O(\varepsilon)$ and similarly $\|\zeta\langle Yf^\varepsilon\rangle\|_{p/2,K} = O(\varepsilon)$. Next, by the Cartan identity (see e. g. [15])

$$\begin{aligned}\zeta\langle Zf^\varepsilon\rangle = \zeta\big\langle[X,Y]f^\varepsilon\big\rangle &= X\zeta\langle Yf^\varepsilon\rangle - Y\zeta\langle Xf^\varepsilon\rangle - d\zeta\langle Xf^\varepsilon, Yf^\varepsilon\rangle,\\ \zeta\big\langle\widehat{d}f\langle Z\rangle\big\rangle = \zeta\big\langle[Xf,Yf]\big\rangle &= -d\zeta\langle Xf, Yf\rangle.\end{aligned}\tag{3.7}$$

From the representation (3.6) and Lemma 3.3 it follows that the first two terms in (3.7) vanish as $\varepsilon\to 0$ in $L^{p/2}(K)$. Since $d\zeta = -dx\wedge dy$, by Hölder inequality

$$d\zeta\langle Xf^\varepsilon, Yf^\varepsilon\rangle = Xf_2^\varepsilon\,Yf_1^\varepsilon - Yf_2^\varepsilon\,Xf_1^\varepsilon \to Xf_2\,Yf_1 - Yf_2\,Xf_1 = d\zeta\langle Xf, Yf\rangle$$

as $\varepsilon\to 0$ in $L^{p/2}(K)$. Finally,

$$\begin{aligned}\zeta\langle Tf^\varepsilon\rangle &= \zeta\langle[X,Z]f^\varepsilon\rangle = X\zeta\langle Zf^\varepsilon\rangle - Z\zeta\langle Xf^\varepsilon\rangle - d\zeta\langle Xf^\varepsilon, Zf^\varepsilon\rangle\\ &= X(X\zeta\langle Yf^\varepsilon\rangle - Y\zeta\langle Xf^\varepsilon\rangle - d\zeta\langle Xf^\varepsilon, Yf^\varepsilon\rangle) - Z\zeta\langle Xf^\varepsilon\rangle - d\zeta\langle Xf^\varepsilon, Zf^\varepsilon\rangle\\ &= XX\zeta\langle Yf^\varepsilon\rangle - 2XY\zeta\langle Xf^\varepsilon\rangle + YX\zeta\langle Xf^\varepsilon\rangle - Xd\zeta\langle Xf^\varepsilon, Yf^\varepsilon\rangle - d\zeta\langle Xf^\varepsilon, Zf^\varepsilon\rangle.\end{aligned}$$

From (3.6) and Lemma 3.3 it follows that the first three terms are $o\big(\frac{1}{\varepsilon}\big)$ as $\varepsilon\to 0$ in $L^{p/2}(K)$. By Lemma 3.2 and the Hölder inequality the last two terms are also $o\big(\frac{1}{\varepsilon}\big)$ as $\varepsilon\to 0$ in $L^{p/2}(K)$. The lemma is proved. ▷

Lemma 3.5. *Let $f\in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p\geqslant 3$, be a contact mapping. Then on any compact $K\subset\Omega$*

$$\|\tau\langle Xf^\varepsilon\rangle\|_{p/3,K} = O(\varepsilon^2),\quad \|\tau\langle Zf^\varepsilon\rangle\|_{p/3,K} = O(\varepsilon),$$
$$\|\tau\langle Yf^\varepsilon\rangle\|_{p/3,K} = O(\varepsilon^2),\quad \tau\langle Tf^\varepsilon\rangle \to \tau\langle\widehat{d}f\langle T\rangle\rangle$$

as $\varepsilon\to 0$ in $L^{p/3}(K)$.

◁ Since f is contact, we have

$$0 = \zeta\langle Xf\rangle = Xf_3 - f_1\,Xf_2,\quad 0 = \tau\langle Xf\rangle = Xf_4 - f_1\,Xf_3 + \frac{1}{2}f_1^2\,Xf_2.$$

It follows $Xf_3 = f_1\,Xf_2$ and $Xf_4 = \frac{1}{2}f_1^2\,Xf_2$. Next, for the mollification f^ε by Lemma 3.1

$$\begin{aligned}
&\tau\langle Xf^\varepsilon\rangle = Xf_4^\varepsilon - f_1^\varepsilon\,Xf_3^\varepsilon + \frac{1}{2}\big(f_1^\varepsilon\big)^2\,Xf_2^\varepsilon = Xf_4{*}\chi_{11,\varepsilon} + Yf_4{*}\chi_{12,\varepsilon}\\
&-f_1^\varepsilon\big(Xf_3{*}\chi_{11,\varepsilon} + Yf_3{*}\chi_{12,\varepsilon}\big) + \frac{1}{2}\big(f_1^\varepsilon\big)^2\big(Xf_2{*}\chi_{11,\varepsilon} + Yf_2{*}\chi_{12,\varepsilon}\big)\\
&= \left[\left(\frac{1}{2}f_1^2Xf_2\right) * \chi_{11,\varepsilon} - f_1^\varepsilon\big(f_1Xf_2\big) * \chi_{11,\varepsilon} + \frac{1}{2}\big(f_1^\varepsilon\big)^2Xf_2 * \chi_{11,\varepsilon}\right]\\
&+\left[\left(\frac{1}{2}f_1^2Yf_2\right) * \chi_{12,\varepsilon} - f_1^\varepsilon\big(f_1Yf_2\big) * \chi_{12,\varepsilon} + \frac{1}{2}\big(f_1^\varepsilon\big)^2Yf_2 * \chi_{12,\varepsilon}\right].
\end{aligned}\tag{3.8}$$

Be Lemma 3.3 we have $\|\tau\langle Xf^\varepsilon\rangle\|_{p/3,K} = O(\varepsilon^2)$ and similarly $\|\tau\langle Yf^\varepsilon\rangle\|_{p/3,K} = O(\varepsilon^2)$ as $\varepsilon\to 0$. Next, by the Cartan identity (see e. g. [15]) we have

$$\tau\big\langle Zf^\varepsilon\big\rangle = \tau\big\langle [X,Y]f^\varepsilon\big\rangle = X\tau\big\langle Yf^\varepsilon\big\rangle - Y\tau\big\langle Xf^\varepsilon\big\rangle - d\tau\big\langle Xf^\varepsilon, Yf^\varepsilon\big\rangle. \tag{3.9}$$

From the representation (3.8) and Lemma 3.3 it follows that the first two terms are $O(\varepsilon)$ in $L^{p/3}(K)$, as $\varepsilon\to 0$. Next, since $d\tau = -dx\wedge\zeta$, we have

$$d\tau\big\langle Xf^\varepsilon, Yf^\varepsilon\big\rangle = -Xf_1^\varepsilon\,\zeta\big\langle Yf^\varepsilon\big\rangle + Yf_1^\varepsilon\,\zeta\big\langle Xf^\varepsilon\big\rangle. \tag{3.10}$$

Applying Lemmas 3.2, 3.4 and the Hölder inequality we obtain $\|d\tau\langle Xf^\varepsilon, Yf^\varepsilon\rangle\|_{p/3,K} = O(\varepsilon)$, as $\varepsilon\to 0$. Next,

$$\tau\big\langle Tf^\varepsilon\big\rangle = \tau\big\langle [X,Z]f^\varepsilon\big\rangle = X\tau\big\langle Zf^\varepsilon\big\rangle - Z\tau\big\langle Xf^\varepsilon\big\rangle - d\tau\big\langle Xf^\varepsilon, Zf^\varepsilon\big\rangle.$$

Let us estimate each term. From (3.9) and (3.10) we have

$$\begin{aligned}
X\tau\big\langle Zf^\varepsilon\big\rangle &= XX\tau\big\langle Yf^\varepsilon\big\rangle - XY\tau\big\langle Xf^\varepsilon\big\rangle\\
+XXf_1^\varepsilon\,\zeta\big\langle Yf^\varepsilon\big\rangle &+ Xf_1^\varepsilon\,X\zeta\big\langle Yf^\varepsilon\big\rangle - XYf_1^\varepsilon\,\zeta\big\langle Xf^\varepsilon\big\rangle - Yf_1^\varepsilon\,X\zeta\big\langle Xf^\varepsilon\big\rangle.
\end{aligned}\tag{3.11}$$

From (3.8) and Lemma 3.3 it follows that the first two summands in (3.11) and also $Z\tau\langle Xf^\varepsilon\rangle$ vanish in $L^{p/3}(K)$. Applying Lemmas 3.2, 3.4 and the Hölder inequality we can conclude that the last four summands, in (3.11) also vanish in $L^{p/3}(K)$. Finally, using Lemmas 3.2, 3.4 and the Hölder inequality we conclude

$$-d\tau\big\langle Xf^\varepsilon, Zf^\varepsilon\big\rangle = dx\wedge\zeta\big\langle Xf^\varepsilon, Zf^\varepsilon\big\rangle = Xf_1^\varepsilon\,\zeta\big\langle Zf^\varepsilon\big\rangle - Zf_1^\varepsilon\,\zeta\big\langle Xf^\varepsilon\big\rangle \to Xf_1\,\zeta\big\langle \widehat{d}f\langle Z\rangle\big\rangle - 0$$

as $\varepsilon\to 0$ in $L^{p/3}(K)$. The only thing left to note is that

$$\tau\Big\langle \widehat{d}f\langle T\rangle\Big\rangle = \tau\Big\langle \Big[Xf, \widehat{d}f\langle Z\rangle\Big]\Big\rangle = -d\tau\Big\langle Xf, \widehat{d}f\langle Z\rangle\Big\rangle = dx\wedge\zeta\Big\langle Xf, \widehat{d}f\langle Z\rangle\Big\rangle = Xf_1\,\zeta\Big\langle \widehat{d}f\langle Z\rangle\Big\rangle.$$

Thus, $\tau\big\langle Tf^\varepsilon\big\rangle \to \tau\big\langle \widehat{d}f\langle T\rangle\big\rangle$, as $\varepsilon\to 0$ in $L^{p/3}(K)$. The lemma is proved. $\rhd$

Define *homogeneous weight* of the basic left-invariant 1-form by

$$\sigma(dx) = \sigma(dy) = 1, \quad \sigma(\zeta) = 2, \quad \sigma(\tau) = 3.$$

$\lhd$ Proof of Theorem 1.1. For the forms of the weight 1 (i. e. looking like $a\,dx + b\,dy$, $a, b\in\mathbb{R}$) the statement of the Theorem immediately follows from Lemma 3.2, for the forms of the weight 2 ($c\zeta$, $c\in\mathbb{R}$) it follows from Lemma 3.4, and for the forms of the weight 3 ($c\tau$, $c\in\mathbb{R}$) from Lemma 3.5. $\rhd$

4. Applications

Let $\Omega, \Omega' \subseteq \mathbb{E}$ be open, $f : \Omega \to \Omega'$ be a contact mapping of the class $W^{1,3}_{\rm loc}(\Omega)$, and ω be a k-form on Ω'. Define the pull-back $f^{\#}\omega$ as

$$f^{\#}\omega(g)\langle \xi_1, \dots, \xi_k\rangle = \omega\bigl(f(g)\bigr)\Bigl\langle \widehat{d}f(g)\langle\xi_1\rangle, \dots, \widehat{d}f(g)\langle\xi_k\rangle\Bigr\rangle,$$

$g \in \Omega$, $\xi_1, \dots, \xi_k \in T_g\mathbb{E}$. Note that $f^{\#}(\omega_1 \wedge \omega_2) = f^{\#}\omega_1 \wedge f^{\#}\omega_2$, and for basic left-invariant forms

$$\begin{aligned} f^{\#}dx &= Xf_1\, dx + Yf_1\, dy, \quad f^{\#}\zeta = \widehat{d}f\langle Z\rangle\, \zeta,\\ f^{\#}dy &= Xf_2\, dx + Yf_2\, dy, \quad f^{\#}\tau = \widehat{d}f\langle T\rangle\, \tau. \end{aligned}$$

The notion of the homogeneous weight can be extended on k-forms by the rule $\sigma(\omega_1 \wedge \omega_2) = \sigma(\omega_1) \cdot \sigma(\omega_2)$. Next, we consider differential forms with terms of the maximal homogeneous weight. On Engel group such forms are

$$\begin{aligned} \omega^1(g) &= a(g)\, \tau,\\ \omega^2(g) &= a(g)\, \zeta \wedge \tau,\\ \omega^3(g) &= \bigl(a_1(g)\, dx + a_2(g)\, dy\bigr) \wedge \zeta \wedge \tau,\\ \omega^4(g) &= a(g)\, dx \wedge dy \wedge \zeta \wedge \tau, \end{aligned} \tag{4.1}$$

and have the weights $\sigma(\omega_1) = 3$, $\sigma(\omega_2) = 5$, $\sigma(\omega_3) = 6$, $\sigma(\omega_4) = 7$.

An analogue of the next theorem was proved for 2-step the Carnot groups in [2, Theorem 3.5] and for arbitrary the Carnot groups in [6, Theorem 4.3].

Theorem 4.1. *Let $\Omega, \Omega' \subseteq \mathbb{E}$ be open, ω be a k-form on Ω' of the form* (4.1) *with the coefficents of the class $C(\Omega') \cap L^\infty(\Omega')$, $k = 1, \dots, 4$, and $f : \Omega \to \Omega'$ be a contact mapping of the class $W^{1,p}_{\rm loc}(\Omega)$, $p \geqslant \sigma(\omega)$. Then*

$$(f^\varepsilon)^*\omega \to f^{\#}\omega,$$

as $\varepsilon \to 0$ in $L^{p/\sigma(\omega)}_{\rm loc}(\Omega)$.

$\lhd$ It suffices to prove the theorem for the forms $\omega(y) = a(y)\xi(y)$, where $a \in C(\Omega') \cap L^\infty(\Omega')$, and ξ is a basic k-form. For basic 1-forms we have

$$\begin{aligned} (f^\varepsilon)^*dx &= df_1^\varepsilon = Xf_1^\varepsilon\, dx + Yf_1^\varepsilon\, dy + Zf_1^\varepsilon\, \zeta + Tf_1^\varepsilon\, \tau,\\ (f^\varepsilon)^*dy &= df_2^\varepsilon = Xf_2^\varepsilon\, dx + Yf_2^\varepsilon\, dy + Zf_2^\varepsilon\, \zeta + Tf_2^\varepsilon\, \tau,\\ (f^\varepsilon)^*\zeta &= \zeta\bigl\langle Xf^\varepsilon\bigr\rangle\, dx + \zeta\bigl\langle Yf^\varepsilon\bigr\rangle\, dy + \zeta\bigl\langle Zf^\varepsilon\bigr\rangle\, \zeta + \zeta\bigl\langle Tf^\varepsilon\bigr\rangle\, \tau,\\ (f^\varepsilon)^*\tau &= \tau\langle Xf^\varepsilon\rangle\, dx + \tau\langle Yf^\varepsilon\rangle\, dy + \tau\langle Zf^\varepsilon\rangle\, \zeta + \tau\langle Tf^\varepsilon\rangle\, \tau. \end{aligned}$$

Therefore, by Theorem 1.1 on each compact $K \subset \Omega$, as $\varepsilon \to 0$ we have

$$\begin{aligned} (f^\varepsilon)^*dx &= f^{\#}dx + o(1)\, dx + o(1)\, dy + o\Bigl(\frac{1}{\varepsilon}\Bigr)\, \zeta + o\Bigl(\frac{1}{\varepsilon^2}\Bigr)\, \tau \quad \text{in } L^p(K),\\ (f^\varepsilon)^*dy &= f^{\#}dy + o(1)\, dx + o(1)\, dy + o\Bigl(\frac{1}{\varepsilon}\Bigr)\, \zeta + o\Bigl(\frac{1}{\varepsilon^2}\Bigr)\, \tau \quad \text{in } L^p(K),\\ (f^\varepsilon)^*\zeta &= f^{\#}\zeta + O(\varepsilon)\, dx + O(\varepsilon)\, dy + o(1)\, \zeta + o\Bigl(\frac{1}{\varepsilon}\Bigr)\, \tau \quad \text{in } L^{p/2}(K),\\ (f^\varepsilon)^*\tau &= f^{\#}\tau + O\bigl(\varepsilon^2\bigr)\, dx + O\bigl(\varepsilon^2\bigr)\, dy + O(\varepsilon)\, \zeta + o(1)\, \tau \quad \text{in } L^{p/3}(K). \end{aligned}$$

Thus, as $\varepsilon \to 0$

$$(f^\varepsilon)^*\tau \to f^\#\tau \ \text{ in } \ L^{p/3}(K),$$
$$(f^\varepsilon)^*(\zeta\wedge\tau) \to f^\#(\zeta\wedge\tau) \ \text{ in } \ L^{p/5}(K),$$
$$(f^\varepsilon)^*(dx\wedge\zeta\wedge\tau) \to f^\#(dx\wedge\zeta\wedge\tau) \ \text{ in } \ L^{p/6}(K),$$
$$(f^\varepsilon)^*(dy\wedge\zeta\wedge\tau) \to f^\#(dy\wedge\zeta\wedge\tau) \ \text{ in } \ L^{p/6}(K),$$
$$(f^\varepsilon)^*(dx\wedge dy\wedge\zeta\wedge\tau) \to f^\#(dx\wedge dy\wedge\zeta\wedge\tau) \ \text{ in } \ L^{p/7}(K).$$

From that for the basic k-form ξ of the weight $\sigma(\xi)=\sigma(\omega)$ we get $(f^\varepsilon)^*\xi \to f^\#\xi$ in $L^{p/\sigma(\omega)}(K)$ as $\varepsilon\to 0$. Sinse $a(y)$ is continuous and bounded, the composition $a\circ f^\varepsilon$ is uniformly bounded and converges to $a\circ f$ a. e. on K as $\varepsilon \to 0$. Hence, by Lebesgue theorem $(f^\varepsilon)^*\omega \to f^\#\omega$ in $L^{p/\sigma(\omega)}(K)$ as $\varepsilon\to 0$. $\rhd$

Horizontal vector field on $\Omega\subseteq\mathbb{E}$ is a mapping $V:\Omega\to H\mathbb{E}$, $V=v_1X+v_2Y$. *Weak* (*horizontal*) *divergence* $\mathrm{div}_H V$ of the horizontal vector field $V\in L^1_{\mathrm{loc}}(\Omega)$ is a function $h\in L^1_{\mathrm{loc}}(\Omega)$, such that for every $\varphi\in C_0^\infty(\Omega)$

$$\int\limits_\Omega V\varphi(g)\,dg = -\int\limits_\Omega h(g)\varphi(g)\,dg.$$

It follows that $\mathrm{div}_H V = Xv_1 + Yv_2$ pointwise for $V\in C^1(\Omega)$ and distributionally for $V\in W^{1,p}_{\mathrm{loc}}(\Omega)$. If to the vector field $V=v_1X+v_2Y$ we assign the dual 3-form

$$\omega = (v_1\,dy - v_2\,dx)\wedge\zeta\wedge\tau, \tag{4.2}$$

then

$$d\omega = \mathrm{div}_H V\,dx\wedge dy\wedge\zeta\wedge\tau = \mathrm{div}_H V\,dx\wedge dy\wedge dz\wedge dt.$$

An analogue of the next theorem is proved for 2-step the Carnot groups in [5, Corollary 2.15] and for arbitrary the Carnot groups in [6, Theorem 4.24].

Theorem 4.2. *Let $\Omega,\Omega'\subseteq\mathbb{E}$ be open, ω be a horizontal 3-form on Ω' of the form (4.2) with the coefficents $v_1,v_2\in W^{1,\infty}(\Omega')$. If $f:\Omega\to\Omega'$ is a contact mapping of the class $W^{1,7}_{\mathrm{loc}}(\Omega)$, and $\overline{f(\Omega)}\subset\Omega'$, then*

$$f^\# d\omega = df^\#\omega$$

in the weak sense.

$\lhd$ Step 1. If $\omega\in C^1(\Omega')$, then for each $\varphi\in C_0^\infty(\Omega)$ we have

$$\int\limits_\Omega \big(f^\varepsilon\big)^* d\omega\cdot\varphi = \int\limits_\Omega d\big(f^\varepsilon\big)^*\omega\cdot\varphi = (-1)^{k+1}\int\limits_\Omega \big(f^\varepsilon\big)^*\omega\wedge d\varphi.$$

By Theorem 4.1 as $\varepsilon\to 0$ we obtain

$$\int\limits_\Omega f^\# d\omega\cdot\varphi = (-1)^{k+1}\int\limits_\Omega f^\#\omega\wedge d\varphi = \int\limits_\Omega d\big(f^\#\omega\big)\cdot\varphi.$$

Step 2. Now let $\omega = \big(v_1\,dy - v_2\,dx\big)\wedge\zeta\wedge\tau$ be as in the conditions of the theorem. Define $V^\varepsilon = v_1^\varepsilon X + v_2^\varepsilon Y$ and $\omega^\varepsilon = \big(v_1^\varepsilon\,dy - v_2^\varepsilon\,dx\big)\wedge\zeta\wedge\tau$. By step 1 for each $\varphi\in C_0^\infty(\Omega)$ we have

$$\int\limits_\Omega f^\# d\omega^\varepsilon\cdot\varphi = (-1)^{k+1}\int\limits_\Omega f^\#\omega^\varepsilon\wedge d\varphi. \tag{4.3}$$

Since v_j are continuous and bounded, compositions $v_j \circ f^\varepsilon$ are uniformly bounded and converge to $v_j \circ f$ a. e. on Ω. Therefore, be Lebesgue theorem

$$\int\limits_\Omega f^\# \omega^\varepsilon \wedge d\varphi \to \int\limits_\Omega f^\# \omega \wedge d\varphi$$

as $\varepsilon \to 0$. On the other hand

$$f^\# d\omega^\varepsilon = f^\# \big(\mathrm{div}_H V_\varepsilon \, dx \wedge dy \wedge \zeta \wedge \tau\big) = \mathrm{div}_H V_\varepsilon \circ f \cdot \det \widehat{d}f \cdot dx \wedge dy \wedge \zeta \wedge \tau.$$

By the properties of convolutions $\mathrm{div}_H V_\varepsilon(y)$ is bounded uniformly in $\varepsilon > 0$ and

$$\mathrm{div}_H V_\varepsilon(y) = X v_1^\varepsilon(y) + Y v_2^\varepsilon(y) \to X v_1(y) + Y v_1(y) = \mathrm{div}_H V(y)$$

as $\varepsilon \to 0$ for $y \in f(\Omega) \setminus \Sigma$, where Σ is some null set. By the change of variables formula [11, Theorem 5.4] $\det \widehat{d}f = 0$ a. e. on $f^{-1}(\Sigma)$. Therefore,

$$(X v_1^\varepsilon + Y v_2^\varepsilon) \circ f(x) \cdot \det \widehat{d}f(x) \to (X v_1 + Y v_2) \circ f(x) \cdot \det \widehat{d}f(x)$$

for a. e. $x \in \Omega$. Hence, by Lebesgue theorem

$$\int\limits_\Omega f^\# d\omega^\varepsilon \cdot \varphi = \int\limits_\Omega \varphi(x) \, \mathrm{div}_H V^\varepsilon(f(x)) \, \det \widehat{d}f(x) \, dx \to \int\limits_\Omega f^\# d\omega \cdot \varphi$$

as $\varepsilon \to 0$. All in all, we can go to the limit as $\varepsilon \to 0$ in both sides of the equation (4.3). The theorem is proved. $\rhd$

Theorem 4.2 extends on Engel group the theorems [5, Theorem 2.6, 2.14] proved for 2-step case. In [5, Remark 2.19] it is noted the these are the only theorems in the paper that rely on 2-step structure of the Carnot group. Thus, results of the paper [5] can be translated on Engel group without changes.

The mapping $f : \mathbb{G} \supseteq \Omega \to \mathbb{G}$ on a Carnot group $\mathbb{G}$ is the *mapping with bounded distortion* if $f \in W^{1,\nu}_{\mathrm{loc}}(\Omega)$ and for some $K > 0$

$$|d_H f(x)|^\nu \leqslant K \det \widehat{d}f(x)$$

for a. e. $x \in \Omega$. The least constant K is called the *outer distortion coefficent* and is denoted $K_O(f)$.

Corollary 4.1 [5, Theorem 4.10]. *Let $\Omega \subseteq \mathbb{E}$, $f : \Omega \to \mathbb{E}$ be a mapping with bounded distortion. Then*

1) *f is locally Hölder continuous;*

2) *f is Pansu differentiable* a. e.;

3) *f has Luzin $\mathscr{N}$-property;*

4) *a certain change of variable formula holds: if $D \subset \Omega$ is a compact, $|\partial D| = 0$, and u is a measurable function on $\mathbb{E}$, then function $u(y)\mu(y, f, D)$ is integrable on $\mathbb{G}$ iff is $u(f(x))J(x, f)$ integrable on D, moreover*

$$\int\limits_D u(f(x)) J(x, f) \, dx = \int\limits_{\mathbb{G}} u(y) \mu(y, f, D) \, dy.$$

Next, let

$$G(x) = \begin{cases} \big(\det \widehat{d}f(x)\big)^{\frac{2}{\nu}} \big(d_H f(x)^T d_H f(x)\big)^{-1} & \text{if } \det \widehat{d}f(x) > 0, \\ \mathrm{Id}, & \text{otherwise.} \end{cases}$$

The matrix $G(x)$ is symmetric and characterizes the local deviation of f from a conformal mapping. The matrix $G(x)$ defines the mapping

$$\mathscr{A}(x,\xi) = \big\langle G(x)\xi, \xi \big\rangle^{\frac{\nu-2}{2}} G(x)\,\xi, \quad x \in \Omega,\ \xi \in H\mathbb{G},$$

satisfying the conditions

$$\frac{1}{C_\nu K_O(f)} |\xi|^\nu \leqslant \big\langle \mathscr{A}(x,\xi), \xi \big\rangle \leqslant C_\nu K_O(f)^{\nu-1} |\xi|^\nu,$$

where C_ν is a constant independent of f.

Corollary 4.2 [5, Corollary 4.8]**.** *Let $\Omega \subseteq \mathbb{E}$, $f : \Omega \to \mathbb{E}$ be a mapping with bounded distortion. If w is a $W^{1,\infty}_{\mathrm{loc}}$-solution to the equation*

$$-\mathrm{div}_H\Big(\big|\nabla_H w\big|^5 \nabla_H w\Big) = 0$$

in an open domain $W \subseteq \mathbb{E}$, then $v = w \circ f$ is a weak solution to the equation

$$-\mathrm{div}_H \mathscr{A}\big(x, \nabla_H v\big) = 0$$

on $f^{-1}(W) \cap \Omega$.

References

1. *Dairbekov, N. S.* The Morphism Property for Mappings with Bounded Distortion on the Heisenberg Group, *Siberian Mathematical Journal*, 1999, vol. 40, no. 4, pp. 682–649. DOI: 10.1007/BF02675669.
2. *Dairbekov, N. S.* Mappings with Bounded Distortion of Two-Step Carnot Groups, *Proceedings on Analysis and Geometry*, ed. S. K. Vodopyanov, Sobolev Institute Press, Novosibirsk, 2000, pp. 122–155.
3. *Reshetnyak, Yu. G.* Certain Geometric Properties of Functions and Mappings with Generalized Derivatives, *Siberian Mathematical Journal*, 1966, vol. 7, no. 4, pp. 704–732. DOI: 10.1007/BF00973267.
4. *Reshetnyak, Yu. G. Space Mappings with Bounded Distortion,* Translation of Mathematical Monographs, vol. 73, American Mathematical Society, Providence, RI, 1989.
5. *Vodopyanov, S. K.* Foundations of the Theory of Mappings with Bounded Distortion on Carnot Groups, *Contemporary Mathematics*, 2007, vol. 424, pp. 303–344. DOI: 10.13140/RG.2.1.3112.8480.
6. *Kleiner, B., Muller, S. and Xie, X.* Pansu Pullback and Exterior Differentiation for Sobolev Maps on Carnot Groups, 2021, arxiv.org/abs/2007.06694v2.
7. *Rotschild, L. P. and Stein, E. M.* Hypoelliptic Differential Operators and Nilpotent Groups, *Acta Mathematica*, 1976, vol. 137, pp. 247–320. DOI: 10.1007/BF02392419.
8. *Folland, G. B. and Stein, E. M. Hardy Spaces on Homogeneous Groups*, Princeton Mathematical Notes, vol. 28, Princeton University Press, Princeton, N. J., 1982.
9. *Bonfiglioli, A., Lanconelli, E. and Uguzonni, F. Stratified Lie Groups and Potential Theory for their Sub-Laplacians*, Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg, 2007. DOI: 10.1007/978-3-540-71897-0.
10. *Vodop'yanov, S. K. and Ukhlov, A. D.-O.* Approximately Differentiable Transformations and Change of Variables on Nilpotent Groups, *Siberian Mathematical Journal*, 1996, vol. 37, no. 1, pp. 62–78. DOI: 10.1007/BF02104760.
11. *Vodop'yanov, S. K.* $\mathscr{P}$-Differentiability on Carnot Groups in Different Topologies and Related Topics, *Proceedings on Analysis and Geometry*, ed. S. K. Vodopyanov, Sobolev Institute Press, Novosibirsk, 2000.

12. *Franchi, B., Lu, G. and Wheeden, R. L.* A Relationship between Poincaré-Type Inequalities and Representation Formulas In Spaces of Homogeneous Type, *International Mathematics Research Notices*, 1996, vol. 1, pp. 1–14. DOI:10.1155/S1073792896000013.
13. *Hajłasz, P.* Sobolev Spaces on an Arbitrary Metric Space, *Potential Analysis*, 1996, vol. 5, pp. 403–415. DOI: 10.1007/BF00275475.
14. *Vodop'yanov, S. K.* Monotone Functions and Quasiconformal Mappings on Carnot Groups, *Siberian Mathematical Journal*, 1996, vol. 37, no. 6, pp. 1113–1136. DOI: 10.1007/BF02106736.
15. *Warner, F. W. Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York, 1983. DOI: 10.1007/978-1-4757-1799-0.

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УСРЕДНЕНИЯ КОМПАКТНЫХ ОТОБРАЖЕНИЙ ГРУППЫ ЭНГЕЛЯ

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Аннотация. На группппе Энгеля, снабженной левоинвариантной субримановой метрикой, исследуются контактные отображения, принадлежащие метрическим классам Соболева. В евклидовом пространстве одним из основных методов работы с негладкими отображениями является сглаживание — свертка с гладким ядром. Дополнительная трудность работы с контактными отображениями групп Карно состоит в том, что сглаживание контактного отображения, как правило, не контактно. Тем не менее, в рассматриваемом нами случае величину отклонения от контактности оказывается возможным оценить в достаточной мере, чтобы получить полезные результаты. Мы получаем оценки на сходимость (или в некоторых случаях расходимость) компонент дифференциала сглаженного отображения к соответствующим компонентам дифференциала Пансю контактного отображения. В качестве приложения этого результата к квазиконформному анализу приведены альтернативные доказательства сходимости усредненных горизонтальных внешних форм и перестановочности переноса внешней формы дифференциалом Пансю с внешним дифференциалом в слабом смысле. Эти результаты, в свою очередь, позволяют получить такие базовые свойства отображений с ограниченным искажением, как непрерывность по Гёльдеру, дифференцируемость в смысле Пансю почти всюду, $\mathscr{N}$-свойство Лузина.

Ключевые слова: группа Карно, группа Энгеля, квазиконформные отображения, ограниченное искажение.

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