ЮЖНЫЙ МАТЕМАТИЧЕСКИЙ ИНСТИТУТ

ЮМИ

ISSN-1683-3414 (Print)
ISSN-1814-0807 (Online)

ВЛАДИКАВКАЗСКИЙ МАТЕМАТИЧЕСКИЙ ЖУРНАЛ

http://www.vlmj.ru

Том 21, выпуск 4 2019

SOUTHERN MATHEMATICAL INSTITUTE

ISSN-1683-3414 (Print)
ISSN-1814-0807 (Online)

VLADIKAVKAZ MATHEMATICAL JOURNAL

http://www.vlmj.ru

Volume 21, Issue 4

2019

Адрес редакции: 362027, Владикавказ, Маркуса, 22
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Журнал основан в 1999 г. Выходит четыре раза в год
Электронная версия: **www.vlmj.ru**

Зарегистрирован в Федеральной службе по надзору в сфере связи,
информационных технологий и массовых коммуникаций:
свид. ПИ № ФС77-70008 от 31 мая 2017 г.;
свид. ЭЛ № ФС77-70171 от 21 июня 2017 г.

Editorial Office: 22 Markusa St., Vladikavkaz 362027,
the Republic of North Ossetia-Alania, Russia
Phone: (8672) 50-18-06; E-mail: rio@smath.ru
Managing editor: V. V. BOZROVA

The journal was founded in 1999. It is published four times a year.
ELECTRONIC VERSION: **www.vlmj.ru**

Registered with the Federal Service for Supervision in the Sphere of Telecom,
Information Technologies and Mass Communications:
ПИ № ФС77-70008 dated May 31, 2017; ЭЛ № ФС77-70171 dated June 21, 2017.

РОССИЙСКАЯ АКАДЕМИЯ НАУК
ВЛАДИКАВКАЗСКИЙ НАУЧНЫЙ ЦЕНТР
ЮЖНЫЙ МАТЕМАТИЧЕСКИЙ ИНСТИТУТ

ВЛАДИКАВКАЗСКИЙ МАТЕМАТИЧЕСКИЙ ЖУРНАЛ

Том 21, выпуск 4 октябрь–декабрь, 2019

СОДЕРЖАНИЕ

Владикавказ
2019

RUSSIAN ACADEMY OF SCIENCES
VLADIKAVKAZ SCIENTIFIC CENTER
SOUTHERN MATHEMATICAL INSTITUTE

VLADIKAVKAZ MATHEMATICAL JOURNAL

Volume 21, issue 4 October–December, 2019

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Vladikavkaz
2019

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УДК 517.98
DOI 10.23671/VNC.2019.21.44595

2-LOCAL ISOMETRIES OF NON-COMMUTATIVE LORENTZ SPACES

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Dedicated to E. I. Gordon on the occasion of his 70th birthday

Abstract. Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal finite trace τ, and let $S(\mathcal{M},\tau)$ be an $*$-algebra of all τ-measurable operators affiliated with $\mathcal{M}$. For $x \in S(\mathcal{M},\tau)$ the generalized singular value function $\mu(x) : t \to \mu(t;x)$, $t > 0$, is defined by the equality $\mu(t;x) = \inf\{\|xp\|_{\mathcal{M}} : p^2 = p^* = p \in \mathcal{M},\ \tau(\mathbf{1}-p) \leqslant t\}$. Let ψ be an increasing concave continuous function on $[0,\infty)$ with $\psi(0) = 0$, $\psi(\infty) = \infty$, and let $\Lambda_\psi(\mathcal{M},\tau) = \big\{x \in S(\mathcal{M},\tau) :\ \|x\|_\psi = \int_0^\infty \mu(t;x)d\psi(t) < \infty\big\}$ be the non-commutative Lorentz space. A surjective (not necessarily linear) mapping $V : \Lambda_\psi(\mathcal{M},\tau) \to \Lambda_\psi(\mathcal{M},\tau)$ is called a surjective 2-local isometry, if for any $x, y \in \Lambda_\psi(\mathcal{M},\tau)$ there exists a surjective linear isometry $V_{x,y} : \Lambda_\psi(\mathcal{M},\tau) \to \Lambda_\psi(\mathcal{M},\tau)$ such that $V(x) = V_{x,y}(x)$ and $V(y) = V_{x,y}(y)$. It is proved that in the case when $\mathcal{M}$ is a factor, every surjective 2-local isometry $V : \Lambda_\psi(\mathcal{M},\tau) \to \Lambda_\psi(\mathcal{M},\tau)$ is a linear isometry.

Key words: measurable operator, Lorentz space, isometry.

Mathematical Subject Classification (2010): 46L52, 46B04.

For citation: *Alimov, A. A. and Chilin, V. I.* 2-Local Isometries of Non-Commutative Lorentz Spaces, *Vladikavkaz Math. J.*, 2019, vol. 21, no. 4, pp. 5–10. DOI: 10.23671/VNC.2019.21.44595.

1. Introduction

Let $\mathcal{H}$ be a complex separable infinite-dimensional Hilbert space, let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a Banach ideal of compact linear operators in $\mathcal{H}$ generated by symmetric sequence space $(E, \|\cdot\|_E) \subset c_0$, and let V be a surjective 2-local isometry on $\mathcal{C}_E$, that is, $V : \mathcal{C}_E \to \mathcal{C}_E$ is a surjective (not necessarily linear) mapping such that for any $x, y \in \mathcal{C}_E$ there exists a surjective linear isometry $V_{x,y}$ on $\mathcal{C}_E$ for which $V(x) = V_{x,y}(x)$ and $V(y) = V_{x,y}(y)$. In the papers [1, 2] it is shown that in the case when $\mathcal{C}_E$ is separable or has the Fatou property, $\mathcal{C}_E \neq \mathcal{C}_{l_2}$, every surjective 2-local isometry on $\mathcal{C}_E$ is a linear isometry. In the proof of this statement is essentially used explicit description of all surjective linear isometries on $\mathcal{C}_E$ [1, 3].

Banach ideals $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ of compact linear operators are examples of non-commutative symmetric spaces $\mathcal{E}(\mathcal{M},\tau)$ of measurable operators affiliated with a von Neumann algebra $\mathcal{M}$

equipped with a faithful normal semifinite trace τ (see, for example, [4, Ch. 2, § 2.5]). It is natural to expect that for these non-commutative symmetric spaces with the Fatou property, every surjective 2-local isometry $V : \mathcal{E}(\mathcal{M},\tau) \to \mathcal{E}(\mathcal{M},\tau)$ is a linear map. Unfortunately, the method of proof of a similar statement for Banach ideals $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ can not be applied here, since there is no description of surjective linear isometries $V : \mathcal{E}(\mathcal{M},\tau) \to \mathcal{E}(\mathcal{M},\tau)$. At the same time, in the case of non-commutative Lorentz and Marcinkiewicz spaces, such a description of surjective linear isometries was obtained in the paper [5]. Using this description, we obtain the following description of surjective 2-local isometries of non-commutative Lorentz spaces.

Theorem 1. *Let $\mathcal{M}$ be an arbitrary factor with a faithful normal finite trace τ, and let $(\Lambda_\psi(\mathcal{M},\tau), \|\cdot\|_\psi)$ be a non-commutative Lorentz space. Then every surjective 2-local isometry $V : \Lambda_\psi(\mathcal{M},\tau) \to \Lambda_\varphi(\mathcal{M},\tau)$ is a linear isometry.*

2. Preliminaries

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space, let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators in $\mathcal{H}$, and let $\mathbf{1}$ be the unit in $\mathcal{B}(\mathcal{H})$. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra on Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ (see, for example, [6]). A linear operator $x : \mathfrak{D}(x) \to \mathcal{H}$, where the domain $\mathfrak{D}(x)$ of x is a linear subspace of $\mathcal{H}$, is said to be *affiliated* with $\mathcal{M}$ if $yx \subseteq xy$ for all $y \in \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. A linear operator $x : \mathfrak{D}(x) \to \mathcal{H}$ is termed *measurable* with respect to $\mathcal{M}$ if x is closed, densely defined, affiliated with $\mathcal{M}$ and there exists a sequence $\{p_n\}_{n=1}^\infty$ in the lattice $\mathcal{P}(\mathcal{M})$ of all projections of $\mathcal{M}$, such that $p_n \uparrow \mathbf{1}$, $p_n(\mathcal{H}) \subseteq \mathfrak{D}(x)$ and $\mathbf{1} - p_n$ is a finite projection (with respect to $\mathcal{M}$) for all n. The collection $S(\mathcal{M})$ of all measurable operators with respect to $\mathcal{M}$ is a unital $*$-algebra with respect to strong sums and products.

Let x be a self-adjoint operator affiliated with $\mathcal{M}$ and let $\{e^x\}$ be a spectral measure of x. It is well known that if x is a closed operator affiliated with $\mathcal{M}$ with the polar decomposition $x = u|x|$, then $u \in \mathcal{M}$ and $e \in \mathcal{M}$ for all projections $e \in \{e^{|x|}\}$. Moreover, $x \in S(\mathcal{M})$ if and only if x is closed, densely defined, affiliated with $\mathcal{M}$ and $e^{|x|}(\lambda,\infty)$ is a finite projection for some $\lambda > 0$.

An operator $x \in S(\mathcal{M})$ is called τ-measurable if there exists a sequence $\{p_n\}_{n=1}^\infty$ in $\mathcal{P}(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(\mathcal{H}) \subseteq \mathfrak{D}(x)$ and $\tau(\mathbf{1} - p_n) < \infty$ for all n. The collection $S(\mathcal{M},\tau)$ of all τ-measurable operators is a unital $*$-subalgebra of $S(\mathcal{M})$. It is well known that a linear operator x belongs to $S(\mathcal{M},\tau)$ if and only if $x \in S(\mathcal{M})$ and there exists $\lambda = \lambda(x) > 0$ such that $\tau(e^{|x|}(\lambda,\infty)) < \infty$.

The generalized singular value function $\mu(x) : t \to \mu(t;x)$, $t > 0$, of the operator $x \in S(\mathcal{M},\tau)$ is defined by setting [7]

$$\mu(t;x) = \inf\Big\{\|xp\| : p \in \mathcal{P}(\mathcal{M}),\ \tau(\mathbf{1} - p) \leqslant t\Big\} = \inf\Big\{s > 0 : \tau(e^{|x|}(s,\infty)) \leqslant t\Big\}.$$

A non-zero linear subspace $\mathcal{E}(\mathcal{M},\tau) \subset S(\mathcal{M},\tau)$ with the Banach norm $\|\cdot\|_{\mathcal{E}(\mathcal{M},\tau)}$ is called *a symmetric space* if the conditions

$$x \in \mathcal{E}(\mathcal{M},\tau), \quad y \in S(\mathcal{M},\tau), \quad \mu_t(y) \leqslant \mu_t(x) \ \text{ for all } \ t > 0,$$

imply that $y \in \mathcal{E}(\mathcal{M},\tau)$ and $\|y\|_{\mathcal{E}(\mathcal{M},\tau)} \leqslant \|x\|_{\mathcal{E}(\mathcal{M},\tau)}$.

It is known that in the case $\tau(\mathbf{1}) < \infty$ it is true

$$S(\mathcal{M}) = S(\mathcal{M},\tau) \quad \text{and} \quad \mathcal{M} \subseteq \mathcal{E}(\mathcal{M},\tau) \subseteq L_1(\mathcal{M},\tau)$$

for each symmetric space $\mathcal{E}(\mathcal{M},\tau)$, where

$$L_1(\mathcal{M},\tau)=\left\{x\in S(\mathcal{M},\tau):\ \|x\|_1=\int\limits_0^\infty \mu_t(x)\,dt<\infty\right\}.$$

In addition,

$$\mathcal{M}\cdot\mathcal{E}(\mathcal{M},\tau)\cdot\mathcal{M}\subseteq\mathcal{E}(\mathcal{M},\tau),$$

and

$$\|axb\|_{\mathcal{E}(\mathcal{M},\tau)}\leqslant\|a\|_{\mathcal{M}}\cdot\|b\|_{\mathcal{M}}\cdot\|x\|_{\mathcal{E}(\mathcal{M},\tau)}$$

for all $a,b\in\mathcal{M}$, $x\in\mathcal{E}(\mathcal{M},\tau)$.

Let ψ be an increasing concave continuous function on $[0,\infty)$ with $\psi(0)=0$, $\psi(\infty)=\lim\limits_{t\to\infty}\psi(t)=\infty$, and let

$$\Lambda_\psi(\mathcal{M},\tau)=\left\{x\in S(\mathcal{M},\tau):\ \|x\|_\psi=\int\limits_0^\infty \mu(t;x)\,d\psi(t)<\infty\right\}$$

be the non-commutative *Lorentz space*. It is known that $(\Lambda_\psi(\mathcal{M},\tau),\|\cdot\|_\psi)$ is a symmetric space [8], and the norm $\|\cdot\|_\psi$ has the Fatou property, that is, the conditions $0\leqslant x_k\in\Lambda_\psi(\mathcal{M},\tau)$ for all k, and $\sup_{k\geqslant1}\|x_k\|_\psi<\infty$, imply that there exists $0\leqslant x\in\Lambda_\psi(\mathcal{M},\tau)$ such that $x_k\uparrow x$ and $\|x\|_\psi=\sup_{k\geqslant1}\|x_k\|_\psi$.

Denote by $M_\psi(\mathcal{M},\tau)$ the set of all $x\in S(\mathcal{M},\tau)$ for which

$$\|x\|_{M_\psi}=\sup_{t>0}\frac{1}{\psi(t)}\int\limits_0^t\mu(s;x)\,ds$$

is finite. The set $M_\psi(\mathcal{M},\tau)$ with the norm $\|\cdot\|_{M_\psi}$ is a symmetric space which is called a *Marcinkiewicz space*.

Denote by $M^0_\psi(\mathcal{M},\tau)$ the closure of $\mathcal{M}$ in $M_\psi(\mathcal{M},\tau)$. It is known [9] that the conjugate space of $(\Lambda_\psi(\mathcal{M},\tau),\|\cdot\|_\psi)$ is identified with $(M_\psi(\mathcal{M},\tau),\|\cdot\|_{M_\psi})$, and the conjugate space of $(M^0_\psi(\mathcal{M},\tau),\|\cdot\|_{M_\psi})$, under the condition $\lim\limits_{t\to0}\frac{t}{\psi(t)}=0$, is identified with$(\Lambda_\psi(\mathcal{M},\tau),\|\cdot\|_\psi)$. The duality in these pairs of spaces is realized via the bilinear form $(x,y)=\tau(xy)$. It should be pointed out that the spaces $(\Lambda_\psi(\mathcal{M},\tau),\|\cdot\|_\psi)$, $(M_\psi(\mathcal{M},\tau),\|\cdot\|_{M_\psi})$ and $(M^0_\psi(\mathcal{M},\tau),\|\cdot\|_{M_\psi})$ are symmetric spaces [4, Ch. 2, § 2.6], [8].

3. Isometries of Non-Commutative Lorentz Spaces

Let $\mathcal{M}\subseteq\mathcal{B}(\mathcal{H})$ be a von Neumann algebra on Hilbert space $\mathcal{H}$. A linear bijective mapping $\Phi\colon\mathcal{M}\to\mathcal{M}$ is called *a Jordan isomorphism* if $\Phi(x^2)=(\Phi(x))^2$ and $\Phi(x^*)=(\Phi(x))^*$ for all $x\in\mathcal{M}$.

If $\Phi\colon\mathcal{M}\to\mathcal{M}$ is a Jordan isomorphism, then there exists a central projection $z\in\mathcal{M}$ such that $\Phi_z(x)=\Phi(x)\cdot z$, $x\in\mathcal{M}$, is an $*$-homomorphism, and $\Phi_{z^\perp}(x)=\Phi(x)\cdot(\mathbf{1}-z)$, $x\in M$, is an $*$-antihomomorphism (see, for example, [10, Ch. 3, § 3.2.1]). Consequently, if $\mathcal{M}$ is a factor then a Jordan isomorphism $\Phi:\mathcal{M}\to\mathcal{M}$ is an $*$-homomorphism or $*$-antihomomorphism.

If τ is a faithful normal finite trace on von Neumann algebra $\mathcal{M}$ then a Jordan isomorphism $\Phi\colon\mathcal{M}\to\mathcal{M}$ is continuous with respect to measure topology t_τ generated by trace τ (see, for example, [11, Ch. 5, § 3, Proposition 1]). Therefore, Φ extends to a t_τ-continuous Jordan isomorphism $\widetilde{\Phi}\colon S(\mathcal{M},\tau) \to S(\mathcal{M},\tau)$. In addition, if $\tau(\Phi(x)) = \tau(x)$ for all $x \in \mathcal{M}$ then $\mu(t;\widetilde{\Phi}(x)) = \mu(t;x)$ for all $x \in S(\mathcal{M},\tau)$, in particular, $\widetilde{\Phi}(\mathcal{E}(\mathcal{M},\tau) = \mathcal{E}(\mathcal{M},\tau))$ and $\|\widetilde{\Phi}(x)\|_{\mathcal{E}(\mathcal{M},\tau)} = \|x\|_{\mathcal{E}(\mathcal{M},\tau)}$ for all $x \in \mathcal{E}(\mathcal{M},\tau)$, that is, $\widetilde{\Phi}\colon \mathcal{E}(\mathcal{M},\tau) \to \mathcal{E}(\mathcal{M},\tau)$ is a surjective linear isometry for any symmetric space $(\mathcal{E}(\mathcal{M},\tau), \|\cdot\|_{\mathcal{E}(\mathcal{M},\tau)})$.

Thus, it is true the following

Proposition 1. *Let $\mathcal{M}$ be an arbitrary von Neumann algebra with a faithful normal finite trace τ, and let $\Phi\colon \mathcal{M} \to \mathcal{M}$ be a Jordan isomorphism such that $\tau(\Phi(x)) = \tau(x)$ for all $x \in \mathcal{M}$. Then for every symmetric space $(\mathcal{E}(\mathcal{M},\tau), \|\cdot\|_{\mathcal{E}(\mathcal{M},\tau)})$ the mapping $V\colon \mathcal{E}(\mathcal{M},\tau) \to \mathcal{E}(\mathcal{M},\tau)$ given by the equality $V(x) = u \cdot \widetilde{\Phi}(x) \cdot v$, $x \in \mathcal{E}(\mathcal{M},\tau)$, u, v are unitary operators in $\mathcal{M}$, is a surjective linear isometry.*

We need the following description of surjective linear isometries of the spaces $(\Lambda_\psi(\mathcal{M},\tau), \|\cdot\|_\psi)$ and $(M^0_\psi(\mathcal{M},\tau), \|\cdot\|_{M_\psi})$ [5, Theorems 5.1, 6.1].

Theorem 2. *Let $\mathcal{M}$ be an arbitrary von Neumann algebra with a faithful normal finite trace τ, and let $V : \Lambda_\psi(\mathcal{M},\tau) \to \Lambda_\psi(\mathcal{M},\tau)$ (respectively, $V : M^0_\psi(\mathcal{M},\tau) \to M^0_\psi(\mathcal{M},\tau)$) be a surjective linear isometry. Then there exist uniquely an unitary operator $u \in \mathcal{M}$ and a Jordan isomorphism $\Phi : \mathcal{M} \to \mathcal{M}$ such that $V(x) = u \cdot \Phi(x)$ and $\tau(\Phi(x)) = \tau(x)$ for all $x \in \mathcal{M}$.*

4. Local Isometries of Non-Commutative Lorentz Spaces

Let $(X, \|\cdot\|_X)$ be an arbitrary Banach space over the field $\mathbb{K}$ of complex or real numbers. A surjective (not necessarily linear) mapping $T\colon X \to X$ is called a surjective 2-local isometry [2], if for any $x, y \in X$ there exists a surjective linear isometry $V_{x,y}\colon X \to X$ such that $T(x) = V_{x,y}(x)$ and $T(y) = V_{x,y}(y)$. It is clear that every surjective linear isometry on X is a surjective 2-local isometry on X. In addition,

$$T(\lambda x) = V_{x,\lambda x}(\lambda x) = \lambda V_{x,\lambda x}(x) = \lambda T(x)$$

for any $x \in X$ and $\lambda \in \mathbb{K}$.

Consequently, in order to establish linearity of a 2-local isometry T, it is sufficient to show that $T(x + y) = T(x) + T(y)$ for all $x, y \in X$.

Since

$$\|T(x) - T(y)\|_X = \|V_{x,y}(x) - V_{x,y}(y)\|_X = \|x - y\|_X \quad \text{for all } x, y \in X,$$

it follows that T is continuous map on $(X, \|\cdot\|_X)$. In addition, in the case a real Banach space X ($\mathbb{K} = \mathbb{R}$), every surjective 2-local isometry on X is a linear map (see Mazur–Ulam Theorem [12, Ch. 1, § 1.3, Theorem 1.3.5.]). In the case a complex Banach space X ($\mathbb{K} = \mathbb{C}$), this fact is not true.

Using the description of all surjective linear isometries on a separable Banach symmetric ideal $\mathcal{C}_E$ [3] (respectively, on a Banach symmetric ideal $\mathcal{C}_E$ with Fatou property [1]), $\mathcal{C}_E \neq \mathcal{C}_{l_2}$, in the papers [1, 2] it is proved that every surjective 2-local isometry $T : \mathcal{C}_E \to \mathcal{C}_E$ is a linear isometry.

The following Theorem is a version of the above results for the spaces $\Lambda_\psi(\mathcal{M},\tau)$ and $M^0_\psi(\mathcal{M},\tau)$.

Theorem 3. *Let $\mathcal{M}$ be an arbitrary factor with a faithful normal finite trace τ, and let $T : \Lambda_\psi(\mathcal{M},\tau) \to \Lambda_\psi(\mathcal{M},\tau)$ (respectively, $T : M^0_\psi(\mathcal{M},\tau) \to M^0_\psi(\mathcal{M},\tau)$) be a surjective 2-local isometry. Then T is a linear isometry.*

◁ Fix $x, y \in \mathcal{M}$ and let $V_{x,y} : \Lambda_\psi(\mathcal{M}, \tau) \to \Lambda_\psi(\mathcal{M}, \tau)$ be a surjective isometry such that $T(x) = V_{x,y}(x)$ and $T(y) = V_{x,y}(y)$. By Theorem 2, there exist uniquely an unitary operator $u \in \mathcal{M}$ and a Jordan isomorphism $\Phi : \mathcal{M} \to \mathcal{M}$ such that $V_{x,y}(a) = u \cdot \Phi(a)$ and $\tau(\Phi(a)) = \tau(a)$ for all $a \in \mathcal{M}$. Since $\mathcal{M}$ is a factor it follows then $\Phi : \mathcal{M} \to \mathcal{M}$ is an $*$-isomorphism or Φ is an $*$-anti-isomorphism.

We assume that Φ is an $*$-isomorphism (in the case when Φ is an $*$-anti-isomorphism, the proof is similar).

We have

$$\tau(T(x) \cdot (T(y))^*) = \tau(V_{x,y}(x) \cdot (V_{x,y}(y))^*)$$
$$= \tau(u \cdot \Phi(x) \cdot (u \cdot \Phi(y))^*) = \tau(u \cdot \Phi(xy^*) \cdot u^*) = \tau(\Phi(xy^*)) = \tau(xy^*).$$

Consequently, $\tau(T(x) \cdot (T(y))^*) = \tau(xy^*)$ for all $x, y \in \mathcal{M}$.

If $x, y, z \in \mathcal{M}$, then

$$\tau(T(x+y) \cdot (T(z))^*) = \tau((x+y)z^*), \quad \tau(T(x) \cdot T(z)^*) = \tau(xz^*),$$
$$\tau(T(y) \cdot T(z)^*) = \tau(y \cdot z^*).$$

Therefore

$$\tau((T(x+y) - T(x) - T(y)) \cdot (T(z))^*) = 0$$

for all $z \in \mathcal{M}$. Taking $z = x + y$, $z = x$ and $z = y$, we obtain

$$\tau((T(x+y) - T(x) - T(y) \cdot ((T(x+y) - T(x) - T(y))^*) = 0,$$

that is, $T(x+y) = T(x) + T(y)$ for all $x, y \in \mathcal{M}$.

Since the Lorentz space $\Lambda_\psi(0, \infty)$ of measurable functions on a semi-axis $[0, \infty)$ is separable space [13, Ch. 2I, § 5], it follows that the non-commutative Lorentz $(\Lambda_\psi(\mathcal{M}, \tau), \| \cdot \|_\psi)$ has an order continuous norm [14, Proposition 3.6], that is, $\|x_n\|_\psi \downarrow 0$ whenever $x_n \in \Lambda_\psi(\mathcal{M}, \tau)$ and $x_n \downarrow 0$. Consequently, the factor $\mathcal{M}$ is dense in the space $\Lambda_\psi(\mathcal{M}, \tau)$. Since T is a continuous mapping on $\Lambda_\psi(\mathcal{M}, \tau)$ it follows that $T(x+y) = T(x) + T(y)$ for all $x, y \in \Lambda_\psi(\mathcal{M}, \tau)$, that is, T is a surjective linear isometry.

For the space $M^0_\psi(\mathcal{M}, \tau)$, the proof of the linearity of the surjective 2-local isometry $T : M^0_\psi(\mathcal{M}, \tau) \to M^0_\psi(\mathcal{M}, \tau)$ repeats the previous proof. ▷

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Received 20 June, 2019

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Владикавказский математический журнал
2019, Том 21, Выпуск 4, С. 5–10

2-ЛОКАЛЬНЫЕ ИЗОМЕТРИИ НЕКОММУТАТИВНЫХ ПРОСТРАНСТВ ЛОРЕНЦА

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Аннотация. Пусть $\mathcal{M}$ алгебра фон Неймана с точным нормальным конечным следом τ, и пусть $S(\mathcal{M},\tau)$ инволютивная алгебра всех τ-измеримых операторов, присоединенных к алгебре $\mathcal{M}$. Для оператора $x \in S(\mathcal{M},\tau)$ невозрастающая перестановка $\mu(x) : t \to \mu(t;x)$, $t > 0$, определяется с помощью равенства $\mu(t;x) = \inf\{\|xp\|_{\mathcal{M}} : p^2 = p^* = p \in \mathcal{M}, \tau(\mathbf{1} - p) \leqslant t\}$. Пусть ψ возрастающая вогнутая непрерывная функция на $[0,\infty)$, для которой $\psi(0) = 0$, $\psi(\infty) = \infty$. Пусть $\Lambda_\psi(\mathcal{M},\tau) = \big\{x \in S(\mathcal{M},\tau) : \|x\|_\psi = \int_0^\infty \mu(t;x)d\psi(t) < \infty\big\}$ некоммутативное пространство Лоренца. Сюръективное (не обязательно линейное) отображение $V : \Lambda_\psi(\mathcal{M},\tau) \to \Lambda_\psi(\mathcal{M},\tau)$ называется сюръективной 2-локальной изометрией, если для любых $x, y \in \Lambda_\psi(\mathcal{M},\tau)$ существует такая сюръективная линейная изометрия $V_{x,y} : \Lambda_\psi(\mathcal{M},\tau) \to \Lambda_\psi(\mathcal{M},\tau)$, что $V(x) = V_{x,y}(x)$ и $V(y) = V_{x,y}(y)$. Доказано, что в случае, когда $\mathcal{M}$ есть фактор, каждая сюръективная 2-локальная изометрия $V : \Lambda_\psi(\mathcal{M},\tau) \to \Lambda_\psi(\mathcal{M},\tau)$ есть линейная изометрия.

Ключевые слова: измеримый оператор, пространство Лоренца, изометрия.

Mathematical Subject Classification (2010): 46L52, 46B04.

Образец цитирования: *Aminov B. R., Chilin V. I.* Isometries of Real Subspaces of Self-Adjoint Operators in Banach Symmetric Ideals // Владикавк. мат. журн.—2019.—Т. 21, № 4.—С. 5–10 (in English). DOI: 10.23671/VNC.2019.21.44595.

Vladikavkaz Mathematical Journal
2019, Volume 21, Issue 4, P. 11–24

УДК 517.98
DOI 10.23671/VNC.2019.21.44607

ISOMETRIES OF REAL SUBSPACES OF SELF-ADJOINT OPERATORS IN BANACH SYMMETRIC IDEALS

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Dedicated to E. I. Gordon on the occasion of his 70th birthday

Abstract. Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a Banach symmetric ideal of compact operators, acting in a complex separable infinite-dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{C}_E^h = \{x \in \mathcal{C}_E : x = x^*\}$ be the real Banach subspace of self-adjoint operators in $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$. We show that in the case when $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a separable or perfect Banach symmetric ideal ($\mathcal{C}_E \neq \mathcal{C}_2$) any skew-Hermitian operator $H : \mathcal{C}_E^h \to \mathcal{C}_E^h$ has the following form $H(x) = i(xa - ax)$ for same $a^* = a \in \mathcal{B}(\mathcal{H})$ and for all $x \in \mathcal{C}_E^h$. Using this description of skew-Hermitian operators, we obtain the following general form of surjective linear isometries $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$. Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a separable or a perfect Banach symmetric ideal with not uniform norm, that is $\|p\|_{\mathcal{C}_E} > 1$ for any finite dimensional projection $p \in \mathcal{C}_E$ with $\dim p(\mathcal{H}) > 1$, let $\mathcal{C}_E \neq \mathcal{C}_2$, and let $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a surjective linear isometry. Then there exists unitary or anti-unitary operator u on $\mathcal{H}$ such that $V(x) = uxu^*$ or $V(x) = -uxu^*$ for all $x \in \mathcal{C}_E^h$.

Key words: symmetric ideal of compact operators, skew-Hermitian operator, isometry.

Mathematical Subject Classification (2010): 46L52, 46B04.

For citation: *Aminov, B. R. and Chilin, V. I.* Isometries of Real Subspaces of Self-Adjoint Operators in Banach Symmetric Ideals, *Vladikavkaz Math. J.*, 2019, vol. 21, pp. 11–24. DOI: 10.23671/VNC.2019.21.44607.

1. Introduction

The study of linear isometries on classical Banach spaces was initiated by S. Banach. In [1, Ch. XI], he described all isometries on the space $L_p[0,1]$ with $p \neq 2$. In [2], J. Lamperti characterized all linear isometries on the L_p-space $L_p(\Omega, \mathcal{A}, \mu)$, where $(\Omega, \mathcal{A}, \mu)$ is a measure space with a complete σ-finite measure μ. Both S. Banach and J. Lamberti used a method for description of linear isometries on L_p-spaces that was independent of the choice of a scalar field. For studying linear isometries on the broader class of function symmetric spaces $E(\Omega, \mathcal{A}, \mu)$, different approaches are required that depend on a scalar field. If $E(\Omega, \mathcal{A}, \mu)$ is a complex symmetric space then G. Lumer's method [3] based on the theory of Hermitian operators can be effectively applied. For example, M. G. Zaidenberg [4, 5] used this method for description of all surjective linear isometries on the complex symmetric space $E(\Omega, \mathcal{A}, \mu)$, where μ is a continuous measure. For the symmetric space $E = E(0, 1)$ of real-valued measurable functions on the segment $[0,1]$ with a Lebesgue measure μ, where E is a separable space or has the Fatou property, a description of surjective linear isometries on E was given by N. J. Kalton and B. Randrianantoanina [6]. They used methods of the theory of positive numerical operators. For real symmetric sequence spaces, a general form of surjective linear isometries was described by M. Sh. Braverman and E. M. Semenov [7, 8]. They used methods based on the theory of finite groups. For complex separable symmetric sequence spaces (symmetric sequence spaces with the Fatou property), a general form of surjective linear isometries was described in [9] (respectively, in [10]).

Naturally, the next step is to describe surjective linear isometries in the noncommutative situation, when symmetric sequence spaces are replaced by symmetric ideals of compact operators.

Assume $(\mathcal{H},(\cdot,\cdot))$ is an infinite-dimensional complex separable Hilbert space. Let $\mathcal{B}(\mathcal{H})$ (respectively, $\mathcal{K}(\mathcal{H})$) be the C^*-algebra of all bounded (respectively, compact) linear operators on $\mathcal{H}$. For a compact operator $x \in \mathcal{K}(\mathcal{H})$, we denote by $\mu(x) := \{\mu(n,x)\}_{n=1}^{\infty}$ the singular value sequence of x, that is, the decreasing rearrangement of the eigenvalue sequence of $|x| = (x^*x)^{\frac{1}{2}}$. We let Tr denote the standard trace on $\mathcal{B}(\mathcal{H})$. For $p \in [1,\infty)$ ($p=\infty$), we let

$$\mathcal{C}_p := \Big\{x \in \mathcal{K}(\mathcal{H}) \,:\, \mathrm{Tr}\big(|x|^p\big) < \infty\Big\} \quad (\text{respectively, } \mathcal{C}_\infty = \mathcal{K}(\mathcal{H}))$$

denote the p-th Schatten ideal of $\mathcal{B}(\mathcal{H})$, with the norm

$$\|x\|_p := \mathrm{Tr}\big(|x|^p\big)^{\frac{1}{p}} \quad (\text{respectively, } \|x\|_\infty := \sup_{n\geqslant 1} |\mu(n,x)|).$$

In 1975, J. Arazy [11], [12, Ch. 11, § 2, Theorem 11.2.5] gave the following description of all the surjective isometries of Schatten ideals $\mathcal{C}_p$.

Theorem 1. *Let $V : \mathcal{C}_p \to \mathcal{C}_p$, $1 \leqslant p \leqslant \infty$, $p \neq 2$, be an surjective isometry. Then there exist unitary operators u_1 and u_2 or anti-unitary operators v_1 and v_2 on $\mathcal{H}$ such that either $Vx = u_1 x u_2$ or $Vx = v_1 x^* v_2$ for all $x \in \mathcal{C}_p$.*

Recall that a mapping $v : \mathcal{H} \to \mathcal{H}$ is an anti-unitary operator if

$$v(\lambda h + f) = \overline{\lambda} v(h) + v(f) \quad \text{and} \quad \|v(h)\|_{\mathcal{H}} = \|h\|_{\mathcal{H}}$$

for every complex number λ and $h, f \in \mathcal{H}$. If v is an anti-unitary operator then there exists an anti-unitary operator v^* such that $(h, v(f)) = (f, v^*(h))$ for all $h, f \in \mathcal{H}$ (see, for example, [12, Ch. 11, § 2]).

The Schatten ideals $\mathcal{C}_p$ are examples of Banach symmetric ideals $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ of compact operators associated with symmetric sequence spaces $(E, \|\cdot\|_E)$ (see Section 2.2 below). In 1981 A. Sourour [13] proved a version of Theorem 1 for separable Banach symmetric ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ such that $\mathcal{C}_E \neq \mathcal{C}_2$. Recently [14], a variant of Theorem 1 was obtained for any perfect Banach symmetric ideals $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$, $\mathcal{C}_E \neq \mathcal{C}_2$ (recall that $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a perfect ideals, if $\mathcal{C}_E = \mathcal{C}_E^{\times\times}$ [15] (see Section 2.2 below)).

It is clear that for any unitary or anti-unitary operator u the linear operators $V_1(x) = uxu^*$ and $V_2(x) = -uxu^*$ acting in a real Banach space $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$ are surjective isometries, where $\mathcal{C}_E^h = \{x \in \mathcal{C}_E : x = x^*\}$.

Our main result states that if $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a separable or a perfect Banach symmetric ideal of compact operators such that $\mathcal{C}_E \neq \mathcal{C}_2$, there are no other isometries in $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$:

Theorem 2. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a separable or a perfect Banach symmetric ideal with not uniform norm, $\mathcal{C}_E \neq \mathcal{C}_2$, and let $V\colon \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a surjective isometry. Then there exists unitary or anti-unitary operator u on $\mathcal{H}$ such that V can be written in the form $V(x) = uxu^*$ ($x \in \mathcal{C}_E^h$) or in the form $V(x) = -uxu^*$ ($x \in \mathcal{C}_E^h$).*

An analogous result for the space of self-adjoint traceless operators on a finite dimensionalal Hilbert space was obtained by G. Nagy [16].

2. Preliminaries

2.1. Symmetric Sequence Spaces. Let ℓ_∞ (respectively, c_0) be the Banach lattice of all bounded (respectively, converging to zero) sequences $\{\xi_n\}_{n=1}^{\infty}$ of real numbers with respect to the uniform norm $\|\{\xi_n\}_{n=1}^{\infty}\|_\infty = \sup_{n\in\mathbb{N}} |\xi_n|$, where $\mathbb{N}$ is the set of natural numbers. If $2^{\mathbb{N}}$ is the σ-algebra of all subsets of $\mathbb{N}$ and $\mu(\{n\}) = 1$ for each $n \in \mathbb{N}$, then $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ is a σ-finite measure space, $\mathcal{L}_\infty(\mathbb{N}, 2^{\mathbb{N}}, \mu) = \ell_\infty$,

$$\mathcal{L}_1(\mathbb{N}, 2^{\mathbb{N}}, \mu) = \ell_1 = \left\{ \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{R} : \ \|\{\xi_n\}\|_1 = \sum_{n=1}^{\infty} |\xi_n| < \infty \right\},$$

where $\mathbb{R}$ is the field of real numbers. If $\xi = \{\xi_n\}_{n=1}^{\infty} \in \ell_\infty$, then the non-increasing rearrangement $\xi^* : (0, \infty) \to (0, \infty)$ of ξ is defined by

$$\xi^*(t) = \inf\{\lambda : \ \mu(\{|\xi| > \lambda\}) \leqslant t\}, \quad t > 0,$$

(see, for example, [17, Ch. 2, Definition 1.5]).

Therefore the non-increasing rearrangement ξ^* is identified with the sequence $\xi^* = \{\xi_n^*\}$, where

$$\xi_n^* = \inf_{\substack{F \subset \mathbb{N}, \\ \mathrm{card}(F) < n}} \ \sup_{n \notin F} |\xi_n|.$$

A non-zero linear subspace $E \subseteq \ell_\infty$ with a Banach norm $\|\cdot\|_E$ is called *symmetric* sequence space if conditions $\eta \in E$, $\xi \in \ell_\infty$, $\xi^* \leqslant \eta^*$ imply that $\xi \in E$ and $\|\xi\|_E \leqslant \|\eta\|_E$.

If $(E, \|\cdot\|_E)$ is a symmetric sequence space, then $\ell_1 \subset E \subset \ell_\infty$, in addition, $\|\xi\|_E \leqslant \|\xi\|_1$ for all $\xi \in \ell_1$ and $\|\xi\|_\infty \leqslant \|\xi\|_E$ for all $\xi \in E$ [17, Ch. 2, § 6, Theorem 6.6]. If there exists $\xi \in (E \setminus c_0)$ then $\xi^* \geqslant \alpha \mathbf{1}$ for some $\alpha > 0$, and therefore $\mathbf{1} \in E$, where $\mathbf{1} = \{1, 1, \ldots\}$. Consequently, for any symmetric sequence space E we have that $E \subseteq c_0$ or $E = \ell_\infty$.

2.2. Banach Symmetric Ideal of Compact Operators. Let $(\mathcal{H}, (\cdot, \cdot))$ be an infinite-dimensional complex separable Hilbert space, let $\mathcal{B}(\mathcal{H})$ (respectively, $\mathcal{K}(\mathcal{H}), \mathcal{F}(\mathcal{H})$) be the $*$-algebra of all bounded (respectively, compact, finite rank) linear operators in $\mathcal{H}$, and let $\mathcal{P}(\mathcal{H}) = \{p \in \mathcal{B}(\mathcal{H}) : p = p^* = p^2\}$. It is known that $*$-algebras $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ are C^*-algebras with respect to the uniform operator norm, which we shall denote by $\|\cdot\|_\infty$. For a subset $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, we set $\mathcal{A}^h = \{x \in \mathcal{A} : x = x^*\}$.

It is well known that $\mathcal{F}(\mathcal{H}) \subset \mathcal{I} \subset \mathcal{K}(\mathcal{H})$ for any proper two-sided ideal $\mathcal{I}$ in $\mathcal{B}(\mathcal{H})$ (see for example, [18, Proposition 2.1]).

If $(E, \|\cdot\|_E) \subset c_0$ is a symmetric sequence space, then the set

$$\mathcal{C}_E := \{x \in \mathcal{K}(\mathcal{H}) : \{\mu(n, x)\}_{n=1}^{\infty} \in E\}$$

is a proper two-sided ideal in $\mathcal{B}(\mathcal{H})$ (see [18, Theorem 2.5]). In addition, $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a Banach space with respect to the norm $\|x\|_{\mathcal{C}_E} = \|\{\mu(n, x)\}\|_E$ [19] (see also [20, Ch. 3, § 3.5]), and the norm $\|\cdot\|_{\mathcal{C}_E}$ has the following properties:

1) $\|xzy\|_{\mathcal{C}_E} \leqslant \|x\|_\infty \|y\|_\infty \|z\|_{\mathcal{C}_E}$ for all $x, y \in \mathcal{B}(\mathcal{H})$ and $z \in \mathcal{C}_E$;

2) $\|x\|_{\mathcal{C}_E} = \|x\|_\infty$ if $x \in \mathcal{F}(\mathcal{H})$ is of rank 1.

In this case we say that $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a *Banach symmetric ideal* (cf. [18, Ch. 1, §1.7], [21, Ch. III]). It is known that $\mathcal{C}_1 \subset \mathcal{C}_E \subset \mathcal{K}(\mathcal{H})$ and $\|x\|_{\mathcal{C}_E} \leqslant \|x\|_1$, $\|y\|_\infty \leqslant \|y\|_{\mathcal{C}_E}$ for all $x \in \mathcal{C}_1$, $y \in \mathcal{C}_E$.

If $(E, \|\cdot\|_E)$ is a symmetric sequence space (respectively, $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a Banach symmetric ideal), then the Köthe dual $E^\times$ (respectively, $\mathcal{C}_E^\times$) is defined as

$$E^\times = \Big\{\xi = \{\xi_n\}_{n=1}^\infty \in \ell_\infty : \xi\eta = \{\xi_n\eta_n\}_{n=1}^\infty \in \ell_1 \ \text{ for all } \ \eta = \{\eta_n\}_{n=1}^\infty \in E\Big\},$$

$$\Big(\text{respectively,} \ \ \mathcal{C}_E^\times = \Big\{x \in \mathcal{B}(\mathcal{H}) : \ xy \in \mathcal{C}_1 \ \text{ for all } \ y \in \mathcal{C}_E\Big\}\Big),$$

and

$$\|\xi\|_{E^\times} = \sup\Big\{\sum_{n=1}^\infty |\xi_n\eta_n| : \eta = \{\eta_n\}_{n=1}^\infty \in E, \ \|\eta\|_E \leqslant 1\Big\}, \quad \xi \in E^\times,$$

$$\Big(\text{respectively,} \ \ \|x\|_{\mathcal{C}_E^\times} = \sup\Big\{\mathrm{Tr}\big(|xy|\big) : y \in \mathcal{C}_E, \ \|y\|_{\mathcal{C}_E} \leqslant 1\Big\}, \ x \in \mathcal{C}_E^\times\Big).$$

It is known that $(E^\times, \|\cdot\|_{E^\times})$ is a symmetric sequence space [22, Ch. II, § 4, Theorems 4.3, 4.9] and $\ell_1^\times = \ell_\infty$. In addition, if $E \neq \ell_1$ then $E^\times \subset c_0$. Therefore, if $E \neq \ell_1$, the space $(\mathcal{C}_E^\times, \|\cdot\|_{\mathcal{C}_E^\times})$ is a symmetric ideal of compact operators.

A Banach symmetric ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is said to be *perfect* if $\mathcal{C}_E = \mathcal{C}_E^{\times\times}$ (see, for example, [15]). It is clear that $\mathcal{C}_E$ is perfect if and only if $E = E^{\times\times}$.

A symmetric sequence space $(E, \|\cdot\|_E)$ (a Banach symmetric ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$) is said to possess *Fatou property* if the conditions

$$0 \leqslant \xi_k \leqslant \xi_{k+1}, \ \xi_k \in E \ \ (\text{respectively,} \ 0 \leqslant \ x_k \leqslant x_{k+1}, \ x_k \in \mathcal{C}_E) \ \ \text{for all} \ \ k \in \mathbb{N}$$

and $\sup_{k\geqslant 1} \|\xi_k\|_E < \infty$ (respectively, $\sup_{k\geqslant 1} \|x_k\|_{\mathcal{C}_E} < \infty$) imply that there exists an element $\xi \in E$ (respectively, $x \in \mathcal{C}_E$) such that $\xi_k \uparrow \xi$ and $\|\xi\|_E = \sup_{k\geqslant 1} \|\xi_k\|_E$ (respectively, $x_k \uparrow x$ and $\|x\|_{\mathcal{C}_E} = \sup_{k\geqslant 1} \|x_k\|_{\mathcal{C}_E}$).

It is known that $(E, \|\cdot\|_E)$ (respectively, $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$) has the Fatou property if and only if $E = E^{\times\times}$ [23, Vol. II, Ch. 1, Section a] (respectively, $\mathcal{C}_E = \mathcal{C}_E^{\times\times}$ [24, Theorem 5.14]). Therefore $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a perfect Banach symmetric ideal if and only if $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ has the Fatou property.

If $y \in \mathcal{C}_E^\times$, then a linear functional $f_y(x) = \mathrm{Tr}\big(x \cdot y\big)$, $x \in \mathcal{C}_E$, is continuous on $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$, in addition, $\|f_y\|_{\mathcal{C}_E^*} = \|y\|_{\mathcal{C}_E^\times}$, where $(\mathcal{C}_E^*, \|\cdot\|_{\mathcal{C}_E^*})$ is the dual of the Banach space $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ (see, for example, [15]). Identifying an element $y \in \mathcal{C}_E^\times$ and the linear functional f_y, we may assume that $\mathcal{C}_E^\times$ is a closed linear subspace in $\mathcal{C}_E^*$. Since $\mathcal{F}(\mathcal{H}) \subset \mathcal{C}_E^\times$, it follows that $\mathcal{C}_E^\times$ is a total subspace in $\mathcal{C}_E^*$, that is, the conditions $x \in \mathcal{C}_E$, $f(x) = 0$ for all $f \in \mathcal{C}_E^\times$ imply $x = 0$. Thus, the weak topology $\sigma(\mathcal{C}_E, \mathcal{C}_E^\times)$ is a Hausdorff topology, in addition $\mathcal{F}(\mathcal{H})$ (respectively, $\mathcal{F}(\mathcal{H})^h$) is $\sigma(\mathcal{C}_E, \mathcal{C}_E^\times)$-dense in $\mathcal{C}_E$ (respectively, $\mathcal{C}_E^h$).

3. Skew-Hermitian Operators in Banach Symmetric Ideals

Let X be a linear space over the field $\mathbb{K}$ of real or complex numbers. *A semi-inner product* on a space X is a $\mathbb{K}$-valued form $[\cdot,\cdot]\colon X \times X \to \mathbb{K}$ which satisfies

(i) $[\alpha x + y, z] = \alpha \cdot [x, z] + [y, z]$ for all $\alpha \in \mathbb{K}$ and $x, y, z \in X$;

(ii) $[x, \alpha y] = \overline{\alpha} \cdot [x, y]$ for all $\alpha \in \mathbb{K}$ and $x, y \in X$;

(iii) $[x, x] \geqslant 0$ for all $x \in X$ and $[x, x] = 0$ implies that $x = 0$;

(iv) $|[x,y]|^2 \leqslant [x,x]\cdot[y,y]$ for all $x,y \in X$

(see, for example, [25, Ch. 2, § 1]).

The function $\|x\| = \sqrt{[x,x]}$ is the norm on a linear space X. Conversely, if $(X, \|\cdot\|_X)$ is a normed linear space, then there exists semi-inner product $[\cdot,\cdot]$ on X *compatible with the norm* $\|\cdot\|_X$, that is, $\|x\|_X = \sqrt{[x,x]}$ [25, Ch. 2, § 1]. In particular, the semi-inner product (compatible with the norm $\|\cdot\|_X$) can be defined using the equation $[x,y] = \varphi_y(x)$, where $\varphi_y \in X^*$, $\|\varphi_y\|_{X^*} = \|y\|_X$ and $\varphi_y(y) = \|y\|_X^2$ (such functional is called a *support functional* at $y \in X$) [25, Ch. 2, §1, Theorem 10].

Let $(X, \|\cdot\|_X)$ be Banach space over field $\mathbb{K}$, and let $[\cdot,\cdot]$ be a semi-inner product on X which is compatible with the norm $\|\cdot\|_X$. A linear bounded operator $H\colon X \to X$ is said to be *skew-Hermitian*, if $\mathrm{Re}([H(x),x]) = 0$ for all $x \in X$, where $\mathrm{Re}(\alpha)$ is the real part of number $\alpha \in \mathbb{K}$ [12, Ch. 9, § 4]. In particular, if $\mathbb{K} = \mathbb{R}$ then $\varphi_x(H(x)) = [H(x),x] = 0$ for every $x \in X$.

The following Proposition is well known [12, Ch. 9, § 4, Proposition 9.4.2].

Proposition 1. *Let $(X, \|\cdot\|_X)$ be a real Banach space and let H be a skew-Hermitian operator on X. If $V\colon X \to X$ is a surjective isometry then an operator $V\cdot H\cdot V^{-1}$ is a skew-Hermitian.*

It is clear that in the case $(X, \|\cdot\|_X) = (\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$ every linear operator $H : \mathcal{C}_E^h \to \mathcal{C}_E^h$ defined by $H(x) = i(xa - ax), x \in \mathcal{C}_E^h$, where $a \in \mathcal{B}(H)^h$, $i^2 = -1$ is a skew-Hermitian operator.

The following Theorem gives a description of skew-Hermitian operators acting on $\mathcal{C}_E^h$ when $\mathcal{C}_E$ is a separable or perfect Banach symmetric ideal other than $\mathcal{C}_2$.

Theorem 3. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a separable or perfect Banach symmetric ideal, and let $\mathcal{C}_E \neq \mathcal{C}_2$. Then for any skew-Hermitian operator $H : \mathcal{C}_E^h \to \mathcal{C}_E^h$ there exists $a \in \mathcal{B}(H)^h$ such that $H(x) = i(xa - ax)$ for all $x \in \mathcal{C}_E^h$.*

$\triangleleft$ We slightly modify the original proof of Sourour [13]. For vectors $\xi,\eta \in \mathcal{H}$, denote by $\xi\otimes\eta$ the rank one operator on $\mathcal{H}$ given $(\xi\otimes\eta)(h) = (h,\eta)\xi$, $h \in \mathcal{H}$. It is easily seen $\langle x, \xi\otimes\eta\rangle := \mathrm{Tr}((\eta\otimes\xi)\cdot x) = (x(\eta),\xi)$ for any $x \in \mathcal{B}(\mathcal{H})^h$ and $\xi,\eta \in \mathcal{H}$. If $y = \xi\otimes\xi$, $\|\xi\|_{\mathcal{H}} = 1$, then y is an one dimensional projection on $\mathcal{H}$ and $\|y\|_{\mathcal{C}_E} = \|y\|_\infty = 1$. Thus for a linear functional $f_y(x) := \langle x,y\rangle = \mathrm{Tr}(y^*x), x \in \mathcal{C}_E^h$, we have that $f_y(y) = \mathrm{Tr}(y^2) = \mathrm{Tr}(y) = (\xi,\xi) = 1 = \|y\|_{\mathcal{C}_E}^2$. In addition, if $x \in \mathcal{C}_E^h$ and $\|x\|_{\mathcal{C}_E} \leqslant 1$ then $|f_y(x)| = |\mathrm{Tr}(yx)| = |(x(\xi),\xi)| \leqslant \|x\|_\infty \leqslant \|x\|_{\mathcal{C}_E} \leqslant 1$. Consequently, $\|f_y\|_{(\mathcal{C}_E^h)^*} = 1 = \|y\|_{\mathcal{C}_E}$. This means that f_y is a support functional at $y \in \mathcal{C}_E^h$, and $[x,y] = f_y(x)$ is a semi-inner product on $\mathcal{C}_E^h$ compatible with the norm $\|\cdot\|_{\mathcal{C}_E^h}$ ([25, Ch. 2, § 1, Theorem 10].

Step 1. If $\xi,\eta \in \mathcal{H}$, $(\eta,\xi) = 0$, then $\langle H(\eta\otimes\eta), \xi\otimes\xi\rangle = 0$.

We can assume that $\|\eta\|_{\mathcal{H}} = \|\xi\|_{\mathcal{H}} = 1$. Since $p = \eta\otimes\eta$ is one dimensional projections and H is a skew-Hermitian operator, it follows that

$$0 = [H(p),p] = f_p(H(p)) = \langle H(p),p\rangle. \tag{1}$$

By Lemma 9.2.7 ([12, Ch. 9, §9.2], see also the proof of Lemma 11.3.2 [12, Ch. 9, §11.3]), there exists a vector $\xi = \{\xi_1,\xi_2\} \in (\mathbb{R}^2, \|\cdot\|_E)$, $\xi_1 > 0, \xi_2 > 0$, $\|\xi\|_E = 1$, such that the functional $f(\{\eta_1,\eta_2\}) = \eta_1\xi_1 + \eta_2\xi_2$, $\{\eta_1,\eta_2\} \in \mathbb{R}^2$, is a support functional at ξ for space $(\mathbb{R}^2, \|\cdot\|_E)$.

Let us show that the linear functional

$$\varphi(y) = \langle y,x\rangle, \quad y \in \mathcal{C}_E^h, \ x = \xi_1 p + \xi_2 q,$$

is a support functional at x for $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$.

Since f is support functional at ξ for $(\mathbb{R}^2, \|\cdot\|_E)$ and $\|\xi\|_E = 1$, it follows that $\xi_1^2 + \xi_2^2 = f(\{\xi_1, \xi_2\}) = f(\xi) = \|\xi\|_E^2 = 1$. Furthermore, by $\|f\| = \|\xi\|_E = 1$, we have that $|f(\{\eta_1, \eta_2\})| = |\xi_1\eta_1 + \xi_2\eta_2| \leqslant 1$ for every $\{\eta_1, \eta_2\} \in \mathbb{R}^2$ with $\|\{\eta_1, \eta_2\}\|_E \leqslant 1$.

Further, by [21, Ch. II, § 4, Lemma 4.1], we have

$$|(y(\eta), \eta)| \leqslant \mu(1, y), \quad |(y(\xi), \xi)| \leqslant \mu(1, y), \quad |(y(\eta), \eta)| + |(y(\xi), \xi)| \leqslant \mu(1, y) + \mu(2, y),$$

that is, $\{(y(\eta), \eta), (y(\xi), \xi)\} \prec\prec \{\mu(1, y), \mu(2, y)\}$. Since $(E, \|\cdot\|_E)$ is a fully symmetric sequence space, it follows that

$$\|\{(y(\eta), \eta), (y(\xi), \xi)\}\|_E \leqslant \|\{\mu(1, y), \mu(2, y)\}\|_E \leqslant \|y\|_{\mathcal{C}_E}.$$

Consequently, if $y \in \mathcal{C}_E^h$ and $\|y\|_{\mathcal{C}_E} \leqslant 1$, then

$$|\varphi(y)| = |\langle y, x\rangle| = |\xi_1 \mathrm{Tr}(py) + \xi_2 \mathrm{Tr}(qy)| = \big|f\big(\big\{(y(\eta), \eta), (y(\xi), \xi)\big\}\big)\big| \leqslant 1,$$

that is, $\|\varphi\|_{(\mathcal{C}_E^h, \|\cdot\|_E)^*} \leqslant 1$. Since $\|x\|_{\mathcal{C}_E} = \|\xi\|_E = 1$ and

$$\varphi(x) = \langle x, x\rangle = \langle \xi_1 p + \xi_2 q, \xi_1 p + \xi_2 q\rangle = \mathrm{Tr}(\xi_1 p + \xi_2 q)(\xi_1 p + \xi_2 q) = \xi_1^2 + \xi_2^2 = 1,$$

it follows that $\|\varphi\|_{(\mathcal{C}_E^h, \|\cdot\|_E)^*} = 1 = \|x\|_{\mathcal{C}_E}$ and $\varphi(x) = \|x\|_{\mathcal{C}_E}^2$. This means that φ is a support functional at x for space $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$.

Hence,

$$0 = [H(x), x] = \varphi(H(x)) = \big\langle H(x), x\big\rangle = \big\langle \xi_1 H(p) + \xi_2 H(q), \xi_1 p + \xi_2 q\big\rangle.$$

Since $\langle H(p), p\rangle = \langle H(q), q\rangle = 0$ (see (1)), it follows that

$$\big\langle H(p), q\big\rangle = -\big\langle H(q), p\big\rangle. \tag{2}$$

We extend $\eta_1 = \eta$, $\eta_2 = \xi$ up to an orthonormal basis $\{\eta_i\}_{i=1}^\infty$, and let $p_i = \eta_i \otimes \eta_i$. Now we replace our operator H with another skew-Hermitian operator H_0. Let u be a unitary operator such that $u(\eta_1) = \eta_2$, $u(\eta_2) = \eta_1$ and $u(\eta_k) = \eta_k$ if $k \neq 1, 2$. It is clear that $u^* = u^{-1} = u$, $up_1u = p_2$, $up_2u = p_1$, $up_iu = p_i$, $i \neq 1, 2$, and $V(x) = uxu^* = uxu$ is an surjective isometry on $\mathcal{C}_E^h$, in addition, $V^{-1} = V$.

By Proposition 1, a linear operator $H_1 = VHV^{-1}$ is a skew-Hermitian operator, in particular, $\langle H_1(p_k), p_k\rangle = 0$ for all $k \in \mathbb{N}$ (see (1)).

If $i, j \neq 1, 2$, then

$$\begin{aligned}\big\langle H_1(p_i), p_j\big\rangle &= \big\langle uH(p_i)u, p_j\big\rangle = \mathrm{Tr}(p_j uH(p_i)u) = (uH(p_i)u(\eta_j), \eta_j)\\ &= (H(p_i)u(\eta_j), u^*(\eta_j)) = (H(p_i)(\eta_j), \eta_j) = \mathrm{Tr}(p_j H(p_i)) = \big\langle H(p_i), p_j\big\rangle.\end{aligned}$$

If $i = 1$, $j \neq 1, 2$, then

$$\begin{aligned}\langle H_1(p_1), p_j\rangle &= \langle uH(p_2)u, p_j\rangle = \mathrm{Tr}(p_j uH(p_2)u) = (uH(p_2)u(\eta_j), \eta_j)\\ &= (H(p_2)u(\eta_j), u^*(\eta_j)) = (H(p_2)(\eta_j), \eta_j) = \mathrm{Tr}(p_j H(p_2)) = \langle H(p_2), p_j\rangle.\end{aligned}$$

Similarly, we get the following equalities

(*i*) $\langle H_1(p_2), p_j\rangle = \langle H(p_1), p_j\rangle$ if $i = 2$, $j \neq 1, 2$;

(*ii*) $\langle H_1(p_i), p_1\rangle = \langle H(p_i), p_2\rangle$ if $j = 1$, $i \neq 1, 2$;

(*iii*) $\langle H_1(p_1), p_2\rangle = \langle H(p_2), p_1\rangle$ if $i = 1,\ j = 2$;

(*iv*) $\langle H_1(p_2), p_1\rangle = \langle H(p_1), p_2\rangle$ if $i = 2,\ j = 1$.

It is clear that $H_0 = \frac{1}{2}(H - H_1)$ is a skew-Hermitian operator, and if $i, j \neq 1, 2$, then $\langle H_0(p_i), p_j\rangle = \frac{1}{2}(\langle H(p_i), p_j\rangle - \langle H_1(p_i), p_j\rangle) = 0$. Similarly, if $i = 1,\ j \neq 1, 2$ (respectively, $i = 2,\ j \neq 1, 2$) we get

$$\big\langle H_0(p_1), p_j\big\rangle = \frac{1}{2}\big(\big\langle H(p_1), p_j\big\rangle - \big\langle H(p_2), p_j\big\rangle\big)$$

$$\Big(\text{respectively, } \big\langle H_0(p_2), p_j\big\rangle = \frac{1}{2}\big(\big\langle H(p_2), p_j\big\rangle - \big\langle H(p_1), p_j\big\rangle\big)\Big),$$

that is, $\langle H_0(p_1), p_j\rangle + \langle H_0(p_2), p_j\rangle = 0$ in the case $j \neq 1, 2$.

Similarly, $\langle H_0(p_j), p_1\rangle + \langle H_0(p_j), p_2\rangle = 0$ if $j \neq 1, 2$. Since

$$\big\langle H_0(p_1), p_2\big\rangle = \frac{1}{2}\big(\big\langle H(p_1), p_2\big\rangle - \big\langle H(p_2), p_1\big\rangle\big), \quad \big\langle H(p_1), p_2\big\rangle = -\big\langle H(p_2), p_1\big\rangle$$

(see (2)), it follows that $\langle H_0(p_1), p_2\rangle = \langle H(p_1), p_2\rangle$. Similarly, we get that $\langle H_0(p_2), p_1\rangle = -\langle H(p_1), p_2\rangle$. Finally, since H_0 is a skew-Hermitian operator, we have $\langle H_0(p_k), p_k\rangle = 0$ for all $k \in \mathbb{N}$ (see (1)).

Let n be the smallest natural number such that the norm $\|\cdot\|_E$ is not Euclidian on $\mathbb{R}^n$. Then there exist (see, [10, Lemma 5.4]) linear independent vectors $\xi = (\xi_1, \xi_2, \ldots, \xi_n),\ \eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n,\ \|\xi\|_E = 1$, such that

$$\|\xi\|_E = \|f_\eta\|_{E^*} = f_\eta(\xi) = 1, \tag{3}$$

where $f_\eta(\zeta) = \sum_{i=1}^n \zeta_i\eta_i,\ \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{R}^n$. By rearranging the coordinates we may assume that $\xi_1\eta_2 \neq \xi_2\eta_1$.

Let $x = \sum_{j=1}^n \xi_j p_j,\ y = \sum_{j=1}^n \eta_j p_j$, and let $\varphi_y(z) = \langle z, y\rangle = \sum_{j=1}^n \eta_j \cdot \mathrm{Tr}(p_j z),\ z \in \mathcal{C}_E^h$.

Let us show that φ_y is a support functional at x for $(\mathcal{C}_E^h, \|\cdot\|_E)$. Since $\|f_\eta\|_{E^*} = 1$ (see (3)), it follows that $|f_\eta(\zeta)| = |\sum_{i=1}^n \eta_i\zeta_i| \leqslant 1$ for every $\zeta = \{\zeta_i\}_{i=1}^n \in \mathbb{R}^n$ with $\|\zeta\|_E \leqslant 1$. Note that $\|x\|_{\mathcal{C}_E} = \|\xi\|_E = 1$.

We should show that $\|\varphi_y\| = \|x\|_{\mathcal{C}_E} = 1$ and $\varphi_y(x) = \|x\|^2_{\mathcal{C}_E} = 1$. Indeed,

$$\varphi_y(x) = \langle x, y\rangle = \left\langle \sum_{j=1}^n \xi_j p_j, \sum_{j=1}^n \eta_j p_j \right\rangle = \sum_{j=1}^n \xi_j\eta_j = f_\eta(\xi) = 1 = \|x\|^2_{\mathcal{C}_E}.$$

If $z \in \mathcal{C}_E^h,\ \|z\|_{\mathcal{C}_E} \leqslant 1$ then $|\varphi_y(z)| = \big|\sum_{j=1}^n \eta_j(z(\eta_j), \eta_j)\big| \leqslant 1$. The last inequality follows from

$$\big\{(z(\eta_1), \eta_1), (z(\eta_2), \eta_2), \ldots, (z(\eta_n), \eta_n)\big\} \prec \big\{\mu(1, z), \mu(2, z), \ldots, \mu(n, z)\big\}$$

(see [21, Ch. II, § 4, Lemma 4.1]). Therefore $\|\varphi_y\| = \|x\|_{\mathcal{C}_E} = 1$ and $\varphi_y(x) = \|x\|^2_{\mathcal{C}_E} = 1$. This means that φ_y is a support functional at x for $(\mathcal{C}_E^h, \|\cdot\|_E)$.

Consequently,

$$\begin{aligned} 0 = \big\langle H_0(x), y\big\rangle &= \big\langle \xi_1 H_0(p_1) + \ldots + \xi_n H_0(p_n), \eta_1 p_1 + \ldots + \eta_n p_n\big\rangle \\ &= (\xi_1\eta_2 - \xi_2\eta_1)\big\langle H_0(p_1), p_2\big\rangle + (\xi_1\eta_3 - \xi_2\eta_3)\big\langle H_0(p_1), p_3\big\rangle \\ &+ \ldots + (\xi_1\eta_n - \xi_2\eta_n)\big\langle H_0(p_1), p_n\big\rangle + (\xi_3\eta_1 - \xi_3\eta_2)\big\langle H_0(p_3), p_1\big\rangle \\ &+ \ldots + (\xi_n\eta_1 - \xi_n\eta_2)\big\langle H_0(p_n), p_1\big\rangle. \end{aligned} \tag{4}$$

Let now $\tilde{x} = \xi_1 p_1 + \xi_2 p_2 - \xi_3 p_3 - \ldots - \xi_n p_n$ and $\tilde{y} = \eta_1 p_1 + \eta_2 p_2 - \eta_3 p_3 - \ldots - \eta_n p_n$. As above, we have that $\varphi_{\tilde{y}}(\cdot) = \langle \cdot, \tilde{y}\rangle$ is a support functional at $\tilde{x}$. Consequently,

$$\begin{aligned} 0 = \big\langle H_0(\tilde{x}), \tilde{y}\big\rangle &= (\xi_1\eta_2 - \xi_2\eta_1)\big\langle H_0(p_1), p_2\big\rangle + (-\xi_1\eta_3 + \xi_2\eta_3)\langle H_0(p_1), p_3\rangle \\ &+ \ldots + (-\xi_1\eta_n + \xi_2\eta_n)\big\langle H_0(p_1), p_n\big\rangle + (-\xi_3\eta_1 + \xi_3\eta_2)\big\langle H_0(p_3), p_1\big\rangle \\ &+ \ldots + (-\xi_n\eta_1 + \xi_n\eta_2)\big\langle H_0(p_n), p_1\big\rangle. \end{aligned} \tag{5}$$

Summing (4) and (5) we obtain $2(\xi_1\eta_2 - \xi_2\eta_1)\langle H_0(p_1), p_2\rangle = 0$, that is, $\langle H(p_1), p_2\rangle = \langle H_0(p_1), p_2\rangle = 0$.

Step 2. Let $\eta \in \mathcal{H}$, $\|\eta\|_{\mathcal{H}} = 1$, $p = \eta \otimes \eta$, $x \in \mathcal{K}(\mathcal{H})^h$, and let $\operatorname{Tr}(xq) = 0$ for any one dimensional projection q with $qp = 0$. Then there exists $f \in \mathcal{H}$ such that $x = \eta \otimes f + f \otimes \eta - (\eta \otimes \eta)(f \otimes \eta)$, $\|f\|_{\mathcal{H}} \leqslant \|x\|_\infty$.

Indeed, if q is an one dimensional projection with $qp = 0$ then $qxq = \alpha q$ for some $\alpha \in \mathbb{R}$, and $0 = \operatorname{Tr}(xq) = \operatorname{Tr}(qxq) = \operatorname{Tr}(\alpha q) = \alpha$, that is, $\alpha = 0$ and $qxq = 0$. Let $e \in \mathcal{P}(\mathcal{H})$, $\dim e(\mathcal{H}) = 1$, $ep = 0$, $eq = 0$, $y = (q+e)x(q+e)$. If $y \neq 0$ then there exists $r \in \mathcal{P}(\mathcal{H})$, $\dim r(\mathcal{H}) = 1$ such that $r \leqslant q + e$ and $rxr = ryr = \beta r$ for some $0 \neq \beta \in \mathbb{R}$. Since $rp = 0$, it follows that $0 = \operatorname{Tr}(xr) = \operatorname{Tr}(rxr) = \beta \neq 0$. Thus $y = 0$. Continuing this process, we construct a sequence of finite-dimensional projections $g_n \uparrow (I - p)$ such that $g_n x g_n = 0$ for all $n \in \mathbb{N}$, where $I(h) = h$, $h \in \mathcal{H}$. Consequently, $(I-p)x(I-p) = 0$.

If $f = x(\eta)$ then $xp = f \otimes \eta$ and $px = \eta \otimes f$. In addition,

$$(I-p)xp(h) = (I-p)x((h,\eta)\eta)) = (h,\eta)(I-p)f, \quad h \in \mathcal{H},$$

that is, $(I-p)xp = (I-p)f \otimes \eta$. Therefore,

$$x = px + (I-p)xp = \eta \otimes f + (I-p)f \otimes \eta \ \text{ and } \ \|f\|_{\mathcal{H}} \leqslant \|x\|_\infty.$$

Step 3. Let $\eta \in \mathcal{H}$, $\|\eta\|_{\mathcal{H}} = 1$, $p = \eta \otimes \eta$. Then there exists $f \in \mathcal{H}$ such that

$$H(\eta \otimes \eta) = \eta \otimes f + f \otimes \eta, \quad \|f\|_{\mathcal{H}} \leqslant \|H\|.$$

Indeed, if $x = H(\eta \otimes \eta)$, $\xi \in \mathcal{H}$, $(\eta, \xi) = 0$, $q = \xi \otimes \xi$, then by Step 1 we obtain that $(x(\xi), \xi) = \langle x, \xi \otimes \xi\rangle = \operatorname{Tr}(x \cdot \xi \otimes \xi) = 0$. Using Step 2, we have that there exists $f \in \mathcal{H}$ such that $H(\eta \otimes \eta) = x = \eta \otimes f + f \otimes \eta - (\eta \otimes \eta)(f \otimes \eta)$. Since H is a skew-Hermitian operator, it follows that

$$\begin{aligned} 0 = \big\langle H(\eta\otimes\eta), \eta\otimes\eta\big\rangle &= \big\langle \eta\otimes f + f\otimes\eta - (\eta\otimes\eta)(f\otimes\eta), \eta\otimes\eta\big\rangle \\ &= \operatorname{Tr}\big((\eta\otimes\eta)\big(\eta\otimes f + f\otimes\eta - (\eta\otimes\eta)(f\otimes\eta)\big)\big) \\ &= \operatorname{Tr}\big((\eta\otimes\eta)(\eta\otimes f)\big) = \big((\eta\otimes f)(\eta), \eta\big) = (\eta, f). \end{aligned}$$

Thus $(\eta, f) = 0$ and $x = \eta \otimes f + f \otimes \eta - (\eta \otimes \eta)(f \otimes \eta) = \eta \otimes f + f \otimes \eta$. In addition,

$$\|f\|_{\mathcal{H}} \leqslant \|x\|_\infty \leqslant \|x\|_{\mathcal{C}_E} = \|H(\eta\otimes\eta)\|_{\mathcal{C}_E} \leqslant \|H\| \cdot \|\eta\otimes\eta\|_{\mathcal{C}_E} = \|H\| \cdot \|\eta\otimes\eta\|_\infty = \|H\|.$$

Step 4. There exists $a \in \mathcal{B}(\mathcal{H})$ such that $H(x) = ax + xa^*$ for every $x \in \mathcal{C}_E^h$.

Let $\{p_i\}_{i=1}^\infty = \{\eta_i \otimes \eta_i\}_{i=1}^\infty$ be a basis in real linear space $\mathcal{F}(\mathcal{H})^h$, where $\{\eta_i\}_{i=1}^\infty$ is an orthonormal basis of $\mathcal{H}$. For every $\eta_i \in \mathcal{H}$ there exists $f_i \in \mathcal{H}$ such that $H(\eta_i \otimes \eta_i) = \eta_i \otimes f_i + f_i \otimes \eta_i$, and $\|f_i\|_{\mathcal{H}} \leqslant \|H\|$ for all $i \in \mathbb{N}$ (see Step 3). Define a linear operator $a\colon \mathcal{H} \to \mathcal{H}$ setting $a(\eta_i) = f_i$. Since $\|f_i\|_{\mathcal{H}} \leqslant \|H\|$ for all $i \in \mathbb{N}$, it follows that $a \in \mathcal{B}(\mathcal{H})$, in addition,

$H(p_i) = \eta_i \otimes a(\eta_i) + a(\eta_i) \otimes \eta_i$. Since $\eta_i \otimes a(\eta_i) = (\eta_i \otimes \eta_i)a^*$ and $a(\eta_i) \otimes \eta_i = a(\eta_i \otimes \eta_i)$, it follows that $H(x) = ax + xa^*$ for all $x \in \mathcal{F}(\mathcal{H})^h$.

If $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a separable space then $\mathcal{F}(H)^h$ is dense in $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$. Consequently, $H(x) = ax + xa^*$ for all $x \in \mathcal{C}_E^h$.

Let now $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a perfect Banach symmetric ideal. Repeating the proof of Theorem 4.4 [14] that establishes the $\sigma(\mathcal{C}_E, \mathcal{C}_E^\times)$-continuity of the Hermitian operators acting in $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$, we obtain that the skew-Hermitian operator H also $\sigma(\mathcal{C}_E^h, (C_E^\times)^h)$-continuous. Since the space $\mathcal{F}(\mathcal{H})^h$ is $\sigma(\mathcal{C}_E^h, (C_E^\times)^h)$-dense in $\mathcal{C}_E^h$, it follows that $H(x) = ax + xa^*$ for all $x \in \mathcal{C}_E^h$.

Step 5. $a = ib$ for some $b \in \mathcal{B}(\mathcal{H})^h$.

Indeed, if $a = a_1 + ia_2$, $a_1, a_2 \in \mathcal{B}(\mathcal{H})^h$, then

$$H(x) = ax + xa^* = a_1x + xa_1 + i(a_2x - xa_2) = S_1(x_1) + S_2(x),$$

where $S_1(x) = a_1x + xa_1$, $S_2(x) = i(a_2x - xa_2)$, $x \in \mathcal{C}_E^h$. Since H and S_2 are skew-Hermitian, it follows that $S_1 = H - S_2$ is also skew-Hermitian.

If $p \in \mathcal{P}(\mathcal{H})$, $\dim p(\mathcal{H}) = 1$, then the lineal functional $\varphi(y) = \langle y, p\rangle = \mathrm{Tr}(yp)$, $y \in \mathcal{C}_E^h$, is support functional at p. Thus $\mathrm{Tr}(pa_1p + pa_1) = \mathrm{Tr}(S_1(p)p) = 0$, that is, $-\mathrm{Tr}(pa_1) = \mathrm{Tr}(pa_1p) = \mathrm{Tr}(pa_1)$. This means that $\mathrm{Tr}(pa_1) = 0$ for all $p \in \mathcal{P}(\mathcal{H})$ with $\dim p(\mathcal{H}) = 1$. Consequently, $\mathrm{Tr}(xa_1) = 0$ for all $x \in \mathcal{F}(\mathcal{H})$, and by [26, Lemma 2.1] we have $a_1 = 0$. Therefore, $a = ia_2$. $\rhd$

4. The Proof of Theorem 2

Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a Banach symmetric ideal. We say that a bounded linear operator $T\colon \mathcal{C}_E^h \to \mathcal{C}_E^h$ has the property **(P)** if for any $a \in \mathcal{B}(\mathcal{H})^h$ there are operators $b \in \mathcal{B}(\mathcal{H})^h$ and $c \in \mathcal{B}(\mathcal{H})^h$ such that $T(i(bx - xb)) = i(aT(x) - T(x)a)$ and $T(i(ax - xa)) = i(cT(x) - T(x)c)$ for all $x \in \mathcal{C}_E^h$.

It is clear that a bounded linear bijection $T\colon \mathcal{C}_E^h \to \mathcal{C}_E^h$ has the property **(P)** if and only if T^{-1} has the property **(P)**.

Lemma 1. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a separable or a perfect Banach symmetric ideal other than $\mathcal{C}_2$, and let $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a surjective isometry. Then an isometry V has the property* **(P)**.

$\lhd$ If $a \in \mathcal{B}(\mathcal{H})^h$ then the linear operator $H : \mathcal{C}_E^h \to \mathcal{C}_E^h$ defined by $H(x) = i(xa - ax)$, $x \in \mathcal{C}_E^h$, is a skew-Hermitian operator. By the Proposition 1 the operator $V^{-1} \cdot H \cdot V$ is also skew-Hermitian. Using the Theorem 3 we obtain that there exists $b \in \mathcal{B}(\mathcal{H})^h$ such that $V^{-1} \cdot H \cdot V(x) = i(bx - xb)$, that is, $i(aV(x) - V(x)a) = V(i(bx - xb))$ for all $x \in \mathcal{C}_E^h$.

Similarly, $V \cdot H \cdot V^{-1}$ is a skew-Hermitian operator. Hence, there exists an operator $c \in \mathcal{B}(\mathcal{H})^h$ such that $V \cdot H \cdot V^{-1}(y) = i(cy - yc)$ for all $y \in \mathcal{C}_E^h$. If $V^{-1}(y) = x$, then $V(i(ax - xa)) = i(cV(x) - V(x)c)$ for all $x \in \mathcal{C}_E^h$. $\rhd$

Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a Banach symmetric ideal, $0 \neq x \in \mathcal{C}_E^h$, and let $Z(x) = \{x\}' \cap \mathcal{B}(\mathcal{H})^h = \{y \in \mathcal{B}(\mathcal{H})^h : xy = yx\}$. A non-zero operator $x \in \mathcal{C}_E^h$ is said to be a $\mathcal{C}_E^h$-maximal if $Z(x) = Z(y)$ for any $0 \neq y \in \mathcal{C}_E^h$ with $Z(x) \subset Z(y)$ (cf. [27, Definition 1.4]).

Lemma 2. *The following conditions are equivalent:*

(i) $x \in \mathcal{C}_E^h$ is a $\mathcal{C}_E^h$-maximal operator;

(ii) $x = \alpha p$, where $0 \neq p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $0 \neq \alpha \in \mathbb{R}$.

$\lhd$ $(i) \Longrightarrow (ii)$. Since $x \in \mathcal{C}_E^h$, it follows that $x = \sum_{i=1}^t \lambda_i p_i$, $t \in \mathbb{N}$ or $t = \infty$ (the series converges with respect to the norm $\|\cdot\|_\infty$), where $0 \neq p_i \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $p_i p_j = 0$, $i \neq j$, $0 \neq \lambda_i \in \mathbb{R}$, for all $i, j = 1, \ldots, t$. If $y \in Z(x)$ then $y p_i = p_i y$ [28, Ch. 1, §4, p. 17], that is, $Z(x) \subset Z(p_i)$ for all $i = 1, \ldots, t$. Since, x is a $\mathcal{C}_E^h$−maximal operator, it follows that $Z(x) = Z(p_i)$, thus $Z(p_i) = Z(p_k)$ for all $i, k = 1, \ldots, t$.

Suppose that $t \geqslant 2$. As $Z(p_1) = Z(p_2)$, we have

$$\{p_1\}'' = \{p_2\}'' = \{\alpha \cdot p_2 + \beta \cdot (I - p_2) : \alpha, \beta \in \mathbb{C}\},$$

that is, $p_1 = \alpha_0 \cdot p_2 + \beta_0 \cdot (I - p_2)$ for some $\alpha_0, \beta_0 \in \mathbb{C}$. Consequently, $0 = p_1 p_2 = \alpha_0 \cdot p_2$, and $\alpha_0 = 0$. Therefore $p_1 = \beta_0 \cdot (I - p_2)$, which contradicts the inclusion $p_1 \in \mathcal{F}(\mathcal{H})$. Thus $t = 1$ and $x = \lambda_1 p_1$.

$(ii) \Longrightarrow (i)$. Let $x = \alpha p$, where $0 \neq p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $0 \neq \alpha \in \mathbb{R}$. If $0 \neq y \in \mathcal{C}_E^h$ and $Z(x) \subset Z(y)$ then $Z(p) = Z(x) \subset Z(y)$, and $y \in \{y\}'' \subseteq \{p\}'' = \{\alpha \cdot p + \beta \cdot (I - p) : \alpha, \beta \in \mathbb{C}\}$, that is, $y = \alpha_0 \cdot p + \beta_0 \cdot (I - p)$ for some $\alpha_0, \beta_0 \in \mathbb{C}$. Since y is a compact operator, it follows that $\beta_0 = 0$, that is, $y = \alpha_0 \cdot p$ and $Z(x) = Z(y)$. $\rhd$

Lemma 3. *Let $T \colon \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a bounded linear bijective operator with the property* **(P)**. *Then $T(x)$ is a $\mathcal{C}_E^h$-maximal operator for any $\mathcal{C}_E^h$-maximal operator $x \in \mathcal{C}_E^h$.*

$\lhd$ Suppose that $x \in \mathcal{C}_E^h$ is a $\mathcal{C}_E^h$-maximal operator, but $T(x)$ is not $\mathcal{C}_E^h$-maximal, that is, there exists $z \in \mathcal{C}_E^h$ such that $Z(T(x)) \subset Z(z)$ and $Z(T(x)) \neq Z(z)$. Since T is a bijection, $z = T(y)$ for some $y \in \mathcal{C}_E^h$. Hence, $Z(T(x)) \subset Z(T(y))$ and $Z(T(x)) \neq Z(T(y))$.

We show that $Z(x) \subset Z(y)$. Since an operator T has property **(P)**, it follows that for $a \in Z(x)$ there exists $b \in \mathcal{B}(\mathcal{H})^h$ such that

$$T(i(ac - ca)) = i(bT(c) - T(c)b) \tag{6}$$

for all $c \in \mathcal{C}_E^h$. Using equations (6) and $T(i(ax - xa)) = T(0) = 0$, and the injectivity of the mapping T, we obtain that $bT(x) = T(x)b$, that is, $b \in Z(T(x)) \subset Z(T(y))$. Consequently, $T(i(ay - ya)) = 0$ and $ay - ya = 0$ (see (6)), i.e. $a \in Z(y)$. Therefore $Z(x) \subset Z(y)$, and by the $\mathcal{C}_E^h$-maximality of the operator x we obtain that $Z(x) = Z(y)$.

Since $Z(T(x)) \neq Z(T(y))$, there exists an operator $a \in Z(T(y))$ such that $a \notin Z(T(x))$. By the property **(P)** we can choose $b \in \mathcal{B}(\mathcal{H})^h$ such that

$$T(i(bc - cb)) = i(aT(c) - T(c)a) \tag{7}$$

for all $c \in \mathcal{C}_E^h$. Thus $T(i(by - yb)) = 0$, and $by - yb = 0$, that is, $b \in Z(y)$. Besides, $aT(x) - T(x)a \neq 0$ implies that $bx - xb \neq 0$ (see (7)), that is, $b \notin Z(x)$, which contradicts the equality $Z(x) = Z(y)$. $\rhd$

Lemma 4. *Let $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a surjective linear isometry with the property* **(P)**. *Then for every $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}))$ there exists $q_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ such that $V(p) = q_p$ or $V(p) = -q_p$.*

$\lhd$ Let $0 \neq p_i \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $i = 1, 2$, $p_1 p_2 = 0$. Since p_i is a $\mathcal{C}_E^h$-maximal operator (Lemma 2), it follows that $V(p_i)$ is a $\mathcal{C}_E^h$-maximal operator too, $i = 1, 2$ (Lemma 3). Consequently, there exist $0 \neq q_i \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, and $0 \neq \alpha_i \in \mathbb{R}$ such that $V(p_i) = \alpha_i q_i$, $i = 1, 2$ (Lemma 2). Since $p_1 p_2 = 0$, it follows that $(p_1 + p_2) \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ and $V(p_1 + p_2) = \alpha_3 q_3$ for some non-zero projection $q_3 \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ and $0 \neq \alpha_3 \in \mathbb{R}$ (Lemma 2). Therefore $\frac{\alpha_1}{\alpha_3} q_1 + \frac{\alpha_2}{\alpha_3} q_2 = q_3$. By [29] there are four possibilities:

(i) $\frac{\alpha_1}{\alpha_3} = 1$, $\frac{\alpha_2}{\alpha_3} = 1$ if $q_1 q_2 = 0$;

(ii) $\frac{\alpha_1}{\alpha_3} = 1$, $\frac{\alpha_2}{\alpha_3} = -1$ if $q_1q_2 = q_2$;

(iii) $\frac{\alpha_1}{\alpha_3} = -1$, $\frac{\alpha_2}{\alpha_3} = 1$ and $q_1q_2 = q_1$;

(iv) $\frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{\alpha_3} = 1$ and $(q_1 - q_2)^2 = 0$ if $q_1q_2 \neq q_2q_1$.

The case (iv) is impossible because $\|(q_1 - q_2)\|_\infty^2 = \|(q_1 - q_2)^2\|_\infty = 0$, which contradicts the bijectivity of V. In other cases we have $V(p_2) = \alpha q_2$ or $V(p_2) = -\alpha q_2$, where $\alpha = \alpha_1$. Consequently, $V(p) = \alpha q_p$ or $V(p) = -\alpha q_p$ for an arbitrary $0 \neq p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $p_1p = 0$.

Let now $0 \neq e \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ and $p_1e \neq 0$. Then there exists a non-zero finite dimensional projection f, such that $p_1f = 0$ and $ef = 0$. According to above, we have $\alpha_1q_1 = V(p_1) = \alpha_f q_{p_1}$ or $V(p_1) = -\alpha_f q_{p_1}$ and $V(e) = \alpha_f q_e$ or $V(e) = -\alpha_f q_e$ for some non-zero finite dimensional projections q_f, q_e and for non-zero real number α_f. Consequently, $q_1 = q_{p_1}$ and $\alpha_1 = \pm\alpha_f$. In particular, $V(e) = \alpha_1 q_e$ or $V(f) = -\alpha_1 q_e$.

If $e \in \mathcal{P}(\mathcal{H})$ and $\dim e(\mathcal{H}) = 1$, then $1 = \|e\|_{\mathcal{C}_E} = \|V(e)\|_{\mathcal{C}_E} = |\alpha|\|q_e\|_{\mathcal{C}_E} \geqslant |\alpha|\|q_e\|_\infty = |\alpha|$, that is, $|\alpha| \leqslant 1$.

Replacing the isometry V with V^{-1}, we get that $V^{-1}(p) = \beta r_p$ or $V^{-1}(p) = -\beta r_p$ for arbitrary $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, where $r_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ and β does not depend on the projection p. In particular, if $e \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ and $\dim e(\mathcal{H}) = 1$, then $1 = \|e\|_{\mathcal{C}_E} = \|V^{-1}(e)\|_{\mathcal{C}_E} = |\beta|\|r_e\|_{\mathcal{C}_E} \geqslant |\beta|\|r_e\|_\infty = |\beta|$, i. e. $|\beta| \leqslant 1$.

Therefore, for $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ we obtain that $V(p) = \pm\alpha q_p$, and $p = V^{-1}(\pm\alpha q) = \pm(\alpha\beta)r_q$. Hence $|\alpha\beta| = 1$ and $|\alpha| = 1$. $\rhd$

We say that the norm $\|\cdot\|_{\mathcal{C}_E}$ is *a not uniform* if $\|p\|_{\mathcal{C}_E} > 1$ for any $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ with $\dim p(\mathcal{H}) > 1$.

Lemma 5. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a Banach symmetric ideal with not uniform norm, and let $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a surjective isometry with the property* **(P)**. *Then $V(p)$ or $(-V)(p)$ is one dimensional projection for any one dimensional projection p.*

$\lhd$ Let $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $\dim p(\mathcal{H}) = 1$. By Lemma 4 we have that there exists $q_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ such that $V(p) = q_p$ or $V(p) = -q_p$. If $\dim q_p(\mathcal{H}) > 1$ then $1 = \|p\|_{\mathcal{C}_E} = \|V(p)\|_{\mathcal{C}_E} = \|q_p\|_{\mathcal{C}_E} > 1$, what is wrong. $\rhd$

Lemma 6. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ and an isometry V be the same as in the conditions of the Lemma 5. Then*

$$V(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$$

or

$$(-V)(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}).$$

$\lhd$ Let $\mathcal{P}_1(\mathcal{H}) = \{p \in \mathcal{P}(\mathcal{H}) : \dim p(\mathcal{H}) = 1\}$, and let $p, e \in \mathcal{P}_1(\mathcal{H})$. By Lemma 5, there exists $q, r \in \mathcal{P}_1(\mathcal{H})$ such that $V(p) = q$ or $V(p) = -q$ and $V(e) = r$ or $V(e) = -r$. If $V(p) = q$, $V(e) = -r$ then $q - r = V(p + q) = \pm f$ for some $0 \neq f \in \mathcal{P}(\mathcal{H})$ (see Lemma 4), which is not possible because $q, r \in \mathcal{P}_1(\mathcal{H})$. Similarly, the case $V(p) = -q$, $V(e) = r$ is also impossible. Consequently, $V(\mathcal{P}_1(\mathcal{H})) \subseteq \mathcal{P}_1(\mathcal{H})$ or $(-V)(\mathcal{P}_1(\mathcal{H})) \subseteq \mathcal{P}_1(\mathcal{H})$. Since each projector $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ is the final sum of one-dimensional projectors, it follows that $V(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ or $(-V)(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$. $\rhd$

Corollary 1. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ and V be the same as in the conditions of the Lemma 5. Then*

(i) *$V(p)V(e) = 0$ for any $p, e \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ with $pe = 0$;*

(ii) *V is a bijection from $\mathcal{P}_1(\mathcal{H})$ onto $\mathcal{P}_1(\mathcal{H})$.*

$\lhd$ (i). By Lemma 5, $V(p) = q_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ for all $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ or $V(p) = -q_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ for all $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$. In the first case for $p, e \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ with $pe = 0$, we have that $V(p) = q_p$, $V(e) = q_p$, $q_r + q_e = V(r + e) = q_{r+e}$, that is, $V(r)V(e) = q_r q_e = 0$.

The case $V(p) = -q_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ for all $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ is proved similarly.

Item (ii) directly follows from Lemma 5. $\rhd$

$\lhd$ Proof of Theorem 2. We suppose that $V(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ (the case $(-V)(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ is proved by replacing V with $(-V)$). Let

$$x = \sum_{n=1}^{k} \lambda_n p_n \in \mathcal{F}(\mathcal{H})^h, \quad p_n \in \mathcal{P}_1(\mathcal{H}),\ p_n p_m = 0,$$
$$n \neq m,\ 0 \neq \lambda_n \in \mathbb{R},\ n, m = 1, \ldots, k.$$

Since $V(p_n) \cdot V(p_m) = 0,\ n \neq m$ (Corollary 1 (i)), it follows that

$$V(x^2) = V\left(\sum_{n=1}^{k} \lambda_n^2 p_n\right) = \sum_{n=1}^{k} \lambda_n^2 V(p_n) = V(x)^2$$

and

$$\mathrm{Tr}(V(x)) = \sum_{n=1}^{k} \lambda_n \mathrm{Tr}(V(p_n)) = \sum_{n=1}^{k} \lambda_n = \mathrm{Tr}(x).$$

If $p, e, q, f \in \mathcal{P}_1(\mathcal{H})$, $V(p) = q$, $V(e) = f$, then

$$2\,\mathrm{Tr}(pe) = \mathrm{Tr}(pe) + \mathrm{Tr}(ep) = \mathrm{Tr}((p + e)^2 - p - e)$$
$$= \mathrm{Tr}(V((p + e)^2)) - 2 = \mathrm{Tr}(V(p + e))^2 - 2 = \mathrm{Tr}((q + f)^2) - 2 = 2\mathrm{Tr}(qf).$$

Consequently, $\mathrm{Tr}(pe) = \mathrm{Tr}(V(p)V(e))$ for all $p, e \in \mathcal{P}_1(H)$. By [30, Ch. 3, § 3.2, Theorem 3.2.8] we obtain that there exists an unitary or anti-unitary operator u such that $V(p) = upu^*$ for all $p \in \mathcal{P}_1(H)$. Thus $V(x) = u^*xu$ for all $x \in \mathcal{F}(H)^h$.

If $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a separable space then $\mathcal{F}(H)^h$ is dense in $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$. Consequently, $V(x) = u^*xu$ (respectively, $V(x) = -uxu^*$) for all $x \in \mathcal{C}_E^h$.

If $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a perfect Banach symmetric ideal, then V is $\sigma(\mathcal{C}_E, \mathcal{C}_E^\times)$-continuous (see proof of Step 4 in Theorem 4). Since $\mathcal{F}(H)^h$ is $\sigma(\mathcal{C}_E, \mathcal{C}_E^\times)$-dense in $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$, it follows that $V(x) = u^*xu$ (respectively, $V(x) = -uxu^*$) for all $x \in \mathcal{C}_E^h$.

In the case $(-V)(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ we get that $V(x) = -uxu^*$ for all $x \in \mathcal{C}_E^h$. $\rhd$

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Received 13 June, 2019

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Владикавказский математический журнал
2019, Том 21, Выпуск 4, С. 11–24

ИЗОМЕТРИИ ДЕЙСТВИТЕЛЬНЫХ ПОДПРОСТРАНСТВ САМОСОПРЯЖЕННЫХ ОПЕРАТОРОВ В БАНАХОВЫХ СИММЕТРИЧНЫХ ИДЕАЛАХ

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Аннотация. Пусть $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ банахов симметричный идеал компактных операторов, действующих в комплексном сепарабельном бесконечномерном гильбертовом $\mathcal{H}$. Пусть $\mathcal{C}_E^h = \{x \in \mathcal{C}_E : x = x^*\}$ действительное банахово подпространство самосопряженных операторов в $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$. Доказывается, что в случае, когда $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ есть сепарабельный или совершенный банахов симметричный идеал $(\mathcal{C}_E \neq \mathcal{C}_2)$ каждый косоэрмитовый оператор $H : \mathcal{C}_E^h \to \mathcal{C}_E^h$ имеет следующий вид $H(x) = i(xa - ax)$ для некоторого $a^* = a \in \mathcal{B}(\mathcal{H})$ и для всех $x \in \mathcal{C}_E^h$. Используя это описание косоэрмитовых операторов мы получаем следующий общий вид сюръективных линейных изометрий $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$: Пусть $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ сепарабельный или совершенный банахов симметричный идеал с неравномерной нормой, т. е. $\|p\|_{\mathcal{C}_E} > 1$ для всех конечномерных проекторов $p \in \mathcal{C}_E$ с $\dim p(\mathcal{H}) > 1$, пусть $\mathcal{C}_E \neq \mathcal{C}_2$, и пусть $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$ сюръективная линейная изометрия. Тогда существует такой унитарный или антиунитарный оператор u на $\mathcal{H}$, что $V(x) = uxu^*$ или $V(x) = -uxu^*$ для всех $x \in \mathcal{C}_E^h$.

Ключевые слова: симметричный идеал компактных операторов, косоэрмитовый оператор, изометрия.

Mathematical Subject Classification (2010): 46L52, 46B04.

Образец цитирования: *Aminov B. R., Chilin V. I.* Isometries of Real Subspaces of Self-Adjoint Operators in Banach Symmetric Ideals // Владикавк. мат. журн.—2019.—Т. 21, № 4.—С. 11–24 (in English). DOI: 10.23671/VNC.2019.21.44607.

Vladikavkaz Mathematical Journal
2019, Volume 21, Issue 4, P. 25–41

УДК 510.28, 517.13, 517.98
DOI 10.23671/VNC.2019.21.44619

SOME REMARKS ABOUT NONSTANDARD METHODS IN ANALYSIS. I

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In the blessed memory of Evgeniy Alekseevich Gorin,
a wonderful man and an outstanding mathematician

Abstract. This and forthcoming articles discuss two of the most known nonstandard methods of analysis—the Robinson's infinitesimal analysis and the Boolean valued analysis, the history of their origination, common features, differences, applications and prospects. This article contains a review of infinitesimal analysis and the original method of forcing. The presentation is intended for a reader who is familiar only with the most basic concepts of mathematical logic—the language of first-order predicate logic and its interpretations. It is also desirable to have some idea of the formal proofs and the Zermelo–Fraenkel axiomatics of the set theory. In presenting the infinitesimal analysis, special attention is paid to formalizing the sentences of ordinary mathematics in a first-order language for a superstructure. The presentation of the forcing method is preceded by a brief review of C. Godel's result on the compatibility of the Axiom of Choice and the Continuum Hypothesis with Zermelo–Fraenkel's axiomatics. The forthcoming article is devoted to Boolean valued models and to the Boolean valued analysis, with particular attention to the history of its origination.

Key words: boolean valued analysis, nonstandard analysis, forcing.

Mathematics Subject Classification (2010): 03C90, 46A40.

For citation: *Gordon, E. I.* Some Remarks about Nonstandard Methods in Analysis. I, *Vladikavkaz Math. J.*, 2019, vol. 21, no. 4, pp. 25–41. DOI: 10.23671/VNC.2019.21.44619.

1. Introduction

As noted in the book [1] "Nonstandard methods of analysis in the modern sense consist in attracting two different models of set theory—"standard" and "nonstandard" for the study of specific mathematical objects and problems". Currently, the two nonstandard methods are most widely used in analysis—Robinson's infinitesimal analysis and Boolean valued analysis, each of which have become an independent area of analysis.

Application of the methods of mathematical logic for obtaining new results in pure mathematics, started apparently with the article by A. I. Maltsev [2] in which a general method was developed for obtaining local theorems of group theory. This method was based on Maltsev Compactness Theorem proved in his PhD Thesis in 1936. Further penetration of the methods of logic in various areas of mathematics, mainly in algebra, is associated with the development of model theory—a section of mathematical logic that studies algebraic structures from the point of view of their description by first-order logical languages.

The beginning of application of the methods of mathematical logic in analysis is connected with A. Robinson who made a great contribution to the development of model theory. Using Maltsev Compactness Theorem, he constructed an extension of the standard model of analysis which included a slightly modified version of the basic properties of the standard model, but contained also infinitely large and infinitesimal numbers. In this new analysis, which Robinson called the non-standard analysis, many intuitive mathematical formulations that go back to Leibniz and later to Cauchy, such as, for example, the definition of limit: "$\lim_{x\to a} f(x) = L$ means that if x is infinitely close to a but $x \neq a$, then $f(x)$, is infinitely close to L", received the status of rigorous mathematical statements. This made it possible to simplify significantly the proofs of many theorems of standard analysis and even obtain new results in standard mathematics using nonstandard analysis. After the first edition of Robinson's book [3] was published in 1966, many articles appeared in which nonstandard analysis was used to obtain new results in various fields of standard mathematics, especially in functional analysis, stochastic analysis and mathematical physics (see e.g. [4]). Robinson's nonstandard analysis is briefly discussed in Section 2.

The other of the nonstandard method of analysis, named by G. Takeuti "Boolean valued analysis" originated from P. Cohen's method of forcing*, which was developed to prove the independence of the Continuum Hypothesis (CH). The forcing is a quite complicated method. It requires knowledge not only of the foundations of mathematical logic, but also of the very subtle and deep results in axiomatic set theory. It was impossible for a laymen to understand the proof of independence of CH based on the forcing. At the same time, interest in the result itself, marked with the Fields Prize, was very wide. This served as an incentive for D. Scott and R. Solovay to develop a method of Boolean valued models [5, 6, 7]**. An excellent intuitive explanation of the main ideas of this proof is contained in the article [5]. After reading this article, one cannot fully understand the proof of the independence of CH, but it is quite possible to understand its idea on the basis of Boolean valued models.

In the last lines of this article, the hope is expressed that the Boolean valued models of the field of reals will find application in mathematics not only to prove independence, but also by themselves. This hope was justified. Based on the theory of Boolean valued models, the method of Boolean valued analysis was developed, which found applications in various fields of mathematics. Applications to harmonic analysis and von Neumann algebras are primarily related to G. Takeuti and M. Ozawa [9, 10]; and in the theory of vector lattices to A. G. Kusraev and S. S. Kutateladze. See [1, 11, 12] and the references there. Quite recently Boolean valued analysis found application in Mathematical Economics. See the paper of J. M. Zapata in this issue of the VMJ. The history of origination of Boolean valued analysis will be discussed in the forthcoming paper.

Boolean valued analysis is called nonstandard since it uses not two-valued logic, but one in which truth values form a complete Boolean algebra. The truth of a sentence in such a model means that its truth value is equal to the top of this Boolean algebra. Thus, the objects that simulate $\mathbb{R}$ in Boolean models are significantly more complicated than $\mathbb{R}$. The article [13] shows that the class of Boolean valued fields $\mathbb{R}$ coincides with the class of universally complete Kantorovich spaces. Roughly speaking, this allows us, to reduce many problems about complex objects to problems about simpler objects.

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* We use below simply forcing for the method of forcing as it is used in the majority of publications.

** It is mentioned in [8] that [6] is the preliminary version of [7]. However, the article [7] never appeared in print, although it circulated as a preprint and was widely known.

For the most important particular case of Boolean algebras with measure, this characterization of Boolean valued models of $\mathbb{R}$ was actually obtained earlier in [5].

Evgeniy Alekseevich Gorin, who left us a year ago, once attending a seminar, where I gave a talk on Boolean valued analysis, quite accurately characterized this method with his inherent humor: "I understand now what you are doing. You are taking some kind of a theorem about functionals, say something like spells over it and get a theorem about operators."

Those who work in Boolean valued analysis usually do not use the forcing method for the above reasons. There is one more reason. Forcing uses the standard models of ZF. The existence of such a model (SM) cannot be proved in ZF itself by virtue of Gödel's Incompleteness Theorem. For independence proofs the additional hypothesis about the existence of an SM does not matter, because there is a method that allows to convert a deduction using this hypothesis of a contradiction to a deduction of the same contradiction without it. However, if some analysis theorem is proved under the assumption of the existence of a SM, then generally speaking this does not mean that this theorem can be proved without it. For example, the consistency of ZF can be proved in ZF+SM, but it cannot be proved in ZF. There is no such problem in the Boolean valued analysis, since it does not need any SM. Even independence proofs can be carried out in Boolean valued models without resorting to standard models.

The present day books on Boolean valued models do not include even any survey of the method of forcing. However, in my opinion, proofs by the method of forcing are more intuitive than proofs by the method of Boolean valued models, especially in the case of independence proofs. The reader can compare the proofs of independence of CH in the book [11], Section 9.5 and in Section 3.3 of this paper and decide for him/herself which one is more intuitive. CH is a simple case. I think that the it would be very hard to implement a proof of Theorem 8 below in the framework of Boolean valued models. At least I tried to do this, when I worked on Theorem 11 below and understood that it is too hard for me. I am sure that this situation may sometimes occur in Boolean valued analysis as well. That is why Section 3 of this paper contains a survey of forcing.

2. A. Robinson's Nonstandard Analysis

Recall some concepts and facts of model theory, on which the nonstandard analysis is based.

Let σ be a signature of a first order logical language L_σ, and let $\mathcal{M} = (M, \sigma)$ and $\mathcal{M}' = (M', \sigma)$ be σ-structures*.

Definition 1. Let $\imath : \mathcal{M} \to \mathcal{M}'$ be a monomorphism. Say that $(\mathcal{M}', \imath)$ is an elementary extension of $\mathcal{M}$, if for every formula $\varphi(x_1, \dots, x_n)$ of L_σ and for every $m_1, \dots, m_n \in M$ one has

$$\mathcal{M} \models \varphi(m_1, \dots, m_n) \Longleftrightarrow \mathcal{M}' \models \varphi(\imath(m_1), \dots, \imath(m_n)). \tag{1}$$

Remark 1. In what follows we assume WLOG that $\imath$ is the identity monomorphism and so $M \subseteq M'$.

Maltsev Compactness Theorem easily implies the following

Theorem 1. *Each structure $\mathcal{M}$ has an elementary extension of an arbitrary cardinality $\leqslant \max(|\sigma|, \aleph_0)$.*

* This means that the basic functions and predicates in $\mathcal{M}$ and $\mathcal{M}'$ are interpretations of the corresponding symbols in the signature σ. In what follows we use the same notations for the signature symbols and their interpretations.

Recall the definition of the superstructure $\mathcal{S}(X)$ over an arbitrary set X following [14] (see also [15]). We assume that X contain all naturals.

Definition 2. Put $\mathcal{S}_0(X) = X$ and for each $n \in \mathbb{N}$, $n > 0$, $\mathcal{S}_n(X) = \mathcal{S}_{n-1}(X) \cup \mathcal{P}(\mathcal{S}_{n-1}(X))$. Then $\mathcal{S}(X) = \bigcup_{n=0}^{\infty} \mathcal{S}_n(X)$. Define the rank of $x \in X$ as follows. For $x \in \mathcal{S}_0(X)$ $\mathrm{rank}(x) = 0$, for $n \geqslant 1$ $\mathrm{rank}(x) = n$, if $x \in \mathcal{S}_n(X) \setminus \mathcal{S}_{n-1}(X)$.

The superstructure $\mathcal{S}(\mathbb{R})$ contains the mathematical objects that are used in the main areas of mathematics— algebra, geometry, analysis, probability theory and others. For example, the book [4] contains only objects from $\mathcal{S}(\mathbb{R})$ and its elementary extension, which, by the way, is also a subfamily of $\mathcal{S}(\mathbb{R})$ (see below). The signature σ of $\mathcal{S}(\mathbb{R})$ is the following one:*

$$\sigma = \langle \varnothing, +, \cdot, \leqslant, \in \rangle. \tag{2}$$

Here $\varnothing$ is a constant symbol, $+$ and $\cdot$ are binary function symbols and $\leqslant$ and $\in$ are binary predicate symbols. The elements of rank 0—real numbers are considered as individuals (not sets). They are the only elements of $\mathcal{S}(\mathbb{R})$ that are not sets. For example $\mathrm{rank}(\varnothing) = 1$ and in the expressions $x+y$, $x\cdot y$, $x \leqslant y$, $t \in z$ we have $\mathrm{rank}(x) = \mathrm{rank}(y) = 0$ and $\mathrm{rank}(t) < \mathrm{rank}(z)$.

Practically any conventional mathematical statement can be formalized by an appropriate formula of the language L_σ. Such a direct formalization is often quite long and difficult to see even for relatively simple mathematical statements. To simplify it, various notations and abbreviations are usually used.

The following abbreviation is generally accepted in all logical languages.

Let $\varphi(x)$ be a formula that contains a free variable x.

$$\exists!\, x\ f(x) := \exists x \forall y\ (\varphi(x) \wedge (\varphi(y) \longrightarrow x = y)).$$

In conventional language we read this formula of L_σ as “There exists a unique x such that $\Phi(x)$”, where $\Phi(x)$ is a conventional statement about x whose formalization in L_σ is $\varphi(x)$.

Definition 3. We say that an element $A \in \mathcal{S}(\mathbb{R})$ is definable in the superstructure $\mathcal{S}(\mathbb{R})$ if there exists a formula $\varphi(x)$ of L_σ with the only free variable x, such that $\mathcal{S}(\mathbb{R}) \models \exists!\, x\ \varphi(x) \wedge \varphi(A)$.

If φ contains besides x free variables $p_1, \ldots, p_n$, then we say that A is defined in $\mathcal{S}(\mathbb{R})$ via parameters $p_1, \ldots p_n$, provided that

$$\mathcal{S}(\mathbb{R}) \models \forall p_1, \ldots, p_n\, \exists!\, x\ \varphi(x, p_1, \ldots, p_n) \tag{3}$$

and for all $b_1, \ldots, b_n \in \mathcal{S}(\mathbb{R})$ we have

$$\mathcal{S}(\mathbb{R}) \models \varphi(A, b_1, \ldots, b_n). \tag{4}$$

If φ is satisfies (3), then φ defines a function that assign to each n-tuple $\langle b_1, \ldots, b_n \rangle$ the unique element A that satisfies (4). So, the formula φ after assigning an appropriate notation can be added to the signature σ as a function symbol.

For example, consider a formula $\psi(x, x_1, \ldots, x_n)$ and let

$$\varphi(X, x, x_1, \ldots, x_n) := x \in X \ \longleftrightarrow\ \psi(X, x, x_1, \ldots, x_n). \tag{5}$$

Then φ satisfies (4), and so it defines an n-ary function, that is denoted as $X = \{x : \psi(x, x_1, \ldots, x_n)\}$ exactly like in the conventional language.

* We assume that the symbol of equality $=$ is an element of any logic language.

In case, when ψ does not contain the variable x, the formula φ define the n-ary predicate $X(x_1, \ldots, x_n)$, $n \geqslant 0$. In case when the truth domain of X is a set that belongs to $\mathcal{S}(\mathbb{R})$ we write $\langle x_1, \ldots x_n \rangle \in X$, identifying the predicate and its domain. We assign some notation to this predicate and include it in the signature σ as the set constant. Sometimes in this situation we say for brevity that a function or a predicate is defined by the formula ψ, not ϕ.

Example. 1. The formula $\forall\, y\ (y \notin x \wedge x \neq \varnothing)$ defines the unary predicate "x is a real", whose truth domain is the set of all reals $\mathbb{R}$, so we include the constant $\mathbb{R}$ in the signature σ. Usually we use writing $x \in \mathbb{R}$, not $\mathbb{R}(x)$. The truth domain of the predicate $x \notin \mathbb{R}$ consists of all sets in $\mathcal{S}(\mathbb{R})$. So it is not an element of $\mathcal{S}(\mathbb{R})$. Sometimes this predicate is denoted by Set. Here writing Set(x) is preferable.

2. All Boolean operations on sets but complementation are obviously definable in L_σ. As it was mentioned above $\mathcal{S}(\mathbb{R})$ is a set, thus $\mathcal{S}(\mathbb{R}) \setminus x$ is a set for any set $x \in \mathcal{S}(\mathbb{R})$. However, this set is not an element of $\mathcal{S}(\mathbb{R})$, since it necessary contains elements of an arbitrarily large rank. As in conventional mathematics, the complementation can be used, when some universe $\mathcal{U} \in \mathcal{S}(\mathbb{R})$ is fixed and we deal only with its subsets.

3. The predicate $x \subseteq y$ is defined by the formula $\forall\, z\ (z \in x \to z \in y)$. The operation $\mathcal{P}(y)$ is defined by the formula $\mathcal{P}(y) = \{x : x \subseteq y\}$

4. The definition of an ordered pair by Kuratowski:

$$\langle a, b \rangle = \{\{a\}, \{a, b\}\}$$

can be considered as a formula of L_σ written with the abbreviations introduced above. Using the definition of an ordered pairs, we can usually formalize the definitions $X \times Y$, $f : X \to Y$, Y^X, etc.

For the further examples of translations from the conventional language to the formal one see Section 3 of Chapter I in [16] and Section 1 of Chapter 0 in [1].

Consider some proper elementary extension ${}^*\mathcal{S}(\mathbb{R})$ of $\mathcal{S}(\mathbb{R})$. We use the canonical notations for the elementary extension of $\mathcal{S}(\mathbb{R})$ in the nonstandard analysis. Here ${}^* : \mathcal{S}(\mathbb{R}) \to {}^*\mathcal{S}(\mathbb{R})$ is the monomorphism $\imath$ of the Definition 1. The equivalence (1) is called in the nonstandard analysis the *Transfer principle*.

Definition 4. a) An element ${}^*x \in {}^*\mathcal{S}(\mathbb{R})$, the image of $x \in \mathcal{S}(\mathbb{R})$ under the monomorphism *, is said to be standard. We say that $y \in {}^*\mathcal{S}(\mathbb{R})$ is standard (notation St(y)), if $\exists\, x \in \mathcal{S}(\mathbb{R})$ such that $y = {}^*x$.

b) The elements of ${}^*\mathcal{S}(\mathbb{R})$ are called internal elements.

c) The noninternal sets that belong to $\mathcal{S}({}^*\mathbb{R})$ are called external sets.

d) The elements of ${}^*\mathbb{R}$ are called hyperreals numbers or hyperreals.

In what follows $\forall^{st} x \ldots$ stay for $\forall\, x(\mathrm{St}(x) \to \ldots)$ and $\exists^{st} x \ldots$ for $\exists\, x(\mathrm{St}(x) \wedge \ldots)$ respectively.

Remark 2. Let us clarify the difference between ${}^*(\mathcal{S}(\mathbb{R}))$, ${}^*\mathcal{S}(\mathbb{R})$ and $\mathcal{S}({}^*\mathbb{R})$. The first one consists of all standard elements, the second one is an elementary extension of $\mathcal{S}(\mathbb{R})$, i. e. it consists of all internal elements, the third one contains all internal and external elements of $\mathcal{S}({}^*\mathbb{R})$. So,

$$ {}^*(\mathcal{S}(\mathbb{R})) \subseteq {}^*\mathcal{S}(\mathbb{R}) \subseteq \mathcal{S}({}^*\mathbb{R}). \tag{6}$$

Both inclusions in (6) are proper. Notice firstly, that since the elementary extension of $\mathcal{S}(\mathbb{R})$ is proper by definition, then ${}^*(R) \neq {}^*R$. Otherwise, it is easy to see that ${}^*(\mathcal{S}(\mathbb{R})) = \mathcal{S}({}^*\mathbb{R})$, etc. We show now that there are internal sets that are nonstandard. Since there exist internal elements $\alpha < \beta \in {}^*\mathbb{R} \setminus {}^*(\mathbb{R})$ and in $\mathcal{S}(\mathbb{R})$ it is true that for any $x < y \in \mathbb{R}$ there exists a set

$[x, y]$; then, by the Transfer Principle, the same statement is true in $^*\mathcal{S}(\mathbb{R})$ for $^*\mathbb{R}$. Thus, $[\alpha, \beta]$ is an internal set in $^*\mathcal{S}(\mathbb{R})$, that is obviously nonstandard.

It is easy to prove the following

Proposition 1. *If a linearly ordered field $\mathcal{R}$ is an arbitrary proper extension of $\mathbb{R}$, then*

1. *There exits $\rho \in \mathcal{R} \setminus \mathbb{R}$ such that $\forall\, r \in (0, \infty) \subseteq \mathbb{R}\ 0 < |\rho| < r$. Notation: $\rho \approx 0$.*

2. *For every $\beta \in \mathcal{R}$ such that $|\beta| < r$ for some $r \in \mathbb{R}$, there exists $b \in \mathbb{R}$ such that $\beta - b \approx 0$. In this case we write $\beta \approx b$.*

A simple proof of this proposition can be a good exercise for the students who start to study a rigorous course of Analysis. It is just a proposition of standard mathematics. In $^*\mathcal{S}(\mathbb{R})$ we apply Proposition 1 to $^*\mathbb{R}$ for $\mathcal{R}$ and to $^*(\mathbb{R})$ that is an isomorphic copy of $\mathbb{R}$. Here the following definition is used:

Definition 5. a) If $\rho \approx 0$, then ρ is said to be infinitesimal. The set $M_0 = \{\rho \in {}^*\mathbb{R} :\ \rho \approx 0\}$ is called the monad of 0.

b) An element $\Omega \in {}^*\mathbb{R}$ is said to be infinitely large if $\forall^{st}\, r\ |\Omega| > r$.

c) An element $\beta \in {}^*\mathbb{R}$ is said to be limited, or bounded or finite, if $\exists^{st}\, r\ |\beta| < r$. The set of bounded elements is denoted by $^*\mathbb{R}_{fin}$. According to Proposition 1 item 2, in this case there exists a standard b such that $b \approx \beta$. This b is called the standard part or the shadow of β and is denoted by $^\circ\beta$.

Proposition 2. *The monad M_0 is an external set.*

$\lhd$ Suppose that M_0 is an internal set. Since it is bounded from above (e.g. by number 1), it must have $\sup M_0 = \mu$. It is easy to see that both assumptions $\mu \approx 0$ and $\mu \not\approx 0$ lead to contradiction. $\rhd$

The notions defined in Definition 5 are not formulated in L_σ since in their formulation the unary predicate St is present explicitly or implicitly. This predicate is not included in the signature σ. For the formalization of such statements we need to extend the signature σ by adding to it this predicate. Denote the extended signature σ^{st}. The formulas in $L_{\sigma^{st}}$ are called external formulas, while the formulas of L_σ are called internal formulas. The Transfer Principle is not applicable to external formulas, as we just saw in the proof of Proposition 2.

Mathematicians, who begin to study nonstandard analysis with the aim of applying it in their research, often face the following difficulty. They make mistakes related just to the application of the Transfer Principle to external sets. This is because the definitions of internal and external sets (see Definition 4) are very nonconstructive and require a formalization habit, which usually the mathematicians who work in geometry, ODEs and PDEs do not have. The considerations of the previous paragraph imply the following sufficient condition for a set to be internal (external): *A set defined by a formula of the language L_σ is internal, while a set defined by a formula of the language $L_{\sigma^{st}}$ is external.* This condition makes the difficulty mentioned above somewhat easier.

Nevertheless, a certain difficulty in using nonstandard methods remained. The fact is that the Maltsev Compactness Theorem, on which Theorem 1 is based, is a consequence of Gödel's Completeness Theorem, more precisely, on its generalization to signatures of an arbitrary cardinality belonging to Maltsev. Many mathematicians do not like to use in their research the results whose proofs they do not know, or at least do not even imagine the idea. In order to be sure of the correctness of the results obtained using nonstandard analysis, such mathematicians must study the principles of mathematical logic, at least up to Theorem 1 inclusively. This is a rather extensive material that usually is far away from their scientific interests and does not correspond to their way of thinking. But even after studying the proof of Theorem 1, such mathematicians will not feel completely satisfied, since the proof of this

Theorem based on the Compactness Theorem is a pure proof of existence and does not give any construction of elementary extension.

Fortunately, there is another proof of the existence of an elementary extension of an arbitrary structure, in which this extension is constructed as an ultrapower of this structure. This construction does not rely on Gödel's Completeness Theorem and does not require any knowledge of mathematical logic besides the definition of a first-order language and the truth of the formula of this language in its signature. It can be considered as a construction in the conventional mathematics.

DEFINITION 6. 1) Let $\{\mathcal{M}_i : i \in I\}$ be a family of structures of a signature σ and $\mathcal{F}$ be a free ultrafilter on I. Then the ultraproduct of this family is the structure

$$\prod_{\mathcal{F}} \mathcal{M}_i = \left(\prod_i M_i / \sim_{\mathcal{F}}, \sigma\right), \text{ where } \{m_i\} \sim_{\mathcal{F}} \{m_i'\} := \{i : m_i = m_i'\} \in \mathcal{F}. \tag{7}$$

Let f be a function symbol of the signature σ. Consider the sequence $\{f_i : M_i \to \mathcal{M}_i\}$, where f_i is the interpretation of f in $\mathcal{M}_i$. Then the interpretation $f^{\sim_{\mathcal{F}}}$ of f is defined as follows:

$$f^{\sim_{\mathcal{F}}}(m^{\sim_{\mathcal{F}}}) = n^{\sim_{\mathcal{F}}} := \{i : n_i = f_i(m_i)\} \in \mathcal{F} \tag{8}$$

The definitions of interpretation of k-ary functional symbols for arbitrary $k \in \mathbb{N}$ and k-ary predicate symbols are similar.

2) If all $\mathcal{M}_i = \mathcal{M}$ for a certain σ-structure $\mathcal{M}$, then the ultraproduct defined in 1) is called the ultrapower of $\mathcal{M}$ and is denoted $\mathcal{M}^{\mathcal{F}}$. There exists a monomorphism $j : \mathcal{M} \to \mathcal{M}^{\mathcal{F}}$ such that $j(m) = \{m_i : i \in I\}^{\sim_{\mathcal{F}}}$, where $m_i = m$ for all $i \in I$

Theorem 2 (Łoš**).** *Let $\mathcal{M} = (M, \sigma)$ be a structure of the signature σ; $\mathcal{F}$, a free ultrafilter on a set I; $\varphi(x_1, \dots, x_n)$; a formula of L_σ, and $\mu_k = \{m_i^k\}^{\mathcal{F}} \in M^{\mathcal{F}}$, where $k = 1, \dots, n$. Then*

$$\mathcal{M}^{\mathcal{F}} \models \varphi(\mu_1, \dots, \mu_n) \iff \{i \in I : \mathcal{M} \models \varphi(m_i^1, \dots, m_i^n) \in \mathcal{F}.$$

Corollary 1. *The structure $(\mathcal{M}^{\mathcal{F}}, j)$ is an elementary extension of $\mathcal{M}$.*

The elementary extension of $\mathcal{S}(\mathbb{R})$ is the *bounded ultrapower* of $\mathcal{S}(\mathbb{R})$, that is

$$\mathcal{S}(\mathbb{R})_b^{\mathcal{F}} = \bigcup_{n \in \mathbb{N}} \left(\mathcal{S}_n(\mathbb{R})^{\mathcal{F}} \setminus \mathcal{S}_{n-1}(\mathbb{R})^{\mathcal{F}}\right).$$

Corollary 1 is true for the bounded ultrapower. In what follows we keep the notation $({}^*\mathcal{S}(\mathbb{R}), {}^*)$ for the ultrapower nonstandard extension of $\mathcal{S}(\mathbb{R})$, if an ultrafilter $\mathcal{F}$ is fixed.

We cannot say that at least one nonstandard extension built using an ultrafilter is constructive, if only because the very existence of an ultrafilter cannot be proved without the Axiom of Choice. However, this concept is widely used in conventional mathematics: For example, recall the Stone–Čech compactification or the famous paper [18], which essentially uses the construction of a nonstandard hull known to nonstandard analysis, to which the author comes completely independent of nonstandard analysis and related to it mathematical logic.

The internal sets in an ultraproduct nonstandard extension of $\mathcal{S}(\mathbb{R})$ have very clear description: A set in $\mathcal{S}(\mathbb{R})_b^{\mathcal{F}}$ is internal if and only if it is an ultraproduct of a family of sets $\{X_i : i \in I\} \subseteq \mathcal{S}_k(\mathbb{R})$ for some $k \in \mathbb{N}$.

A hyperreal $\rho \in \mathbb{R}^{\mathcal{F}}$ is limited, if $\rho = \{r_i : i \in I\}^{\mathcal{F}}$ is such that the exists $r \in \mathbb{R}$ such that the set $\{i \in I : |r_i| < r\} \in F$. Then we may assume that the set $\{r_i : i \in I\}$ is a bounded subset of $\mathbb{R}$. The limit of a bounded function over free ultrafilter always exists and it is easy to see, that the standard part of ρ

$$^\circ\rho = \lim_{\mathcal{F}} r_i.$$

Using the technique of ultraproducts allowed mathematicians to easily remake the results obtained by nonstandard analysis into standard ones, without even going into the details of the original nonstandard proofs. This led to a certain drop in interest in nonstandard analysis and decrease in the number of publications and conferences related to nonstandard analysis. I believe that the potential of nonstandard analysis is far from exhausted. The justification of this point of view will be contained in another article.

3. Forcing and Independence Proofs

Another outstanding achievement of the mathematical logic of the 1960s was the proof of independence of the Continuum Hypotheses (CH) and the Axiom of Choice (AC) by P. Cohen and his development of the method of forcing for this and many other proofs of independence of the axioms of set theory. This method was ideologically and technically very complex and accessible for the specialists not only in mathematical logic, but also in its very special field—the axiomatic set theory. We discuss briefly Cohen's method. We deal here with the axiomatic due to Zermelo and Fraenkel (ZF). If AC is included in ZF then this system of axiom is denoted ZFC.

3.1. Axiom of constructivity. The consistency of CH. The consistency of CH with ZFC and AC with ZF was proved by K. Gödel in the late 1930s of the last century. We assume that the reader is aware of the axioms of ZF. However, we remind some notions and notations. Recall that a set x is said to be *transitive*, if $\forall\, y \in x\ z \in y\ (z \in x)$.

A set α is said to be an *ordinal* (notation $\alpha \in \mathrm{On}$), if it is linearly ordered by the membership $\in$. By the axiom of regularity $\in$ is a well-ordering of α.

The formula of ZF $\alpha \in \mathrm{On}$ is absolute with respect to any transitive set in the sense of the following

Definition 7. We say that a formula $\varphi(x)$ is absolute with respect to a set M, if for any $a \in M$

$$\varphi(a) \longleftrightarrow M \models \varphi(a). \tag{9}$$

In this context $\models$ on the right hand side means that all quantifiers are restricted to M. We use the notation φ_M for the formula that is obtained by restriction of all quantifiers in φ to M. If M is a set that is definable by a formula $\psi(x)$ in the sense of (5), then all quantifiers in φ are restricted to ψ and the equivalence (9) means

$$ZF \vdash \forall\, x(\psi(x) \to (\varphi(x) \longleftrightarrow \varphi_M(x)). \tag{10}$$

Obviously, if M is a transitive set and all quantifiers are of the form $\forall\, u \in v$, or $\exists\, u \in v$, then φ is absolute with respect to M.

A *cardinal* is an ordinal that is not in one to one correspondence with any of its elements. This definition is not absolute.

Indeed, there are such extensions of some models of set theory, in which ordinals are the same but the scale of cardinals is compressed. This is achieved by adding to the original model the bijective mapping of the ordinal representing a cardinal in the original model onto an ordinal representing a smaller cardinal. This effect is called the collapse of the cardinals. Exactly such an extension is used in the paper [19] discussed below.

It can be deduced from the Axiom of Regularity that the class of all sets V can be represented as follows:

$$V = \bigcup_{\alpha\in\mathrm{On}} V_\alpha, \quad V_0 = \varnothing, \;\; V_{\alpha+1} = \mathcal{P}(V_\alpha), \;\; V_\alpha = \bigcup_{\beta<\alpha} V_\beta, \;\; \text{if } \alpha \text{ is a limit ordinal.} \tag{11}$$

It is easy to see that $x \in V_\alpha$ can be written as a formula $E(x,\alpha)$ of ZF.

To prove the consistency of CH and AC, Gödel studied the class of all sets definable from ordinals. He called such sets *constructive* and denoted by $\mathcal{L}$. The class $\mathcal{L}$ is defined similar to (11):

Definition 8.

$$\mathcal{L} = \bigcup_{\alpha\in\mathrm{On}} \mathcal{L}_\alpha, \quad L_0 = \varnothing, \;\; \mathcal{L}_{\alpha+1} = \mathcal{P}_{def}(\mathcal{L}_\alpha), \;\; \mathcal{L}_\alpha = \bigcup_{\beta<\alpha} \mathcal{L}_\beta, \;\; \text{if } \alpha \text{ is a limit ordinal.} \tag{12}$$

Here $X \in \mathcal{P}_{def}(\mathcal{L}_\alpha)$ if X is definable like in equivalence (5), but ψ is a formula of ZF, with all quantifiers restricted to L_α and each x_i is either an element of $\mathcal{L}_\alpha$ or $\mathcal{L}_\alpha$ itself.

Theorem 3 (Gödel-1)**.** 1. *There exists a function $F(x,\alpha)$ definable in ZF, which establishes a bijection between classes On and $\mathcal{L}$. This function is absolute with respect to $\mathcal{L}$.*
2. $\mathcal{L} \models ZF$*, which means that, if for any axiom φ of ZF one has $ZF \vdash \varphi_{\mathcal{L}}$.*

In what follows we write $x \in \mathcal{L}_\alpha$ for $F(x,\alpha)$ and $x \in \mathcal{L}$ for $\exists\, \alpha \in \mathrm{On}\ x \in \mathcal{L}$.

The absoluteness of the formula $x \in \mathcal{L}$ implies

Corollary 2. $\mathcal{L} \models \forall\, x\ (x \in \mathcal{L})$.

Theorem 3.2 together with Corollary 2 implies the consistency of the statement $\forall\, x\ (x \in \mathcal{L})$ with ZF. This gave Gödel the basis to call it the *Axiom of Constructivity*: Every set is constructive. This axiom is usually written in the form $V = L$

Theorem 4 (Gödel-2)**.** $ZF + V = L \vdash AC + CH$.

The fact that $ZF + V = \mathcal{L} \vdash AC$ follows immediately from Theorem 3.1. The proof of the statement $ZF + V = \mathcal{L} \vdash CH$ is very subtle and is not discussed here. For the proofs of both Gödel Theorems see [20] or Chapter 10 of [21].

3.2. Standard Transitive Models. The natural way to prove the consistency of a certain sentence φ with the axioms of ZF is to present a model M such that $M \models ZF$. If the class M is definable by a formula $\psi(x)$, then it is called an *inner model*. The class L of constructive sets is an inner model for $V = L$, AC and CH.

The first obstacle to proving the independence of these statements is the theorem that there is no internal model, neither for V=L, nor for AC, nor for CH. See the Introduction to Chapter IV of [22]. So, any proof of the consistency of negations of these statements by presenting a standard model for each of them should be a pure existence proof. Such proof is possible only if a model is a set.

We say that a pair (M, E) is a model of ZF, if M is a set, the binary relation $E \subseteq M \times M$ is an interpretation of $\in$, and $(M, E) \models ZF$. Owing to the Gödel Completeness Theorem the existence of a model of ZF is equivalent to consistency of ZF. Thus, it cannot be proved in ZF due to the Gödel Incompleteness Theorem. The vast majority of mathematicians believe in the consistence of ZF.

An example of a set model of ZF+V=L is the set $(\mathcal{L}_\Omega, \in)$, where Ω is an inaccessible cardinal. Recall that a uncountable cardinal α is said to be *inaccessible*, if 1) it is regular and 2) $\forall\, \lambda < \alpha\ 2^\lambda < \alpha$.

The model, in which the inclusion symbol is interpreted as standard membership $\in$ in V is called a standard model. The conjecture about existence of a standard model of ZF and, thus, the conjecture about existence of an inaccessible cardinal are not provable in ZF. Moreover, they are even stronger, than the conjecture on consistency of ZF. However, there are some natural conditions such that a model (M, E) of ZF satisfying them is isomorphic to a standard model. In what follows we deal only with standard set-models of ZF either. It can be proved also that each standard model of ZF has a countable transitive submodel. The proofs and further discussion of the theorems mentioned in this paragraph can be found in the Chapter II of [22] or in Chapter 10 of [21]). The existence of a countable model is important for a proof of independence of CH, since there does not exist an uncountable standard model of ZFC, in which CH fails (see the Introduction to Chapter IV of [22]).

Since every standard transitive model M contains the set of all constructive sets in M as a minimal submodel with the same ordinals, we can assume from the beginning that M is a countable transitive model and $M \models V = L$. It means that

$$M = \bigcup_{\alpha<\lambda} \mathcal{L}_\alpha,$$

where λ is the minimal countable ordinal that is not in M.

Remark 3. Since for every $\alpha \in \mathrm{On}$ there exists a cardinal $\aleph_\alpha$, for every $\alpha \in M \cap \mathrm{On}$, $M \models \exists\, \beta = \aleph_\alpha$. Since $\beta \in M$, β is countable, thus in $V \not\models \aleph_\alpha$. This example illustrates the mentioned above fact that the formula $\beta = \aleph_\alpha$ is not absolute.

3.3. Independence of V=$\mathcal{L}$ and CH. To obtain a model, where $V = \mathcal{L}$ fails we have to extend the model M by adding some $G \subseteq P$, where $P \in M$, but $G \notin M$. We have to consider the minimal standard transitive model $M[G] \supset M \cup \{G\}$. This model consist of all elements constructible from G.

Definition 9. We say that a set x is constructible from a set G, if

$$x \in \bigcup_{\alpha\in\mathrm{On}} \mathcal{L}_\alpha[G],$$

where the definition of $\{\mathcal{L}_\alpha[G] : \alpha \in \mathrm{On}\}$ is similar to the definition of (12), but $\mathcal{L}_0[G] = \{G\}$

In our case, $M[G]$ is transitive, so the predicate $x \in \mathcal{L}$ is absolute with respect to $M[G]$. Hence, $M[G] \cap \mathcal{L} = M$; therefore, G is not constructive and $M[G] \models V \neq \mathcal{L}$. However, not for every G, that is a subset of a set $P \in M$ it is true that $M[G] \models ZFC$.

We formulate here a sufficient condition for G to have this property, which is the key point of the forcing method. To the end of this subsection the model M is fixed.

Definition 10. Let $(P, \leqslant) \in M$ be a partially ordered set (poset). If

1. A set $Q \subseteq P$ is dense in P, if $\forall\, p \in P\, \exists\, q \in Q\ p \leqslant q$.

2. Elements $p, q \in P$ are said to be compatible, if $\exists\, r \in P\ p \leqslant r \wedge q \leqslant r$. If p and q are incompatible, they are said to be disjoint.

3. We say that $(P, \leqslant)$ is a Boolean-type set (BTS),

a) for every two disjoint $p, q \in P\ \exists\, r \in P\ p \leqslant r \wedge q \leqslant r$;

b) for all $p, q \in P$ such that $p \nleqslant q$ there exists $r \leqslant q$ that is incompatible with p.

Remark 4. In P. Cohen's approach to forcing an arbitrary poset P without a top is called a set of forcing conditions. According to the Mac Neille Theorem (see [21]) for any poset $(P, \leqslant)$ that satisfies the condition 3b) of this definition there exist a complete Boolean algebra $\mathbb{B}$ and an embedding $\iota : P \to \mathbb{B}$ that is an anti-monomorphism ($p \leqslant q \longleftrightarrow \iota(q) \leqslant \iota(p)$), and $\iota(P)$ is dense in $\mathbb{B}$. Obviously $\mathbb{B}$ is defined uniquely up to isomorphism. It is called a Dedekind–Mac Neille completion of P and denoted by $RO(P)$. The mapping ι exists even if P does not satisfy 3b), but in this case it is an anti-homomorphism. Condition 3a) follows from the others and from the Mac Neille theorem and is included only for convenience. See [23, § 2.3], and [21, Chapter 16] for proofs and details.

Definition 11. A subset $G \subseteq P$ is said to be M-generic if

1. $\forall p, q \in G \, \exists r \in G \; p \leqslant r \wedge q \leqslant r$;
2. if $p \in G$, $q \leqslant p$, then $q \in G$;
3. for every $Q \subseteq P$, such that $Q \in M$ and Q is dense in P the intersection $G \cap Q \neq \varnothing$.

Theorem 5 (Cohen 1). *If $(P, \leqslant) \in M$ is a Boolean-type set, then there exist an M-generic set $G \subseteq M$, and for every such G, $M[G] \models ZFC + V \neq \mathcal{L}$*

Proofs of existence of M-generic set and of the statement $M[G] \models V \neq \mathcal{L}$ are very simple.

Since M is countable, it has only countably many dense subsets $P_1, \ldots, P_n, \ldots$. So, there exists a sequence $p_1 \leqslant \ldots \leqslant p_n \leqslant \ldots$, such that $p_n \in P_n$ for all $n \in \mathbb{N}$. Let $G_n = \{p \in P : p \leqslant p_n\}$. Then $G = \bigcup\limits_n G_n$ is an M-generic set.

To prove that $M[G] \models V \neq \mathcal{L}$, it is enough to show that $G \notin M$. This is an immediate corollary of the following

Lemma 1. *If an M-generic set $G \in M$, then $P \setminus G \in M$ is dense.*

The proof of this Lemma is an easy exercise.

The proof of the statement $M[G] \models ZFC$, which belongs to P. Cohen [22] is very complicated and technical. It is not discussed here.

Definition 12. An extension $M[G]$ of G, where G is an M-generic set, is called a generic extension of M.

Definitions 10 and 11 are modifications of the general definitions of a *set of forcing conditions* and generic set in Section 7 of Chapter 4 in [22]. These definitions involve the partially ordered set U of forcing conditions and the set S of all sentences (formulas without free variables) of the language of ZF extended by adding to the signature the constant symbols for all constructible sets and for a generic set G. Nothe that $S \in M$. Also the binary relation $\| - \subseteq U \times S$ is defined by induction. The entry $p \| - \varphi$ reads like p forces φ. This relation is crucial for the proof of $M[G] \models ZFC$. In the paper [19] R. Solovay introduced almost the same Definition (I.3) of M-generic sets as Definition 11, which he, keeping in mind to use Boolean valued models, called M-generic filters. The insignificant difference is that Solovay used arbitrary rather than Boolean-type posets.

The concept of an M-generic set and its properties can be considered as definitions and theorems of ZF, if we assume that M is an arbitrary countable family of dense subsets of P. The proof of the existence of an M-generic set and that this set does not belong to M carry over to this case without change. Below the notation $\mathcal{X}$ is used for a countable family in ZF, since the notation M is fixed for the countable standard transitive model $ZF + V = \mathcal{L}$ throughout the entire Section 3.

Given an infinite set Γ put

$$P^\Gamma = \big\{ f : f \text{ is a function, } \operatorname{dom}(f) \in \mathcal{P}^{fin}(\Gamma), \ \operatorname{range}(f) \subseteq \{0, 1\} \big\}.$$

We start with considerations in ZF. Put for p, q and $p \leqslant q := \operatorname{graph}(p) \subseteq \operatorname{graph}(q)$. Let $\mathcal{X}$ be a countable family of dense subsets of P^Γ. It is easy to see that $(P^\Gamma, \leqslant)$ is a Boolean type poset and if G is an $\mathcal{X}$-generic set, then G is a linearly ordered subset of P^Γ. Let $f[G] = \bigcup G$, then $f[G] : \Gamma \to \{0,1\}$. Consider the set $C(\Gamma) = \{0,1\}^\Gamma$ endowed with Tychonoff's topology. This a compact set. For $\Gamma = \mathbb{N}$ this is the Cantor's continuum.

Let $\mathcal{O}_p = \{f \in\colon \operatorname{graph}(p) \subseteq \operatorname{graph}(f)\}$. Then the family $\mathcal{T}(\Gamma) = \{\mathcal{O}_p : p \in P^\Gamma\}$ is a base of a topology of $C(\Gamma)$. For $X \subseteq P$ let $O(X) = \bigcup_{p\in X} \mathcal{O}_p$. Obviously, if X is dense, then $\mathcal{O}(X)$ is dense in C in the sense of Tychonoff's topology. So its complement is nowhere dense and, thus the intersection $\bigcap\{\mathcal{O}(X) : X \in \mathcal{X}\}$ is a comeager set. Elements of this set we call $\mathcal{X}$*-generic*. Since, every function $f \in C(\Gamma)$ is a characteristic functions of a subset of Γ or, in case of $\Gamma = \mathbb{N}$ it can be regarded as a binary fraction, so we can speak about X-generic subsets of Γ or about $\mathcal{X}$-generic reals. The following proposition follows easily from definitions.

Proposition 3. *Any $f \in C(\Gamma)$ is $\mathcal{X}$-generic if and only if $f = f[G]$ for some $\mathcal{X}$-generic set G.*

Denote the bottom and the top of any Boolean algebra by $\mathbf{0}$ and $\mathbf{1}$ respectively. Let $\mathbb{B} = \{\mathbf{0},\mathbf{1}\}$. Then the Boolean algebra $\mathcal{T}(\Gamma)$ has the independent system of generators $\{B_\gamma : \gamma \in \mathcal{P}^\Gamma\}$. Here $B_\gamma = \{\mathcal{O}_p, \mathcal{O}_q\}$, where $\operatorname{dom}(p) = \operatorname{dom}(q) = \{\gamma\}, p(\gamma) = \mathbf{0}$, $q(\gamma) = \mathbf{1}$ is isomorphic to $\mathbb{B}$. In this case we write $\mathcal{T}(\Gamma) = \prod_{\gamma\in\Gamma} B_\gamma$ and say that $\mathcal{T}(\Gamma)$ is a free product of Γ copies of B.

Obviously the mapping $p \mapsto \mathcal{O}_p$ is an inverse isomorphism of posets P^Γ and $\mathcal{T}(\Gamma)$ ($p \leqslant q \longleftrightarrow \mathcal{O}_q \subseteq \mathcal{O}_p$). The poset $\mathcal{T}(\Gamma)$ is the Boolean algebra of clopen sets of $C(\Gamma)$. Its Dedekind–MacNeille (DM) completion ([23, § 2.3], and [21, Ch. 16]) is the quotient algebra of the σ-algebra of Borel sets of $C(\Gamma)$ by the ideal of meager sets [23, § 2.4(2)]. In what follows we denote it by $\mathcal{B}(\Gamma)$ and in case when $\Gamma = \mathbb{N}$, simply by $\mathcal{B}$.

All facts about Boolean algebras used below are taken from the book [23].

Proposition 4. *The algebra $\mathcal{T}(\Gamma)$ and, thus, the algebra $\mathcal{B}(\Gamma)$ satisfy the countable chain condition: any subset of each of them that consists of pairwise disjoint elements is at most countable.*

See [11, § 9.5(5)], for a proof. Assume that $|\Gamma| = \aleph_3$ and $2^{\aleph_2} = \aleph_3$ (GCH). Then

$$\aleph_3 = \left(2^{\aleph_0}\right)^{\aleph_2}. \tag{13}$$

So, $\mathcal{B}(\Gamma)$ is isomorphic to $(\mathcal{B})^{\aleph_3}$. This algebra includes as a dense subalgebra the algebra $\mathcal{T}(\aleph_3)$ that is the free product of $\aleph_2$ copies of the algebra $\mathcal{T}(\mathbb{N}) = \mathcal{T}(\aleph_0)$. A similar algebra was used by P. Cohen in the proof of independence of CH as a poset of forcing conditions.

Let's return to our standard transitive model M of ZFC+V=L to present the main ideas of this proof that is based on the original forcing, with improvements by Solovay [19], Section 2. Another proof of this theorem, which uses only Boolean valued models is contained in the book [11, 9.5].

Denote by $\omega_k \in M$ $(k = 1,2,3)$ a countable ordinal in M such that $M \models \omega_k = \aleph_k$ and let G be an M-generic filter on $\mathcal{T}(\omega_3)$. Since (13) holds in M, we have in M:

$$\mathcal{T}(\omega_3) = \prod_{\lambda\leqslant\omega_2} \mathcal{T}_\lambda(\omega_0) = \mathcal{T}, \tag{14}$$

where $\mathcal{T}_\lambda(\omega_0)$ is the λth copy of $\mathcal{T}(\omega_0)$. The following theorem (see e.g. [21]) is widely used in forcing.

Theorem 6 (Absoluteness of cardinals)**.** *If a poset $P \in M$ satisfies the countable chain conditions* (*see Proposition* 4), *and G is an M-generic set, then the cardinals in M and in $M[\Gamma]$ are the same.*

Theorem 7 (Cohen 2)**.** *If $G \subseteq \mathcal{T}$ is an M-generic set, then $M[G] \models 2^{\omega_0} > \omega_1$.*

By Theorem 6 $M[G] \models \omega_k = \aleph_k$. Notice, that (14) may not be valid in $M[G]$, since the definition of $\mathcal{P}(X)$ is not absolute. However, the right hand side of this equality is absolute. It is easy to see that $G \cap \mathcal{T}_\lambda(\omega_0) = G_\lambda$ is an M-generic set on $\mathcal{T}_\lambda(\omega_0)$ (see [19, § 2]). So, the subset $f(G_\lambda) = f_\lambda \subseteq \mathbb{N}$ is M-generic. The mapping $\Phi : \omega_2 \to 2^\omega$ such that $\Phi(\lambda) = f_\lambda$ is absolutely definable from G and elements of M, so $\Phi \in M[G]$. It is easy to see that Φ is injective. Indeed, if If $\varphi_\lambda = \varphi_\nu$, $\lambda \neq \mu$, then it belongs to the diagonal of $\left(\{0,1\}^{\mathbb{N}}\right)^2$ and (f_λ, f_ν) is M-generic with respect to the M-generic set $\mathcal{T}_\lambda(\omega_0) \cdot \mathcal{T}_\nu(\omega_0) \cap G$. This is impossible, since the complement of the diagonal is a dense open set. Now $M[G] \models \neg CH$ follows from the obvious inequalities:

$$|\Phi(\omega_1)| = \omega_1 < \omega_2 = |\Phi(\omega_2)| \leqslant 2^\omega. \ \triangleright$$

3.4. Solovay's forcing. Random numbers. There is another proof of independence of CH, that starts from the same Boolean algebra of forcing conditions $\mathcal{T}(\Gamma)$. For any set Γ, the product measure μ is defined on the Boolean algebra $\mathcal{T}\ (\Gamma)$ of clopen sets of the compact space $\{0,1\}^\Gamma$ by the formula $\mu(O_p) = 2^{-|\operatorname{dom} p|}$. This measure is extended to a σ-additive measure on the σ-algebra $\mathcal{B}$ of all Borel sets. Let $\mathcal{B}_\mu(\Gamma) = \mathcal{B}/\{A \in \mathcal{B} : \mu(A) = 0\}$. This is a complete Boolean algebra with strictly positive completely additive measure that satisfies countable chain condition. The algebra $\mathcal{B}_\mu(\Gamma)$ is a completion of $\mathcal{T}(\Gamma)$ with respect to the metric $\rho(A_1, A_2) = \mu(A_1 \Delta A_2)$. This completion is not isomorphic to the Dedekind–MacNeille completion and we cannot use the algebra $\mathcal{T}(\Gamma)$ for the set of forcing conditions, since $\mathcal{T}(\Gamma)$ is not orderly dense in $\mathcal{B}_\mu(\Gamma)$. In this case we must use the set $\mathcal{B}_\mu(\Gamma) \setminus \{\mathbf{0}\}$ itself as the set of forcing conditions. In order to do so, we have to make sure that this set is absolute. This is true for the case of $|\Gamma| = \aleph_0$. So, as above put $\Gamma = \mathbb{N}$. Since $\mathcal{T}(\mathbb{N})$ is countable and is defined by an absolute formula, the set of all Borel subsets of $\{0,1\}^{\mathbb{N}}$ has cardinality $2^{\aleph_0}$. So, that they can be coded by elements of $\{0,1\}^{\mathbb{N}}$. Moreover, this coding is absolute. If A_β is a Borel set coded by $\beta \in \{0,1\}^{\mathbb{N}}$, then $\alpha \in A_\beta$, $\alpha \notin \mathbb{A}_\beta$, $A_\alpha \subseteq A_\beta$, $\mu(A_\beta) = 0$, etc. are absolute. See Section II of [19] for details. The crucial role for all absoluteness proofs of theorems about Borel coding plays the following.

Lemma 2 (Shoenfield)**.** *Every formula ZF of the form $\exists\, y\, \forall\, x\, \varphi(x, y, c)$ such that x and c range over $\mathbb{N}^{\mathbb{N}}$ (equivalently over $\{0,1\}^{\mathbb{N}}$) and all quantifiers in φ are of the form $\forall\, n \in \mathbb{N}$ or $\exists\, n \in \mathbb{N}$ is absolute with respect to any transitive model.*

Let M be the same model as above and $D \setminus \{\mathbf{0}\}$ is the set of forcing conditions, where $D \in M$ is an arbitrary complete Boolean algebra. Then $G \subseteq D$ is an M-generic set if and only if G is an ultrafilter on D and for any set $E \subseteq G,\ E \in M$ on has $\bigwedge E \in M$ [21]. An M-generic set in a complete Boolean algebra is said to be an M-generic filter.

The analog of M-generic elements (functions, sets, numbers) of the previous subsection are M-random elements. Consider the algebra $\mathcal{B}_\mu$. Given a Borel set $A_\beta \subseteq \{0,1\}^{\mathbb{N}}$, denote by $[A_\beta]$ its equivalence class.

Definition 13. We say that a function $f \in \{0,1\}^{\mathbb{N}}$ is M-random if $f \in A_\beta,\ \beta \in M$ such that $[A_\beta] = \mathbf{1}$ (i. e. $\mu(A_\beta) = 1$). In other words f is M-random, if it avoids every Borel set of measure zero in M.

Proposition 5. *If f is M-random, if and only if $\{[A_\beta] : \beta \in M,\ f \in A_\beta\}$ is an M-generic filter.*

Corollary 3. *If G is an M-generic filter in $\mathcal{B}_\mu$, then almost all elements of $\{0,1\}^{\mathbb{N}}$ in $M[G]$ are M-random.*

Let $\mathcal{B}_\mu(\kappa) = \prod_{\lambda<\kappa} \mathcal{B}_\mu^\lambda$, where κ is a cardinal. Corollary 3 is true for $\kappa = \omega_0$. This is obvious, since $\omega_0^2 = \omega_0$, and the results about coding the of Borel sets are true for this case.

Corollary is not true for the Boolean algebra $\mathcal{B}_\mu(\kappa)$ with uncountable κ, though it also satisfies the countable chains condition. The proof of the fact that $M[G] \models \neg CH$, where G is an M-generic filter in $\mathbb{B}_\mu(\omega_3)$ repeats the proof of this result for algebra 14 almost without changes.

3.5. Solovay's results. The most impressive independence results after P. Cohen were obtained by R. Solovay [19] and by R. Solovay and S. Tennenbaum [24]. In the first of them the following two theorems were proved.

Theorem 8 [19, Theorem 2]**.** *If the existence of an inaccessible cardinal is consistent with ZFC, then there exists a model of ZF+DC,* in which every set of reals is Lebesgue measurable, has the Baire property, and either is at most countable or contains a perfect subset.*

This theorem follows from

Theorem 9 [19, Theorem 1]**.** *If the existence of an inaccessible cardinal is consistent with ZFC, then there exists a model of ZFC, in which all three statements of the previous Theorem hold for the class of sets definable from a sequence of ordinals.*

The class of sets definable from a sequence of ordinals is very big and important. For example it includes all projective sets. There were two long standing problems in descriptive set theory. About a century ago Suslin constructed an example of a set that is a continuous image of a Borel set, but is not a Borel set. He called such sets A-sets.** P. S. Aleksandroff proved that every uncountable Borel set contains a perfect subset and, thus, has the cardinality of continuum. This implied that every A-set has a perfect subset. In attempts to prove CH N. N. Luzin the teacher of Aleksandroff and Suslin suggested to study the hierarchy: A-sets, their complements (CA-sets), the continuous images of CA-sets (PCA-sets), etc. Luzin showed that at each stage of this hierarchy some new sets appear. The sets of this hierarchy are called projective sets. They are studied in descriptive set theory. The first questions about projective set were about cardinality, Lebesgue measurability and Baire property of these sets. The difficulties started at the very first steps stages. It was known that A-sets and, thus, CA-sets are Lebesgue measurable, but the problem of cardinality of CA-sets (does each uncountable set contain perfect subsets?) and the problem of measurability of PCA-sets remained open. In [20] Gödel announced that $V = L$ implies the existence of an uncountable CA-set without a perfect subset and of a nonmeasurable PCA-set. He did not publish a proof of these statements. They were proved later by P. S. Novikov. Thus, from the results of [19, 25] followed the independence of the problems of cardinality and Lebesgue measurability for all projective sets followed, modulo, of course, the hypothesis of an inaccessible cardinal. That is why the following problem formulated in [19] was so important.

*Is it possible to eliminate the assumption about inaccessible cardinals from the statements of this theorem, concerning the Baire property and Lebesgue measurability?****

This problem was solved by S. Shelah [26] who proved that the elimination is possible for the Baire property but impossible for the Lebesgue measurability.

In his proof Solovay used the model $M[G]$, where G is an M-generic filter in the $(\aleph_0, \Omega)$-algebra Lévy (see Model VI in Chapter 20 of [21]), where $M \models \Omega$ is an inaccessible cardinal. This algebra is the Dedekind–MacNeille completion of the following absolute set of forcing conditions

$$P = \{p : p \subseteq \{\langle \alpha, n, \beta \rangle : \beta, \alpha < \Omega\rangle,\ |p| < \omega_0\}\}.$$

* DC—the axiom of dependent choice.

** Now they are called Suslin's sets.

*** The impossibility of removing this assumption for the statement concerning perfect subsets follows from the earlier paper of R. Solovay [25].

It is easy to see that, if $G \cap P$ is an M-generic set, then $\bigcup G$ consists of bijections of ω_0 on each infinite ordinal $\alpha \in \Omega$. Thus, $M[G] \models \Omega = \aleph_1$.

The following proposition is needed in the future discussion.

Proposition 6. *Let $s \in M[G]$ be a countable sequence of ordinals. Then the following hold:*

1) *Almost all real numbers in $M[G]$ are $M[s]$-random.*

2) *There exists $M[s]$-generic set $H \subseteq P$ such that $M[G] = M[s][H]$.*

The inaccessible cardinal is crucial for the Item 1 of this Proposition.

Without the hypothesis of the existence of inaccessible cardinal a weaker version of the Solovay's Theorem 1 was proved by G. Saks [8].

Theorem 10. *The existence of an extension of the Lebesgue measure to an invariant σ-additive measure on all sets definable from countable sequences of ordinals is consistent with ZF+DC.*

Notice that the simplest and most known example of a non-measurable set—the Vitali set is non-measurable with respect to any extension of the Lebesgue measure that satisfies the conditions of Theorem 10.

The corresponding version of Solovay's Theorem 2 was not even formulated in [8] and it is not even clear whether it can be proved on the way used in [8].

This theorem was proved in my PhD thesis. The result was announced in the article [13]. The detailed proofs of more general versions of both Solovay's Theorems are published in my PhD thesis and in a preprint [27] deposited in the VINITI (All Union Institute of the Science and Technology Information):

Theorem 11. *Let α be an arbitrary ordinal definable in ZF. Denote by* $\mathrm{Base}(X, \beta)$ *and* $\mathrm{Ext}(X, \beta)$ *the statements*

1. *"X is a σ-compact group with the base of topology of cardinality β";*

2. *"In a σ-compact group X the left Haar measure can be extended to a left invariant σ-additive measure defined on all subsets of X definable by a β-sequence of ordinals"*

respectively. Then the following proposition is consistent with ZFC:

$$\forall X \forall \beta < \aleph_\alpha < |\mathbb{R}| \ \ (\mathrm{Base}(X, \beta) \longrightarrow \mathrm{Ext}(X, \beta))$$

Theorem 12. *Let α be an arbitrary ordinal definable in ZF. Denote by* $\mathrm{Base}(X, \beta)$ *and* $\mathrm{Ext}'(X, \beta)$ *the statements*

1. *"X is a σ-compact group with the base of topology of cardinality β";*

2. *"In a σ-compact group X the left Haar measure can be extended to a left invariant σ-additive measure on all subsets of X"*

respectively. Then the following proposition is consistent with ZF+AD+AC_β:

$$\forall X \forall \beta < \aleph_\alpha < |\mathbb{R}| \ \ (\mathrm{Base}(X, \beta) \longrightarrow \mathrm{Ext}(X, \beta))\,,$$

where AC_β is a the Axiom of Choice for a family of cardinality β.

Both Solovey's theorems almost automatically carry over to the case of Haar measures on locally compact separable groups. In the case of non-separable groups, some problems arise with the absolute coding of Borel sets, due to the fact that the Schonfield Absolute Lemma holds only for countable sequences of positive integers. In the proofs of Theorems 11 and 12 these difficulties are overcome.

The independence of Suslin's hypothesis and Martin's Axiom were proved in [24]. This paper was also very important for the Boolean valued analysis, since the iterated forcing and the technique of ascents and descents were introduced there.

Acknowledgments. I am very grateful to my supervisor in postgraduate studies V. A. Lyubetskiy, who taught me the method of forcing, to my friends and colleagues A. G. Kusraev and S. S. Kutateladze for more, than forty years of friendship and fruitful cooperation in the field of non-standard methods of analysis, as well as my friend and co-author of many papers L. Yu. Glebsky for his great help in preparation of this paper.

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Received 8 November, 2019

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Владикавказский математический журнал
2019, Том 21, Выпуск 4, С. 25–41

НЕКОТОРЫЕ ЗАМЕЧАНИЯ О НЕСТАНДАРТНЫХ МЕТОДАХ АНАЛИЗА. I

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Аннотация. В этой и последующей статьях обсуждаются два наиболее известных нестандартных метода математического анализа — инфинитезимальный анализ А. Робинсона и булевозначный анализ, затрагивается история их возникновения, общие черты и различия, приложения и перспективы. В этой статье содержится обзор инфинитизимального анализа и метода вынуждения. Изложение рассчитано на читателя знакомого лишь с самыми начальными понятиями математической логики — языком логики предикатов 1-го порядка и его интерпретациями. Желательно иметь также некоторое представление о формальных доказательствах и аксиоматике теории множеств Цермело — Френкеля. При изложении инфинитезимального анализа особое внимание уделяется формализации предложений обычной математики в языке первого порядка для суперструктуры. Изложение метода форсинга предваряется кратким обзором результата К. Гёделя о совместимости аксиомы выбора и гипотезы континуума с аксиоматикой Цермело — Френкеля. Следующая статья будет посвящена, булевозначным моделям и булевозначному анализу. Особое внимание будет уделено истории их возникновения.

Ключевые слова: булевозначный анализ, нестандартный анализ, метод вынуждения.

Mathematical Subject Classification (2010): 03C90, 46A40.

Образец цитирования: *Gordon E. I.* Some Remarks about Nonstandard Methods in Analysis. I // Владикавк. мат. журн.—2019.—Т. 21, № 4.—С. 25–41 (in English). DOI: 10.23671/VNC.2019.21.44619.

Владикавказский математический журнал
2019, Том 21, Выпуск 4, С. 42–55

УДК 517.98
DOI 10.23671/VNC.2019.21.44621

ЛЕКСИКОГРАФИЧЕСКИЕ СТРУКТУРЫ НА ВЕКТОРНЫХ ПРОСТРАНСТВАХ[#]

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Евгению Израильевичу Гордону
в связи с его 70-летием

Аннотация. Описаны основные свойства отношений архимедовой эквивалентности и мажорируемости в линейно упорядоченном векторном пространстве. Введено и исследовано понятие (пред)лексикографической структуры на векторном пространстве. Лексикографическая структура представляет собой двойственность между векторами и точками, посредством которой абстрактное упорядоченное векторное пространство реализуется в виде изоморфного ему пространства вещественных функций, снабженного лексикографическим порядком. Введены понятия функциональной и базисной лексикографической структуры. Уточнена взаимосвязь между упорядоченным векторным пространством и его функциональным лексикографическим представлением. Приведено новое доказательство теоремы об изоморфном вложении любого линейно упорядоченного векторного пространства в лексикографически упорядоченное пространство вещественных функций с вполне упорядоченными носителями. Получен критерий плотности максимального конуса относительно сильнейшей локально выпуклой топологии. Базисные максимальные конусы описаны в терминах множеств, состоящих из попарно неэквивалентных векторов. Охарактеризован класс векторных пространств, в которых существуют небазисные максимальные конусы.

Ключевые слова: максимальный конус, всюду плотный конус, линейно упорядоченное векторное пространство, архимедова эквивалентность, архимедова мажорируемость, лексикографический порядок, базис Гамеля, локально выпуклое пространство.

Mathematical Subject Classification (2010): 06F20, 46A03.

Образец цитирования: *Гутман А. Е., Емельяненков И. А.* Лексикографические структуры на векторных пространствах // Владикавк. мат. журн.—2019.—Т. 21, № 4.—С. 42–55. DOI: 10.23671/VNC.2019.21.44621.

1. Введение

Подмножество K векторного пространства над полем $\mathbb{R}$ вещественных чисел называется *конусом*, если $K + K \subset K$, $\alpha K \subset K$ для всех $\alpha \in \mathbb{R}^+$, где $\mathbb{R}^+ := \{\alpha \in \mathbb{R} : \alpha \geqslant 0\}$,

[#]Работа выполнена при поддержке программы фундаментальных научных исследований СО РАН № I.1.2, проект № 0314-2019-0005.

и $K \cap (-K) = \{0\}$. Иными словами, конус — это непустое множество, замкнутое относительно линейных комбинаций $\alpha_1 x_1 + \cdots + \alpha_n x_n$ с положительными коэффициентами α_i и содержащее не более одного вектора из каждой пары $x, -x$.

Понятие конуса тесно взаимосвязано с понятием *упорядоченного векторного пространства* — вещественного векторного пространства X, снабженного таким отношением порядка $\leqslant$, что для любых $x, y, z \in X$ и $\alpha \in \mathbb{R}^+$ из $x \leqslant y$ следует $x + z \leqslant y + z$ и $\alpha x \leqslant \alpha y$. А именно, если $(X, \leqslant)$ — упорядоченное векторное пространство, то множество $X^+ := \{x \in X : x \geqslant 0\}$ является конусом; и наоборот: если $K \subset X$ — конус и $x \leqslant_K y \Leftrightarrow y - x \in K$ $(x, y \in X)$, то $(X, \leqslant_K)$ — упорядоченное векторное пространство и $X^+ = K$ (см., например, [1, 3.2]).

Из классической теоремы Хана — Банаха непосредственно следует, что всякий конус, являющийся надграфиком сублинейного функционала, имеет опорную гиперплоскость. В этой связи было бы естественно ожидать, что любой конус в вещественном векторном пространстве X лежит в некотором полупространстве, т. е. во множестве вида $\{x \in X : f(x) \geqslant 0\}$, где $0 \neq f \in X^{\#}$. (Здесь и ниже $X^{\#}$ — векторное пространство линейных функционалов на X с поточечными линейными операциями.) Тем не менее, это не так. Например, если $K := \{x \in L^0(\mathbb{R}) : x \geqslant 0\}$ — конус положительных элементов пространства $X := L^0(\mathbb{R})$ классов эквивалентности измеримых по Лебегу вещественных функций, то не существует ни одного положительного на K линейного функционала $0 \neq f \in X^{\#}$ (см. [2, § 5]). В частности, K не лежит ни в каком полупространстве и, более того, не отделяется ни от одной точки X, т. е. является всюду плотным в самом сильном смысле — относительно сильнейшей локально выпуклой топологии (в которой все линейные функционалы непрерывны). Описанный конус $K \subset L^0(\mathbb{R})$ — «гигантский», но не максимальный, он может быть увеличен до еще большего конуса $K \subset \overline{K} \subset L^0(\mathbb{R})$.

Конус K в векторном пространстве X называется *максимальным*, если в X не существует другого конуса, содержащего K, т. е. K является максимальным элементом упорядоченного по включению множества всех конусов в X.

Следующие свойства конуса K в векторном пространстве X равносильны:

(a) *K — максимальный конус;*

(b) *$(X, \leqslant_K)$ — линейно упорядоченное векторное пространство, т. е. $(\forall\, x, y \in X)(x \leqslant_K y$ или $y \leqslant_K x)$;*

(c) *$(\forall\, x \in X)(x \in K$ или $-x \in K)$.*

С помощью леммы Цорна легко показать, что любой конус может быть увеличен до максимального. Более того, для любого конуса $K \subset X$ и любого выпуклого множества $C \subset X$, не пересекающегося с K, существует максимальный конус $\overline{K} \subset X$, который содержит K и не пересекается с C.

Примером максимального конуса служит следующее подмножество пространства $\mathbb{R}^{\mathbb{N}}_{\text{fin}}$ всех последовательностей $x \colon \mathbb{N} \to \mathbb{R}$ (здесь $\mathbb{N} = \{1, 2, \dots\}$ — множество натуральных чисел) с конечными носителями $[x] := \{n \in \mathbb{N} : x(n) \neq 0\}$:

$$\bigl\{x \in \mathbb{R}^{\mathbb{N}}_{\text{fin}} \backslash \{0\} : x(\max[x]) > 0\bigr\} \cup \{0\}.$$

Этот максимальный конус, состоящий из тех последовательностей $x \in \mathbb{R}^{\mathbb{N}}_{\text{fin}}$, чей последний ненулевой член $x(\max[x])$ положителен, всюду плотен в $\mathbb{R}^{\mathbb{N}}_{\text{fin}}$ относительно сильнейшей локально выпуклой топологии (см. [3, гл. II, § 5]). Аналогичный конус

$$\bigl\{x \in \mathbb{R}^{\mathbb{N}}_{\text{fin}} \backslash \{0\} : x(\min[x]) > 0\bigr\} \cup \{0\}$$

тоже максимален, но не является всюду плотным, поскольку содержится в полупространстве $\bigl\{x \in \mathbb{R}^{\mathbb{N}}_{\text{fin}} : x(1) \geqslant 0\bigr\}$.

Приведенные факты мотивируют постановку следующих задач.

(1) Выяснить, в каких случаях для максимального конуса K в векторном пространстве X существует линейно упорядоченный базис Гамеля $(B, \leqslant_B)$, обеспечивающий представление

$$K = \{x \in X \backslash \{0\} : x_B(\min[x_B]) > 0\} \cup \{0\},$$

где $[x_B] := \{b \in B : x_B(b) \neq 0\}$ — носитель семейства $x_B \in \mathbb{R}^B_{\mathrm{fin}}$ коэффициентов разложения $\sum_{b \in B} x_B(b)b$ вектора x по базису B. (Такой конус K будем называть *базисным*.)

(2) Охарактеризовать векторные пространства, в которых все максимальные конусы являются базисными. (Очевидно, к ним относятся любые конечномерные пространства.)

(3) Привести общие примеры и описать структуру любых, в том числе небазисных, максимальных конусов.

(4) Выяснить, при каких условиях максимальный конус не лежит ни в одном полупространстве, т. е. всюду плотен относительно сильнейшей локально выпуклой топологии.

Данная статья посвящена решению перечисленных задач.

В качестве решения задачи (1) предложен критерий базисности конуса в терминах дискретных множеств — совокупностей векторов, среди которых нет архимедово эквивалентных (см. 4.2–4.4).

Исчерпывающий ответ на вопрос (2) дает теорема 4.7, согласно которой небазисные максимальные конусы существуют в векторных пространствах несчетной размерности, и только в них.

Задача (3) тесно связана с теоремой Хана о вложении — одним из наиболее глубоких результатов теории упорядоченных групп, утверждающим, что всякая линейно упорядоченная группа изоморфно вкладывается в лексикографическое произведение действительных групп [4, гл. IV, теорема 16]. Известен и менее громоздкий аналог этой теоремы для случая линейно упорядоченных векторных пространств [5, теорема 3.1], который фактически дает ответ на вопрос (3). В данной статье мы, в частности, попытались переосмыслить и существенно упростить имеющиеся подходы, переведя их на язык лексикографических и предлексикографических структур (см. § 3). Доказательство основной леммы 3.9, идейно и технически близкое содержанию статьи [5], на наш взгляд, стало если не элементарным, то по меньшей мере значительно более доступным и коротким.

Описание порядка посредством лексикографической структуры позволяет получить очень простой ответ на вопрос (4): максимальный конус всюду плотен тогда и только тогда, когда среди соответствующих архимедовых классов нет наименьшего (см. 3.12).

2. Архимедова эквивалентность

Всюду в этом параграфе X — линейно упорядоченное векторное пространство над $\mathbb{R}$. Отношение порядка по умолчанию обозначается символом $\leqslant$. Модуль $|x|$ вектора $x \in X$, как обычно, полагается равным x при $x \geqslant 0$ и $-x$ при $x < 0$. Символ $\operatorname{lin} Y$ обозначает линейную оболочку подмножества $Y \subset X$. Мы также будем использовать упрощенную запись $\operatorname{lin}(x, Y) := \operatorname{lin}(\{x\} \cup Y)$.

2.1. Введем на множестве $X^{++} := X^{+} \backslash \{0\}$ отношение линейного предпорядка

$$x \preccurlyeq y \Leftrightarrow (\exists\, \alpha > 0)(x \leqslant \alpha y)$$

и соответствующие отношения строгого порядка и эквивалентности

$$x \prec y \Leftrightarrow y \not\preccurlyeq x,$$
$$x \sim y \Leftrightarrow x \preccurlyeq y \ \&\ y \preccurlyeq x.$$

Отношения $\preccurlyeq$ и $\sim$ называют *архимедовым мажорированием* и *архимедовой эквивалентностью.*

Подмножество $D \subset X$ назовем *дискретным*, если $D \subset X^{++}$ и элементы D попарно не эквивалентны: $(\forall\, d, e \in D)(d \neq e \ \Rightarrow\ d \nsim e)$. В любом линейно упорядоченном векторном пространстве X существует максимальное дискретное множество. Всякое такое множество является результатом выбора представителей в классах архимедовой эквивалетности, т. е. имеет вид $\{d_c : c \in X^{++}/{\sim}\}$, где $d_c \in c$ для каждого класса c.

2.2. Как легко видеть, для любых $x, y \in X^{++}$

$$x \prec y \ \Leftrightarrow\ (\forall\, \alpha > 0)(x < \alpha y) \ \Leftrightarrow\ (\forall\, \alpha \in \mathbb{R})(\forall\, \beta > 0)(\alpha x < \beta y). \tag{1}$$

Распространим отношение $\prec$ с $X^{++} \times X^{++}$ на $X \times X^{++}$, принимая (1) в качестве определения выражения $x \prec y$ для произвольных $x \in X$ и $y \in X^{++}$.

Для любого вектора $y \in X^{++}$ множество $\{x \in X : x \prec y\}$ является векторным подпространством X.

$\lhd$ Если $x_1, x_2 \prec y$ и $\alpha_1, \alpha_2 \in \mathbb{R}$, то для всех $\alpha > 0$ мы имеем $\alpha_1 x_1, \alpha_2 x_2 < \frac{\alpha}{2} y$ и, следовательно, $\alpha_1 x_1 + \alpha_2 x_2 < \alpha y$. $\rhd$

2.3. *Всякое дискретное множество линейно независимо.*

$\lhd$ Индукцией по $n \in \mathbb{N}$ покажем, что любое подмножество $E \subset D$ мощности $|E| = n$ линейно независимо. Случай $n = 1$ тривиален. Пусть все подмножества D мощности n линейно независимы и пусть $\sum_{i=1}^{n+1} \alpha_i e_i = 0$, где $e_i \in D$ попарно различны, $\alpha_i \in \mathbb{R}$. Не нарушая общности, можно считать, что $e_1, \ldots, e_n \prec e_{n+1}$. Тогда с учетом 2.2

$$\alpha_{n+1} e_{n+1} = \sum_{i=1}^{n} (-\alpha_i) e_i \prec e_{n+1}, \quad -\alpha_{n+1} e_{n+1} = \sum_{i=1}^{n} \alpha_i e_i \prec e_{n+1}$$

и поэтому $\alpha_{n+1} = 0$. Следовательно, $\sum_{i=1}^{n} \alpha_i e_i = 0$, а значит, $\alpha_1 = \cdots = \alpha_n = 0$ по предположению индукции. $\rhd$

2.4. Если $x, y \in X^{++}$ и $x \sim y$, то $A := \{\alpha > 0 : \alpha y < x\}$ и $B := \{\beta > 0 : x < \beta y\}$ — непустые ограниченные множества, причем точные границы $\sup A$ и $\inf B$ совпадают. Обозначим их общее значение символом $\frac{x}{y}$. Таким образом, если $x \sim y$, то $\frac{x}{y}$ — единственное число, удовлетворяющее условию

$$\big(\tfrac{x}{y} - \alpha\big) y < x < \big(\tfrac{x}{y} + \alpha\big) y \quad \text{для всех } \alpha > 0.$$

Лемма. *Пусть $x, y \in X^{++}$, $x \sim y$, $x_0 := \big|x - \frac{x}{y} y\big|$. Тогда $x_0 \prec y$ и $\operatorname{lin}\{x_0, y\} = \operatorname{lin}\{x, y\}$.*

$\lhd$ Для любого числа $\alpha > 0$ из $x < \big(\frac{x}{y} + \alpha\big) y$ следует $x - \frac{x}{y} y < \alpha y$, а из $\big(\frac{x}{y} - \alpha\big) y < x$ следует $\frac{x}{y} y - x < \alpha y$. Таким образом, $\big|x - \frac{x}{y} y\big| < \alpha y$ для всех $\alpha > 0$, т. е. $x_0 \prec y$. Равенство $\operatorname{lin}\{x_0, x\} = \operatorname{lin}\{x, y\}$ очевидно. $\rhd$

2.5. Лемма. *Пусть Y — конечное подмножество X^{++}. Если $x \in X^{++}$ и $x \notin \operatorname{lin} Y$, то существует такой вектор $\tilde{x} \in X^{++}$, что $\operatorname{lin}(\tilde{x}, Y) = \operatorname{lin}(x, Y)$ и $(\forall\, y \in Y)(\tilde{x} \nsim y)$.*

$\lhd$ Для каждого вектора $z \in Z := \{z \in X^{++} : \operatorname{lin}(z, Y) = \operatorname{lin}(x, Y)\}$ положим

$$y(z) := \begin{cases} \min\{y \in Y : z \sim y\}, & \text{если } (\exists\, y \in Y)(z \sim y); \\ 0, & \text{если } (\forall\, y \in Y)(z \nsim y). \end{cases}$$

Поскольку множество $\{y(z) : z \in Z\}$ конечно, существует такой вектор $\tilde{x} \in Z$, что

$$y(\tilde{x}) = \min\{y(z) : z \in Z\}.$$

Допустим, $y(\tilde{x}) \neq 0$. Тогда $y(\tilde{x}) \in Y$ и $\tilde{x} \sim y(\tilde{x})$. Согласно лемме 2.4 имеется такой вектор $\tilde{x}_0 \in X^+$, что $\tilde{x}_0 \prec y(\tilde{x})$ и $\operatorname{lin}\{\tilde{x}_0, y(\tilde{x})\} = \operatorname{lin}\{\tilde{x}, y(\tilde{x})\}$. Из соотношений $x \notin \operatorname{lin} Y$ и $\operatorname{lin}(\tilde{x}, Y) = \operatorname{lin}(x, Y)$ следует, что $\tilde{x}$ и $y(\tilde{x})$ линейно независимы, и поэтому $\tilde{x}_0 \neq 0$. С другой стороны,

$$\operatorname{lin}(\tilde{x}_0, Y) = \operatorname{lin}(\operatorname{lin}\{\tilde{x}_0, y(\tilde{x})\} \cup Y) = \operatorname{lin}(\operatorname{lin}\{\tilde{x}, y(\tilde{x})\} \cup Y) = \operatorname{lin}(\tilde{x}, Y) = \operatorname{lin}(x, Y),$$

а значит, $\tilde{x}_0 \in Z$ и, следовательно, $y(\tilde{x}) \leqslant y(\tilde{x}_0)$. Таким образом, $\tilde{x}_0 \prec y(\tilde{x}) \leqslant y(\tilde{x}_0) \sim \tilde{x}_0$. Полученное противоречие показывает, что $y(\tilde{x}) = 0$, т. е. $\tilde{x}$ — искомый вектор. ▷

3. Лексикографические структуры

3.1. Пусть S — произвольное множество. Для обозначения характеристической функции подмножества $T \subset S$ будем использовать символ 1_T. Символом $\mathbb{R}^S$ обозначается векторное пространство всех функций $x\colon S \to \mathbb{R}$ с поточечными линейными операциями. Если $x_1, \ldots, x_n \in \mathbb{R}^S$ и $\varphi(\alpha_1, \ldots, \alpha_n)$ — формальная запись какого-либо утверждения о числах, то

$$[\varphi(x_1, \ldots, x_n)] \,:=\, \big\{s \in S : \varphi\big(x_1(s), \ldots, x_n(s)\big)\big\}.$$

В частности, если $x, y \in \mathbb{R}^S$, то $[x \neq y] = \{s \in S : x(s) \neq y(s)\}$. *Носитель* $[x \neq 0]$ функции x условимся обозначать символом $[x]$.

Векторное подпространство $\mathbb{R}^S$, состоящие из функций с конечными носителями, обозначим через $\mathbb{R}^S_{\mathrm{fin}}$. Если $(S, \leqslant)$ — линейно упорядоченное множество, то $\mathbb{R}^S_{\mathrm{wo}}$ — векторное подпространство $\mathbb{R}^S$, состоящее из функций с вполне упорядоченными носителями, а $\mathbb{R}^S_{\min}$ — подмножество $\mathbb{R}^S$, состоящее из нулевой функции и всех функций $x \in \mathbb{R}^S$, носитель $[x]$ которых имеет наименьший элемент $\min[x]$. Очевидно,

$$\mathbb{R}^S_{\mathrm{fin}} \subset \mathbb{R}^S_{\mathrm{wo}} \subset \mathbb{R}^S_{\min} \subset \mathbb{R}^S.$$

Отметим, что подмножество $\mathbb{R}^S_{\min} \subset \mathbb{R}^S$ не всегда является векторным подпространством. Для $x \in \mathbb{R}^S_{\min}$ определим число $x(\min) \in \mathbb{R}$, полагая

$$x(\min) := \begin{cases} x(\min[x]), & \text{если } x \neq 0; \\ 0, & \text{если } x = 0. \end{cases}$$

3.2. Пусть X — векторное пространство над $\mathbb{R}$, S — произвольное множество. Функцию $\langle\cdot\,|\,\cdot\rangle\colon X \times S \to \mathbb{R}$ назовем *двойственностью*, если для любых $x, y \in X$, $s \in S$, $\alpha \in \mathbb{R}$

$$\begin{aligned} &\langle x + y \,|\, s\rangle = \langle x \,|\, s\rangle + \langle y \,|\, s\rangle, \\ &\langle \alpha x \,|\, s\rangle = \alpha\langle x \,|\, s\rangle, \\ &x \neq 0 \;\Rightarrow\; (\exists\, s \in S)\ \langle x \,|\, s\rangle \neq 0. \end{aligned}$$

Для $\langle\cdot\,|\,\cdot\rangle\colon X \times S \to \mathbb{R}$ рассмотрим функции

$$\langle x]\colon S \to \mathbb{R}, \quad [s\rangle\colon X \to \mathbb{R}, \quad \langle x](s) := [s\rangle(x) := \langle x \,|\, s\rangle \quad (x \in X,\ s \in S)$$

и положим

$$\langle X] := \{\langle x] : x \in X\}, \quad [S\rangle := \{[s\rangle : s \in S\}.$$

Как легко видеть, следующие свойства функции $\langle\cdot\,|\,\cdot\rangle\colon X \times S \to \mathbb{R}$ равносильны:

(a) $\langle\cdot\,|\,\cdot\rangle$ является двойственностью;

(b) $x \mapsto \langle x]$ — изоморфизм X на векторное подпространство $\langle X] \subset \mathbb{R}^S$;

(c) $[S\rangle \subset X^\#$ — множество функционалов, разделяющее точки X.

Носитель $\big[\langle x]\big]$ функции $\langle x]\colon S \to \mathbb{R}$ условимся записывать в виде $[x]$.

3.3. Пусть $(S, \leqslant)$ — линейно упорядоченное множество. Векторное подпространство $X \subset \mathbb{R}^S$ будем называть (*пред*)*лексикографическим*, если

(a) $X \subset \mathbb{R}^S_{\text{wo}}$ (соответственно, $X \subset \mathbb{R}^S_{\min}$);

(b) $(\forall\, s \in S)(\exists\, x \in X\backslash\{0\})(s = \min[x])$.

Примером (пред)лексикографического пространства служит любое подпространство $X \subset \mathbb{R}^S$, удовлетворяющее включениям $\mathbb{R}^S_{\text{fin}} \subset X \subset \mathbb{R}^S_{\text{wo}}$ (соответственно, $\mathbb{R}^S_{\text{fin}} \subset X \subset \mathbb{R}^S_{\min}$).

Пусть X — произвольное векторное пространство над $\mathbb{R}$. Тройку $(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)$ назовем (*пред*)*лексикографической структурой* на X, если $\leqslant_S$ — линейный порядок на S, $\langle\cdot\,|\,\cdot\rangle\colon X \times S \to \mathbb{R}$ — двойственность и $\langle X]$ — (пред)лексикографическое подпространство $\mathbb{R}^S$. Вместо $(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)$ условимся писать $(S, \leqslant_S)$, если из контекста ясно, о какой двойственности $\langle\cdot\,|\,\cdot\rangle$ идет речь.

3.4. *Если $(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)$ — предлексикографическая структура на векторном пространстве X, то $[s\rangle \neq [t\rangle$ при $s \neq t$ и $[S\rangle$ — линейно независимое подмножество $X^\#$.*

$\lhd$ Благодаря 3.3 (b) имеется такое семейство элементов $x_s \in X\backslash\{0\}$, что $s = \min[x_s]$ для всех $s \in S$.

Пусть $s, t \in S$, $s <_S t$. Положим $x := x_t$. Из $s <_S t = \min[x]$ следует $[s\rangle(x) = \langle x\,|\,s\rangle = 0$ и $[t\rangle(x) = \langle x\,|\,t\rangle \neq 0$, а значит, $[s\rangle \neq [t\rangle$.

Рассмотрим попарно различные точки $s_1, \dots, s_n \in S$, ненулевые числа $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ и покажем, что $f := \sum_{i=1}^n \alpha_i [s_i\rangle \neq 0$. Пусть для определенности $s_1 <_S \dots <_S s_n$. Положим $x := x_{s_n}$. Поскольку $s_1, \dots, s_{n-1} <_S s_n = \min[x]$, мы имеем $\langle x\,|\,s_1\rangle = \dots = \langle x\,|\,s_{n-1}\rangle = 0$, $\langle x\,|\,s_n\rangle \neq 0$. Следовательно, $f(x) = \sum_{i=1}^n \alpha_i \langle x\,|\,s_i\rangle = \alpha_n \langle x\,|\,s_n\rangle \neq 0$, а значит, $f \neq 0$. $\rhd$

3.5. Если $\leqslant_S$ — линейный порядок на S и X — (пред)лексикографическое подпространство $\mathbb{R}^S$, то (пред)лексикографическая структура $(S, \leqslant_S)$ на X с естественной двойственностью $\langle x\,|\,s\rangle = x(s)$ называется *функциональной*. Как легко видеть, с точностью до изоморфизма всякая (пред)лексикографическая структура является функциональной.

Примерами функциональных лексикографических структур служат

$$(S, \leqslant_{\text{wo}}) \text{ на } \mathbb{R}^S, \quad (S, \leqslant_S) \text{ на } \mathbb{R}^S_{\text{wo}}, \quad (S, \leqslant_S) \text{ на } \mathbb{R}^S_{\text{fin}},$$

где $\leqslant_{\text{wo}}$ — полное упорядочение S, $\leqslant_S$ — произвольный линейный порядок на S. Если $S = \big\{\pm\frac{1}{n} : n \in \mathbb{N}\big\}$, $\leqslant_S$ — стандартный числовой порядок на S и $X = \mathbb{R}^S_{\text{fin}} + \mathbb{R}1_S$, то $(S, \leqslant_S)$ — предлексикографическая, но не лексикографическая функциональная структура на X.

3.6. Пусть $(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)$ — (пред)лексикографическая структура на векторном пространстве X. Для всякого элемента $x \in X$ положим $\langle x\,|\min\rangle := \langle x](\min)$, т. е.

$$\langle x\,|\min\rangle = \begin{cases} \langle x\,|\,\min[x]\rangle, & \text{если } x \neq 0; \\ 0, & \text{если } x = 0. \end{cases}$$

Как легко видеть, множество

$$X(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)^+ := \{x \in X : \langle x\,|\min\rangle \geqslant 0\}$$

представляет собой максимальный конус в X. Соответствующее линейно упорядоченное векторное пространство $(X, \leqslant_X)$ с положительным конусом $(X, \leqslant_X)^+ = X(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)^+$ обозначим символом $X(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)$, а порядок $\leqslant_X$ назовем *(пред)лексикографическим порядком, наведенным структурой* $(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)$.

3.7. *Пусть $(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)$ — предлексикографическая структура на векторном пространстве X. Снабдим пространство X порядком, наведенным структурой $(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)$.*

(a) *Для любых $x, y \in X$*

$$x < y \;\Leftrightarrow\; \langle x\,|\,t\rangle < \langle y\,|\,t\rangle, \text{ где } t = \min\big[\langle x] \neq \langle y]\big] = \min\{s \in S : \langle x\,|\,s\rangle \neq \langle y\,|\,s\rangle\}.$$

(b) *Для любых $x, y \in X^{++}$*

$$x \preccurlyeq y \Leftrightarrow \min[x] \geqslant_S \min[y], \quad x \prec y \Leftrightarrow \min[x] >_S \min[y], \quad x \sim y \Leftrightarrow \min[x] = \min[y].$$

(c) *Для любого максимального дискретного множества $D \subset X$, снабженного порядком $d \leqslant_D e \;\Leftrightarrow\; e \leqslant d$, отображение $d \mapsto \min[d]$ является порядковым изоморфизмом между $(D, \leqslant_D)$ и $(S, \leqslant_S)$.*

3.8. Лемма. *Если $X = (X, \leqslant)$ — линейно упорядоченное векторное пространство и D — произвольное максимальное дискретное подмножество X, снабженное порядком $d \leqslant_D e \;\Leftrightarrow\; e \leqslant d$, то $(X, \leqslant) = X(D, \leqslant_D, \langle\cdot\,|\,\cdot\rangle)$ для некоторой предлексикографической структуры $(D, \leqslant_D, \langle\cdot\,|\,\cdot\rangle)$ на X, причем такой, что $\langle d] = 1_{\{d\}}$ для всех $d \in D$.*

$\lhd$ Рассмотрим произвольный элемент $d \in D$ и заметим, что $d \notin \operatorname{lin}(X_d \cup D_d)$, где $X_d := \{x \in X : x \prec d\}$, $D_d := \{e \in D : d \prec e\}$. Действительно, из 2.2 следует, что X_d — векторное подпространство X, а значит, в случае $d \in \operatorname{lin}(X_d \cup D_d)$ нашелся бы элемент $x \in X_d^{++}$, удовлетворяющий соотношению $d \in \operatorname{lin}(x, D_d)$, которое невозможно, так как множество $\{d, x\} \cup D_d$ дискретно и поэтому линейно независимо (см. 2.3).

Следовательно, для каждого $d \in D$ можно выбрать такой функционал $f_d \in X^{\#}$, что $f_d(d) = 1$, $f_d(x) = 0$ при $x \in X$, $x \prec d$ и $f_d(e) = 0$ при $e \in D$, $d \prec e$. Определим функцию $\langle\cdot\,|\,\cdot\rangle\colon X \times D \to \mathbb{R}$, полагая $\langle x\,|\,d\rangle := f_d(x)$. Очевидно, $\langle d] = 1_{\{d\}}$ для всех $d \in D$.

Пусть $x \in X^{++}$. Поскольку дискретное множество D максимально, $x \sim d$ для некоторого элемента $d \in D$. По лемме 2.4 мы имеем $x - \frac{x}{d}d \leqslant \big|x - \frac{x}{d}d\big| \prec d$, откуда $f_d(x - \frac{x}{d}d) = 0$ и поэтому $\langle x\,|\,d\rangle = \frac{x}{d} > 0$. Кроме того, если $e <_D d$, то $x \sim d \prec e$, а значит, $\langle x\,|\,e\rangle = f_e(x) = 0$. Таким образом, $d = \min[x]$ и $\langle x\,|\,\min[x]\rangle > 0$.

Из сказанного выше следует, что функция $\langle\cdot\,|\,\cdot\rangle$ является двойственностью, а $\langle X]$ — предлексикографическим пространством, так как $\mathbb{R}^S_{\mathrm{fin}} \subset \langle X] \subset \mathbb{R}^S_{\min}$. Кроме того, установленное включение $X^+ \subset X(D, \leqslant_D, \langle\cdot\,|\,\cdot\rangle)^+$ влечет равенство $X^+ = X(D, \leqslant_D, \langle\cdot\,|\,\cdot\rangle)^+$ ввиду максимальности конуса X^+. $\rhd$

3.9. Пусть $(S, \leqslant)$ — линейно упорядоченное множество. Для $x \in \mathbb{R}^S$ и $t \in S$ определим функцию $x\rceil_t \in \mathbb{R}^S$, полагая $x\rceil_t(s) := x(s)$ при $s < t$ и $x\rceil_t(s) := 0$ при $s \geqslant t$. Для $x, y \in \mathbb{R}^S$ будем говорить, что x *и* y *совпадают до* t, и писать $x =_t y$, если $(\forall\, s < t)\ x(s) = y(s)$, т. е. $x\rceil_t = y\rceil_t$.

Лемма. *Если $(S, \leqslant)$ — линейно упорядоченное множество, X — векторное подпространство $\mathbb{R}^S$ и $\mathbb{R}^S_{\mathrm{fin}} \subset X \subset \mathbb{R}^S_{\min}$, то существует линейный оператор $(x \mapsto x^*)\colon X \to \mathbb{R}^S$, обладающий следующими свойствами:*

(a) $\mathbb{R}^S_{\mathrm{fin}} \subset X^* \subset \mathbb{R}^S_{\mathrm{wo}}$, *где* $X^* := \{x^* : x \in X\}$;

(b) $x \mapsto x^*$ — *линейный и порядковый изоморфизм* $X(S, \leqslant)$ *на* $X^*(S, \leqslant)$;

(c) $x^* = x$ *для всех* $x \in \mathbb{R}^S_{\mathrm{fin}}$;

(d) $x^*\rceil_t \in X^*$ *для всех* $x \in X$ *и* $t \in S$.

$\lhd$ Пусть $(S, \leqslant)$ — линейно упорядоченное множество и Y — векторное подпространство $\mathbb{R}^S$, удовлетворяющее включениям $\mathbb{R}^S_{\mathrm{fin}} \subset Y \subset \mathbb{R}^S_{\mathrm{min}}$.

Как легко видеть, лемма Цорна применима к множеству пар (X, F), составленных из пространств $\mathbb{R}^S_{\mathrm{fin}} \subset X \subset Y$ и операторов $F\colon x \in X \mapsto x^* \in \mathbb{R}^S$, удовлетворяющих условиям (a)–(d), относительно порядка $(X_1, F_1) \leqslant (X_2, F_2) \Leftrightarrow (X_1 \subset X_2,\ F_1 = F_2|_{X_1})$. Следовательно, задача будет решена, если мы рассмотрим векторное подпространство $X \subset Y$, содержащее $\mathbb{R}^S_{\mathrm{fin}}$, оператор $F\colon x \in X \mapsto x^* \in \mathbb{R}^S$, удовлетворяющий (a)–(d), фиксируем вектор $y \in Y \backslash X$ и продолжим F на подпространство $X + \mathbb{R}y \subset Y$ с сохранением условий (a)–(d).

Снабдим Y, X и X^* порядками, наведенными функциональной структурой $(S, \leqslant)$.

Предварительно покажем, что для любых $x_1, x_2 \in X$ и $s \in S$

$$x_1 =_s x_2 \Leftrightarrow x_1^* =_s x_2^*. \tag{2}$$

Поскольку при $x_1 \neq x_2$ соотношение $x_1 =_s x_2$ равносильно $s \leqslant \min[x_1 \neq x_2]$, причем $[x_1 \neq x_2] = [\,|x_1 - x_2|\,]$, для обоснования (2) достаточно показать, что $\min[x] = \min[x^*]$ для всех $x \in X^{++}$. Действительно, благодаря 3.7 (b) для $x \in X^{++}$ и $s \in S$ мы имеем $\min[x] = s \Leftrightarrow x \sim 1_{\{s\}} \Leftrightarrow x^* \sim 1^*_{\{s\}} = 1_{\{s\}} \Leftrightarrow \min[x^*] = s$.

Рассмотрим следующее подмножество S:

$$T := \{\min[x \neq y] : x \in X\}.$$

Заметим, что T — начальный фрагмент S, т. е. $(\forall\, s \in S)(\forall\, t \in T)(s \leqslant t \Rightarrow s \in T)$. Действительно, если $x \in X$ и $s < t = \min[x \neq y]$, то $x' := x + 1_{\{s\}} \in X$ и $s = \min[x' \neq y] \in T$.

Для каждой точки $t \in T$ рассмотрим какой-либо элемент $x \in X$, удовлетворяющий равенству $t = \min[x \neq y]$, и положим $x_t := x - x(t)1_{\{t\}} + y(t)1_{\{t\}}$, $t' := \min[x_t \neq y]$. Тогда для всех $t \in T$ имеют место следующие соотношения:

$$x_t \in X, \quad t' \in T, \quad t < t', \quad x_t =_{t'} y.$$

Определим функцию $y^* \in \mathbb{R}^S$, полагая $y^*(t) := x_t^*(t)$ для $t \in T$ и $y^*(s) := 0$ для $s \in S \backslash T$. Покажем, что для всех $t \in T$

$$x_t^* =_{t'} y^*.$$

Действительно, пусть $s < t'$. Из $x_s =_{s'} y$ и $x_t =_{t'} y$ следует $x_s =_r x_t$, где $r := \min\{s', t'\}$, откуда согласно (2) вытекает совпадение $x_s^* =_r x_t^*$. Следовательно, $y^*(s) = x_s^*(s) = x_t^*(s)$, так как $s < r$.

Продолжим изоморфизм $F\colon X \to X^*$ до линейного оператора $\overline{F}\colon X + \mathbb{R}y \to \mathbb{R}^S$, полагая $\overline{F}(x + \alpha y) := x^* + \alpha y^*$, и покажем, что оператор $\overline{F}$ и его образ $X^* + \mathbb{R}y^*$ удовлетворяют условиям (a)–(d).

(a) Если $t \in [y^*]$, то $t \in T$, $t < t'$, $y^* =_{t'} x_t^*$ и поэтому $\{s \in [y^*] : s \leqslant t\} \subset [x_t^*]$ — вполне упорядоченное множество, так как $x_t^* \in X^* \subset \mathbb{R}^S_{\mathrm{wo}}$. Ввиду произвольности $t \in [y^*]$ отсюда вытекает вполне упорядоченность $[y^*]$. Из включений $y^* \in \mathbb{R}^S_{\mathrm{wo}}$ и $\mathbb{R}^S_{\mathrm{fin}} \subset X^* \subset \mathbb{R}^S_{\mathrm{wo}}$ следует, что $\mathbb{R}^S_{\mathrm{fin}} \subset X^* + \mathbb{R}y^* \subset \mathbb{R}^S_{\mathrm{wo}}$.

(b) Для обоснования инъективности оператора $\overline{F}\colon X + \mathbb{R}y \to X^* + \mathbb{R}y^*$ достаточно показать, что $y^* \notin X^*$. Допустим, $x^* = y^*$ для некоторого вектора $x \in X$. Поскольку $x \neq y$, мы можем рассмотреть точку $t := \min[x \neq y] \in T$. Тогда $t < t'$ и $x^* = y^* =_{t'} x_t^*$, откуда в силу (2) следует, что $x =_{t'} x_t$. С другой стороны, $x_t =_{t'} y$, а значит, $x =_{t'} y$ и поэтому $t' \leqslant \min[x \neq y] = t$ вопреки неравенству $t < t'$.

Докажем, что $\overline{F}$ сохраняет порядок. Элементарные выкладки показывают, что для этого достаточно обосновать импликации $x < y \Rightarrow x^* < y^*$ и $x > y \Rightarrow x^* > y^*$ для любых

$x \in X$. Ограничимся доказательством первой импликации, поскольку вторая устанавливается совершенно аналогично.

Итак, допустим, что $x \in X$, $x < y$, но $x^* > y^*$ (равенство $x^* = y^*$ уже исключено). Тогда $x^*(t) > y^*(t)$, где $t := \min[x^* \neq y^*]$. Если бы $t > [y^*]$, то с учетом (d) мы бы имели $y^* = y^*\rceil_t = x^*\rceil_t \in X^*$, что невозможно. Следовательно, $(\exists\, s \in S)\ t \leqslant s \in [y^*] \subset T$, а значит, $t \in T$ и мы располагаем вектором $x_t \in X$ и точкой $t' \in T$, для которых $t < t'$, $x_t =_{t'} y$, $x_t^* =_{t'} y^*$. Далее, $x_t^*(t) = y^*(t) < x^*(t)$, причем $x_t^* =_t y^* =_t x^*$, откуда следует, что $x_t^* < x^*$. Тогда $x_t < x$, поскольку F является порядковым изоморфизмом. Из неравенств $x_t < x < y$ последовательно выводим $0 < x - x_t < y - x_t$, $x - x_t \preccurlyeq y - x_t$, $\min[x_t \neq x] \geqslant \min[x_t \neq y] = t'$, $x_t =_{t'} x$, $x_t^* =_{t'} x^*$, $x_t^*(t) = x^*(t)$ и получаем противоречие с неравенством $x_t^*(t) < x^*(t)$.

Условие (c) сохраняется ввиду включения $\mathbb{R}^S_{\text{fin}} \subset X$.

(d) Если $t \in T$, то $y^*\rceil_t = x_t^*\rceil_t \in X^*$, а если $t \notin T$, то $y^*\rceil_t = y^*$. Следовательно, $(x^* + \alpha y^*)\rceil_t = x^*\rceil_t + \alpha y^*\rceil_t \in X^* + \mathbb{R}y^*$ для любых $x \in X$, $\alpha \in \mathbb{R}$, $t \in S$. ▷

3.10. Замечание. Предложенное нами доказательство леммы 3.9 не является оригинальным и фактически воспроизводит схему доказательства теоремы [5, 3.2]. Значительного упрощения по сравнению с выкладками, приведенными в [5], удается достичь за счет леммы 3.8, благодаря которой абстрактное упорядоченное векторное пространство заменяется его функциональной предлексикографической копией.

3.11. Следующее утверждение, вытекающее из 3.8 и 3.9, представляет собой переформулировку теоремы [5, 3.1], согласно которой во всяком линейно упорядоченном векторном пространстве порядок является лексикографическим.

Теорема. *Пусть $X = (X, \leqslant)$ — линейно упорядоченное векторное пространство. Рассмотрим произвольное максимальное дискретное множество $D \subset X$ и снабдим его порядком $\leqslant_D$, полагая $d \leqslant_D e \Leftrightarrow e \leqslant d$. Тогда $(X, \leqslant) = X(D, \leqslant_D, \langle\cdot\,|\,\cdot\rangle)$ для некоторой лексикографической структуры $(D, \leqslant_D, \langle\cdot\,|\,\cdot\rangle)$ на X, удовлетворяющей следующим дополнительным условиям: $\langle d] = 1_{\{d\}}$ и $\langle x]\rceil_d \in \langle X]$ для всех $x \in X$ и $d \in D$.*

3.12. Пусть X — векторное пространство. Напомним, что сильнейшей локально выпуклой топологией на X является топология Макки τ_X, согласованная с двойственностью $(X, X^{\#})$ (см., например, [6, 8-2-14; 1, 10.4.4]). Относительно этой топологии непрерывны все линейные функционалы. Выпуклое множество $C \subset X$ плотно в X относительно топологии τ_X тогда и только тогда, когда C не лежит ни в каком полупространстве $\{x \in X : f(x) \geqslant \alpha\}$, где $f \in X^{\#}\backslash\{0\}$, $\alpha \in \mathbb{R}$. Конус $K \subset X$ плотен в X относительно τ_X тогда и только тогда, когда нулевой функционал является единственным линейным функционалом, положительным на K: если $f \in X^{\#}$ и $f(x) \geqslant 0$ для всех $x \in K$, то $f = 0$.

Теорема. *Пусть $X = X(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)$, где $(S, \leqslant_S, \langle\cdot\,|\,\cdot\rangle)$ — предлексикографическая структура на X. Конус X^+ плотен в X относительно сильнейшей локально выпуклой топологии тогда и только тогда, когда в $(S, \leqslant_S)$ отсутствует наименьший элемент.*

◁ *Необходимость.* Если в S существует наименьший элемент s, то для всех $x \in X\backslash\{0\}$ из неравенства $s \leqslant_S \min[x]$ следует, что $\langle x\,|\,s\rangle = \langle x\,|\min[x]\rangle$ при $s = \min[x]$ и $\langle x\,|\,s\rangle = 0$ в остальных случаях, а значит, $\langle x\,|\,s\rangle \geqslant 0$ при $x > 0$, т. е. $[s\rangle$ — положительный на X^+ ненулевой линейный функционал (см. 3.4).

Достаточность. Пусть имеется ненулевой функционал $f \in X^{\#}$ такой, что $f(x) \geqslant 0$ для всех $x \in X^+$. Поскольку $f \neq 0$, существует вектор $x \in X^{++}$, для которого $f(x) > 0$. Покажем, что $\min[x] = \min S$.

Допустим вопреки доказываемому, что $s <_S \min[x]$ для некоторой точки $s \in S$. Согласно 3.3 (b) имеется вектор $y \in X^{++}$, для которого $\min[y] = s$. Положим $z := y - \frac{f(y)+1}{f(x)}x$.

Поскольку $\min[y] = s <_S \min[x]$, имеют место равенства $\min[z] = s$ и $\langle x\,|\,s\rangle = 0$. Следовательно, $\langle z\,|\min\rangle = \langle z\,|\,s\rangle = \langle y\,|\,s\rangle \geqslant 0$, т. е. $z \in X^+$ и поэтому $f(z) \geqslant 0$. С другой стороны, $f(z) = f(y) - \frac{f(y)+1}{f(x)} f(x) = -1$. $\rhd$

4. Базисные максимальные конусы

4.1. Если X — произвольное векторное пространство над $\mathbb{R}$ и B — базис Гамеля в X, то значение двойственности $\langle x\,|\,b\rangle$ по умолчанию определяется как коэффициент при $b \in B$ в разложении $\sum_{b\in B}\langle x\,|\,b\rangle b$ вектора $x \in X$ по базису B. В этом случае $(B, \leqslant_B)$ — лексикографическая структура на X для любого линейного порядка $\leqslant_B$ на B. Структуры такого вида будем называть *базисными*.

Максимальный конус в векторном пространстве X назовем *базисным*, если он соответствует базисной лексикографической структуре, т. е. имеет вид $X(B, \leqslant_B)^+$ для некоторого линейно упорядоченного базиса Гамеля $(B, \leqslant_B)$ пространства X.

4.2. Лемма. *Пусть B — базис Гамеля в линейно упорядоченном векторном пространстве X и пусть $\leqslant_B$ — линейный порядок на B. Следующие утверждения равносильны:*

(a) $X = X(B, \leqslant_B)$;

(b) *множество B дискретно и $b \leqslant_B c \Leftrightarrow b \geqslant c$ для всех $b, c \in B$.*

$\lhd$ (a)$\Rightarrow$(b). Пусть $X = X(B, \leqslant_B)$. Тогда для любых $b, c \in B$

$$\begin{aligned} b \leqslant_B c \Leftrightarrow \min_B\{b, c\} = b \Leftrightarrow \langle b - c\,|\min\rangle \geqslant 0 \\ \Leftrightarrow b - c \in X(B, \leqslant_B)^+ = X^+ \Leftrightarrow b \geqslant c. \end{aligned}$$

Если $b \in B$, то $\langle b|\min\rangle = \langle b|b\rangle = 1 \geqslant 0$ и, следовательно, $b \in X(B, \leqslant_B)^+ = X^+$. Рассмотрим произвольные $b, c \in B$, $b \neq c$. Предположим для определенности, что $b \leqslant_B c$. Если $b \sim c$ вопреки доказываемому, то имеется число $\alpha > 0$, для которого $b < \alpha c$. Тогда $\alpha c - b \in X^+ = X(B, \leqslant_B)^+$ и поэтому $\langle \alpha c - b|\min\rangle \geqslant 0$. С другой стороны,

$$\langle \alpha c - b|\min\rangle = \langle \alpha c - b|\min_B\{b, c\}\rangle = \langle \alpha c - b|\,b\rangle = -1.$$

(b)$\Rightarrow$(a). Чтобы установить включение $X^+ \subset X(B, \leqslant_B)^+$, рассмотрим произвольный элемент $x \in X^{++}$, положим $b := \min[x]$, $\alpha := \langle x\,|\,b\rangle$ и покажем, что $\alpha \geqslant 0$. Если множество $C := [x]\backslash\{b\}$ пусто, то $x = \alpha b$ и тогда $\alpha \geqslant 0$. Пусть теперь $C \neq \varnothing$. Предположим вопреки доказываемому, что $\alpha < 0$. Для каждого элемента $c \in C$ мы имеем $b <_B c$, откуда с учетом (b) вытекает $c \prec b$, т. е. $c < \beta b$ для всех $\beta > 0$. Тогда

$$x - \alpha b = \sum_{c\in C}\langle x\,|\,c\rangle c \leqslant \sum_{c\in C}\bigl|\langle x\,|\,c\rangle\bigr| c < \sum_{c\in C}\bigl|\langle x\,|\,c\rangle\bigr| \frac{-\alpha}{\bigl|\langle x\,|\,c\rangle\bigr|\,|C|} b = -\alpha b,$$

где $|C|$ — число элементов множества C. Следовательно, $x < 0$, что противоречит условию $x \in X^+$. Таким образом, $X^+ \subset X(B, \leqslant_B)^+$, а значит, $X^+ = X(B, \leqslant_B)^+$ в силу максимальности конуса X^+. $\rhd$

4.3. Следствие. *Максимальный конус K в векторном пространстве X является базисным тогда и только тогда, когда в $(X, \leqslant_K)$ существует дискретный базис Гамеля.*

4.4. Замечание. Максимальное дискретное (и поэтому линейно независимое) подмножество линейно упорядоченного векторного пространства X не обязано быть базисом Гамеля, даже если в X существует дискретный базис Гамеля. Примером служит максимальное дискретное множество $\{1_{\{n\}} : n \in \mathbb{N}\}$ в пространстве $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}} + \mathbb{R}1_{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$ с порядком, наведенным функциональной лексикографической структурой $(\mathbb{N}, \leqslant_{\mathbb{N}})$, где $\leqslant_{\mathbb{N}}$ — стандартный порядок на $\mathbb{N}$. Дискретным базисом Гамеля в данном случае является, например, множество $\{1_{[n,\infty)} : n \in \mathbb{N}\}$.

4.5. *Если $(S, \leqslant_S)$ — предлексикографическая структура мощности $|S|$ на векторном пространстве X и $|S| < \dim X$, то в $X(S, \leqslant_S)$ не существует дискретного базиса Гамеля.*

$\lhd$ Никакое дискретное множество $D \subset X(S, \leqslant_S)$ не может быть базисом Гамеля в X, так как согласно 3.7 (b) отображение $d \in D \mapsto \min[d] \in S$ инъективно и, следовательно, $|D| \leqslant |S| < \dim X$. $\rhd$

В частности, для любого бесконечного вполне упорядоченного множества $(S, \leqslant_S)$ пространство $\mathbb{R}^S(S, \leqslant_S)$ не имеет дискретного базиса Гамеля и поэтому $\mathbb{R}^S(S, \leqslant_S)^+$ служит примером максимального конуса в $\mathbb{R}^S$, не являющегося базисным (см. 4.3).

4.6. Замечание. Поскольку для всякой предлексикографической структуры $(S, \leqslant_S)$ на X имеет место неравенство $\dim X \leqslant \dim \mathbb{R}^S$ (см. 3.2 (b)), из 4.5 следует, что небазисный максимальный конус существует в любом пространстве, размерность λ которого удовлетворяет условию $\kappa < \lambda \leqslant 2^\kappa$ для какого-либо кардинала κ. Этим свойством обладают кардиналы λ, не являющиеся строго предельными. Напомним, что кардинал λ называется *строго предельным*, если для любого кардинала κ из $\kappa < \lambda$ следует $2^\kappa < \lambda$ (см. [7, § 5]). Наименьшим строго предельным кардиналом является $\aleph_0 = |\mathbb{N}|$. Если κ_1 — произвольный кардинал и $\kappa_{n+1} = 2^{\kappa_n}$ $(n \in \mathbb{N})$, то $\sup\{\kappa_n : n \in \mathbb{N}\}$ — строго предельный кардинал, откуда следует, что строго предельные кардиналы образуют собственный класс.

Таким образом, согласно 4.5 небазисный максимальный конус существует в любом пространстве, размерность которого бесконечна и не является строго предельным кардиналом. Тем не менее это наблюдение не имеет особой ценности, поскольку, как показывает приведенная ниже теорема, небазисный максимальный конус существует в любом пространстве несчетной размерности.

4.7. Теорема. *В векторном пространстве X все максимальные конусы являются базисными тогда и только тогда, когда X имеет конечную или счетную размерность.*

$\lhd$ *Необходимость.* Пусть размерность $\dim X$ векторного пространства X бесконечна и не счетна. Покажем, что в X имеется максимальный конус, не являющийся базисным.

Если $|\mathbb{N}| < \dim X \leqslant |\mathbb{R}|$, то нужный нам факт содержится в 4.6. Пусть $\dim X > |\mathbb{R}|$. Рассмотрим вполне упорядоченное множество $(S, \leqslant_S)$ и два его подмножества $M, N \subset S$ такие, что

$$S = M \cup N,$$
$$M <_S N \text{ (т. е. } m <_S n \text{ для всех } m \in M \text{ и } n \in N),$$
$$|M| = \dim X,$$
$$N \text{ порядково изоморфно } \mathbb{N}.$$

(В качестве S можно взять ординал $\dim X + \omega$ и положить $M := \dim X$, $N := S \backslash M$.) Определим векторные подпространства $Y, Z \subset \mathbb{R}^S$, полагая

$$Y := \{y \in \mathbb{R}^S : [y] \subset N\}, \quad Z := \{z \in \mathbb{R}^S_{\mathrm{fin}} : [z] \subset M\}.$$

Поскольку

$$\dim Y = \dim \mathbb{R}^N = \dim \mathbb{R}^{\mathbb{N}} = |\mathbb{R}|, \quad \dim Z = \dim \mathbb{R}^M_{\mathrm{fin}} = |M| = \dim X > |\mathbb{R}|,$$

размерность суммы $Y+Z\subset\mathbb{R}^S$ совпадает с размерностью X, а значит, можно считать, что $X=Y+Z$. Снабдим X порядком, наведенным функциональной лексикографической структурой $(S,\leqslant_S)$, и покажем, что максимальный конус X^+ не является базисным.

Пусть вопреки доказываемому в X существует дискретный базис гамеля B (см. 4.3). Покажем, что $\operatorname{lin}(B\cap Y)=Y$, для чего возьмем произвольный элемент $y\in Y\backslash\{0\}$ и установим включение $y\in\operatorname{lin}(B\cap Y)$. Рассмотрим разложение $y=\sum_{i=1}^n\alpha_i b_i$, где $b_1,\dots,b_n\in B$ и $\alpha_1,\dots,\alpha_n\in\mathbb{R}\backslash\{0\}$. Учитывая дискретность множества B, можно считать, что $b_1\prec\dots\prec b_n$, т. е. $\min[b_1]>_S\dots>_S\min[b_n]$ (см. 3.7 (b)). Тогда $\min[y]=\min[b_n]$ и поэтому $\min[b_n]\in N$, так как $[y]\subset N$ в силу включения $y\in Y$. Поскольку N — финальный фрагмент S, мы имеем $\min[b_1],\dots,\min[b_n]\in N$ и $[b_1],\dots,[b_n]\subset N$, т. е. $b_1,\dots,b_n\in Y$. Следовательно, $y=\sum_{i=1}^n\alpha_i b_i\in\operatorname{lin}(B\cap Y)$. Таким образом, $\operatorname{lin}(B\cap Y)=Y$, а значит, $B\cap Y$ — дискретный базис Гамеля в Y, что противоречит 4.5, так как пространство Y изоморфно $\mathbb{R}^{\mathbb{N}}(\mathbb{N},\leqslant_{\mathbb{N}})$.

Достаточность установлена, например, в [4, гл. IV, теорема 19]. Мы приведем здесь элементарное доказательство, не задействующее специфические конструкции и факты из теории групп.

Пусть X — векторное пространство размерности $|N|$, где $N=\{1,\dots,m\}$ или $N=\mathbb{N}$, и пусть K — максимальный конус в X. Снабдим X линейным векторным порядком $\leqslant_K$ и рассмотрим произвольный базис Гамеля $(x_n)_{n\in N}\subset X^+$. Построим последовательность $(y_n)_{n\in N}\subset X^{++}$, удовлетворяющую условию

$$(\forall\, i,j\in\{1,\dots,n\})(i\neq j\ \Rightarrow\ y_i\nsim y_j),\quad \operatorname{lin}\{y_1,\dots,y_n\}=\operatorname{lin}\{x_1,\dots,x_n\}$$

для всех $n\in N$, с помощью следующей рекурсивной процедуры: положим $y_1:=x_1$ и выберем в качестве y_{n+1} произвольный вектор $\tilde{x}$, существование которого утверждается в лемме 2.5 для $Y:=\{y_1,\dots,y_n\}$ и $x:=x_{n+1}$. Как легко видеть, $\{y_n:n\in N\}$ — дискретный базис Гамеля в пространстве $(X,\leqslant_K)$, а значит, K — базисный конус согласно 4.3. $\rhd$

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Статья поступила 10 июля 2019 г.

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Vladikavkaz Mathematical Journal
2019, Volume 21, Issue 4, P. 42–55

LEXICOGRAPHIC STRUCTURES ON VECTOR SPACES

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Abstract. Basic properties are described of the Archimedean equivalence and dominance in a totally ordered vector space. The notion of (pre)lexicographic structure on a vector space is introduced and studied. A lexicographic structure is a duality between vectors and points which makes it possible to represent an abstract ordered vector space as an isomorphic space of real-valued functions endowed with a lexicographic order. The notions of function and basic lexicographic structures are introduced. Interrelations are clarified between an ordered vector space and its function lexicographic representation. A new proof is presented for the theorem on isomorphic embedding of a totally ordered vector space into a lexicographically ordered space of real-valued functions with well-ordered supports. A criterion is obtained for denseness of a maximal cone with respect to the strongest locally convex topology. Basic maximal cones are described in terms of sets constituted by pairwise nonequivalent vectors. The class is characterized of vector spaces in which there exist nonbasic maximal cones.

Key words: maximal cone, dense cone, totally ordered vector space, Archimedean equivalence, Archimedean dominance, lexicographic order, Hamel basis, locally convex space.

Mathematical Subject Classification (2010): 06F20, 46A03.

For citation: *Gutman, A. E. and Emelyanenkov, I. A.* Lexicographic Structures on Vector Spaces, *Vladikavkaz Math. J.*, 2019, vol. 21, no. 4, pp. 42–55 (in Russian). DOI: 10.23671/VNC.2019.21.44621.

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Received July 10, 2019

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Владикавказский математический журнал
2019, Том 21, Выпуск 4, С. 56–62

УДК 510.898, 517.98
DOI 10.23671/VNC.2019.21.44624

UNBOUNDED ORDER CONVERGENCE AND THE GORDON THEOREM#

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Dedicated to Professor E. I. Gordon
on occasion of his 70th birthday

Abstract. The celebrated Gordon's theorem is a natural tool for dealing with universal completions of Archimedean vector lattices. Gordon's theorem allows us to clarify some recent results on unbounded order convergence. Applying the Gordon theorem, we demonstrate several facts on order convergence of sequences in Archimedean vector lattices. We present an elementary Boolean-Valued proof of the Gao–Grobler–Troitsky–Xanthos theorem saying that a sequence x_n in an Archimedean vector lattice X is *uo*-null (*uo*-Cauchy) in X if and only if x_n is *o*-null (*o*-convergent) in X^u. We also give elementary proof of the theorem, which is a result of contributions of several authors, saying that an Archimedean vector lattice is sequentially *uo*-complete if and only if it is σ-universally complete. Furthermore, we provide a comprehensive solution to Azouzi's problem on characterization of an Archimedean vector lattice in which every *uo*-Cauchy net is *o*-convergent in its universal completion.

Key words: unbounded order convergence, universally complete vector lattice, Boolean valued analysis.
Mathematical Subject Classification (2010): 03H05, 46S20, 46A40.
For citation: *Emelyanov, E. Y., Gorokhova, S. G., Kutateladze, S. S.* Unbounded Order Convergence and the Gordon Theorem, *Vladikavkaz Math. J.*, 2019, vol. 21, no. 4, pp. 56–62. DOI: 10.23671/VNC.2019.21.44624.

1. Introduction

Throughout the paper, we let X stand for a vector lattice, and all vector lattices are assumed to be real and Archimedean. We refer to [1, 2] for the unexplained terminology and facts on vector lattices and start with recalling some definitions and results. A vector lattice X is said to be *Dedekind* (*σ-Dedekind*) *complete* if each nonempty order bounded (countable)

#The research was partially supported by the Science Support Foundation Program of the Siberian Branch of the Russian Academy of Sciences; № I.1.2, Project № 0314-2019-0005.

subset of X has a supremum. A Dedekind complete (σ-Dedekind complete) vector lattice X is said to be *universally* (σ*-universally*) *complete* if each nonempty pairwise disjoint (countable) subset of X_+ has a supremum. Clearly, each universally complete vector lattice has a weak unit. It is well known that X possesses Dedekind and universal completions unique up to lattice isomorphism which are denoted by X^δ and X^u. We will always suppose that $X \subseteq X^\delta \subseteq X^u$, whereas X^δ is an ideal in X^u.

A sublattice Y of X is said to be *regular* if $y_\alpha \downarrow 0$ in Y implies $y_\alpha \downarrow 0$ in X; while Y is *order dense* in X if for every $0 \neq x \in X_+$ there exists $y \in Y$ satisfying $0 < y \leqslant x$. Obviously, the ideals and order dense sublattices are regular. In what follows, we will freely use the regularity of X in X^u. Note also that X is *atomic* iff X is lattice isomorphic to an order dense sublattice of $\mathbb{R}^C$ (cf. [1, Theorem 1.78]).

A net $(x_\alpha)_{\alpha\in A}$ in X o-converges to x if there exists a net $(z_\gamma)_{\gamma\in\Gamma}$ in X satisfying $z_\gamma \downarrow 0$ and, for each $\gamma \in \Gamma$, there is $\alpha_\gamma \in A$ with $|x_\alpha - x| \leqslant z_\gamma$ for all $\alpha \geqslant \alpha_\gamma$. In this case we write $x_\alpha \xrightarrow{o} x$. This definition is used for instance in [2, 3]. Sometimes (in particular, see [1, 4, 5]) the slightly different definition of o-convergence appears: $(x_\alpha)_{\alpha\in A}$ o-converges to $x \in X$ if there is a net $(z_\alpha)_{\alpha\in A}$ such that $z_\alpha \downarrow 0$ and $|x_\alpha - x| \leqslant z_\alpha$ for all α. These two definitions agree in the case of order bounded nets in Dedekind complete vector lattices (cf. [3, Remark 2.2]). The article [6] contains a more details discussion of the definitions of o-convergence. By [7, Theorem 1] (cf. also [8, Theorem 2]), o-convergence in X is topological iff X is finite dimensional.

A net x_α in X is said to be *uo-convergent* to x if $|x_\alpha - x| \wedge y \xrightarrow{o} 0$ for every $y \in X_+$. We write $x_\alpha \xrightarrow{uo} x$. Following Nakano [9], uo-convergence is investigated as a generalization of almost everywhere convergence (see [3, 4, 10–18] and references therein). Note that o-convergence agrees with eventually order bounded uo-convergence. Furthermore, uo-convergence passes freely between X, X^δ, and X^u [3, Theorem 3.2]. It was shown in [3, Corollary 3.5] that if e is a weak unit of X then $x_\alpha \xrightarrow{uo} x \Leftrightarrow |x_\alpha - x| \wedge e \xrightarrow{o} 0$. By [3, Corollary 3.12] every uo-null sequence in X is o-null in X^u. This is untrue for arbitrary nets. By Theorem 4 below, or independently, by [18, Proposition 15.2], all uo-null nets in X are o-null in X^u if only if $\dim(X) < \infty$. Although uo-convergence is not topological in many important cases (e. g., in $L_1[0,1]$ and in $C[0,1]$), it is topological in atomic vector lattices; see [7, Theorem 2].

A net x_α is said to be *o-Cauchy* (*uo-Cauchy*) if the double net $(x_\alpha - x_\beta)_{(\alpha,\beta)}$ o-converges (uo-converges) to 0. Clearly, every o-Cauchy net is uo-Cauchy. In a Dedekind complete vector lattice with a weak unit e, a net x_α is uo-Cauchy iff $\inf_\alpha \sup_{\beta,\gamma\geqslant\alpha} |x_\beta - x_\gamma| \wedge e = 0$ [13, Lemma 2.7]. It is well known that completeness with respect to o-convergence is equivalent to Dedekind completeness. By [3, Corollary 3.12], a sequence in X is *uo-Cauchy* in X iff it is o-convergent in X^u. As showed in Theorem 4, there is no net-version of the latter fact unless X is finite-dimensional. It was proved in [16, Theorem 3.9] (see also [15, Theorem 28]) that X is σ-universally complete iff X is sequentially uo-complete. In [15, Theorem 17], it was demonstrated that uo-completeness is equivalent to universal completeness. Thus, there is no need in any special investigation of (sequential) uo-completion.

The (always complete) Boolean algebra $\mathfrak{B}(X)$ of all bands of X is called the *base* of X. If X has the projection property (e.g., if X is Dedekind complete), then $\mathfrak{B}(X)$ can be identified with the Boolean algebra $\mathfrak{P}(X)$ of all band projections in X and, if X has also a weak unit e, both $\mathfrak{B}(X)$ and $\mathfrak{P}(X)$ can be identified with the Boolean algebra $\mathfrak{C}(e)$ of all fragments of e (cf. [2, Theorem 1.3.7 (1)]).

2. Boolean-Valued Analysis and Unbounded Order Convergence

The classical Gordon's discovery [19, Theorem 2] (expressing the immanent connection between vector lattices and Boolean-valued analysis) reads shortly as follows: *Each universally* *complete vector lattice is an interpretation of the reals $\mathcal{R}$ in an appropriate Boolean-valued model* $V^{(B)}$. Furthermore, each Archimedean vector lattice is an order dense ideal of the descent of $\mathcal{R}$ within $V^{(B)}$. These facts are combined as follows (see [2, Theorems 8.1.2 and 8.1.6]):

Theorem 1 (Gordon's Theorem)**.** *Let X be an Archimedean vector lattice, while $B = \mathfrak{B}(X)$ and $\mathcal{R}$ is the reals in the Boolean-valued model $V^{(B)}$. Then $\mathcal{R}\downarrow$ is a universally complete vector lattice including X as an order dense sublattice. Moreover,*

$$bx \leqslant by \iff b \leqslant [\![x \leqslant y]\!] \quad (\forall\, b \in B);(\forall\, x, y \in \mathcal{R}\downarrow).$$

By the Gordon Theorem, the universal completion X^u of an Archimedean vector lattice X is the descent $\mathcal{R}\downarrow$ of the reals $\mathcal{R}$ in $V^{(\mathfrak{B}(X))}$, and the uniqueness of X^u up to an order isomorphism follows from the uniqueness of $\mathcal{R}$ in $V^{(\mathfrak{B}(X))}$ (see [2, 8.1.7]).

In [20] Kantorovich introduced Dedekind complete vector lattices and propounded his famous Heuristic Transfer Principle: *The members of every Dedekind complete vector lattice are generalized reals* (see [5] for further historical notes). This Kantorovich's motto was justified by the Gordon Theorem [19] published 42 years later in the same journal. The aim of the present paper, published another 42 years after [19], is to provide another illustration of the fruitfulness of the Gordon Theorem in exploring the theory of *uo*-convergence. To some extent, Archimedean vector lattices are commonly presented in the repertoire of the *Boolean-valued orchestra*, where the musicians are *complete Boolean algebras* and the orchestra director is the *reals*. To our knowledge, the present paper is a first attempt to apply Theorem 1 to *uo*-convergence. For the unexplained terminology and techniques of Boolean-valued analysis we refer the reader to [2, 5, 19, 21–25].

Let us turn to *uo*-convergence in X. Passing to $X^u = \mathcal{R}\downarrow$, which has the weak unit $\mathbf{1}$, $[\![\mathbf{1}$ is the multiplicative unit of $\mathcal{R}]\!] = \mathbb{1}$ we have, by [3, Corollary 3.5],

$$x_\alpha \xrightarrow{uo} 0 \iff |x_\alpha| \wedge \mathbf{1} \xrightarrow{o} 0 \quad (x_\alpha \in X).$$

By [2, 8.1.4], for every net $s = (x_\alpha)_{\alpha \in A}$ in $\mathcal{R}\downarrow$, the *standard name* $A^\wedge$ of A in $V^{(B)}$ (see [2, p. 401]) is also directed and $(s\uparrow) : A^\wedge \to \mathcal{R}$ is a net in $\mathcal{R}$ (within $V^{(B)}$); moreover,

$$b \leqslant [\![\lim(s\uparrow) = x]\!] \iff o-\lim \chi(b) \circ s = \chi(b)x$$

for every $b \in B = \mathfrak{B}(X) = \mathfrak{P}(\mathcal{R}\downarrow)$ and every $x \in \mathcal{R}\downarrow$ [2, 8.1.4 (3)]. Thus,

$$x_\alpha \xrightarrow{uo} x \Leftrightarrow o-\lim_A(|x_\alpha - x| \wedge \mathbf{1}) = 0 \Leftrightarrow \left[\!\!\left[\lim_{A^\wedge}(|x_\alpha - x| \wedge \mathbf{1}) = 0\right]\!\!\right] = \mathbb{1}. \tag{1}$$

In the case of a sequence, $A = \mathbb{N}$, $A^\wedge = \mathbb{N}^\wedge = \mathcal{N}$ [25, p. 330]), and hence

$$\begin{aligned} x_n \xrightarrow{uo} 0 \text{ in } X \iff x_n \xrightarrow{uo} 0 \text{ in } \mathcal{R}\downarrow \iff \left[\!\!\left[\lim_{\mathcal{N}\ni n\to\infty}(|x_n| \wedge \mathbf{1}) = 0\right]\!\!\right] = \mathbb{1} \\ \iff [\![\lim |x_n| = 0]\!] = \mathbb{1} \iff x_n \xrightarrow{o} 0 \text{ in } \mathcal{R}\downarrow = X^u. \end{aligned} \tag{2}$$

Similarly,

$$\begin{aligned} &x_n \text{ is uo-Cauchy in } X \iff x_n \text{ is uo-Cauchy in } \mathcal{R}\downarrow \\ &\iff o-\lim_{k,m\to\infty} |x_k - x_m| \wedge \mathbf{1} = 0 \iff \left[\!\!\left[\lim_{\mathcal{N}^2=(\mathbb{N}\times\mathbb{N})^\wedge\ni(k,m)\to\infty}(|x_k - x_m| \wedge \mathbf{1}) = 0\right]\!\!\right] = \mathbb{1} \\ &\iff \left[\!\!\left[\lim_{\mathcal{N}\ni k,m\to\infty}|x_k - x_m| = 0\right]\!\!\right] = \mathbb{1} \iff [\![x_n \text{ is Cauchy in } \mathcal{R}]\!] = \mathbb{1} \\ &\iff [\![(\exists\, z \in \mathcal{R}) \lim x_n = z]\!] = \mathbb{1} \\ &\iff [\![\lim x_n = z]\!] = \mathbb{1}, \text{ for some } z \in \mathcal{R}\downarrow; \iff x_n \xrightarrow{o} z \in \mathcal{R}\downarrow = X^u. \end{aligned} \tag{3}$$

The last equivalence in (3) is actually due to Gordon [19, Theorem 4] (see also [22]). Clearly, (3) implies that X^u is always sequentially *uo*-complete. The equivalences of (2) are exactly the first part of the following theorem (see [3, Corollary 3.12]), whereas (3) is its second part.

Theorem 2 (Gao–Grobler–Troitsky–Xanthos). *A sequence x_n in an Archimedean vector lattice X is uo-null in X iff x_n is o-null in X^u; while x_n is uo-Cauchy in X iff x_n is o-convergent in X^u.*

The presented proof of Theorem 2 is based on the fundamental fact that the standard name $\mathbb{N}^{\wedge}$ of the naturals is the naturals $\mathcal{N}$ in $V^{(B)}$. It seems to be the main obstacle in obtaining the net versions of this theorem which are indeed impossible due to Theorem 4.

The following theorem, stated and proved in [16, Theorem 3.9] and [15, Theorem 28], is a result of contributions of several authors (cf. also [3, Theorem 3.10], [3, Proposition 5.7], and [13, Proposition 2.8]).

Theorem 3. *X is sequentially uo-complete iff X σ-universally complete.*

$\lhd$ For the "if part" we remark firstly that the fact that every (sequentially) *uo*-complete vector lattice is (σ-) Dedekind complete is already contained in the proof of [3, Proposition 5.7]. Now, the (σ-) lateral completeness of a (sequentially) *uo*-complete vector lattice follows from the *o*-summability of every (countable) order bounded disjoint family in a (σ-) Dedekind complete vector lattice (cf. [2, 1.3.4]).

The "only if part" is exactly [3, Theorem 3.10]. $\rhd$

It could be illustrative to present some Boolean-valued proof of Theorem 3 as well as a Boolean-valued proof of Azouzi's Theorem [15, Theorem 17] which yields the equivalence of *uo*-completeness and universal completeness.

We conclude our paper with the following theorem which provides, among other things, an answer to Azouzi's question [15, Problem 23].

Theorem 4. *Let X be an Archimedean vector lattice. Then the following are equivalent*:

(1) $\dim(X) < \infty$;

(2) *every uo-Cauchy net in X is eventually order bounded in X^u*;

(3) *every uo-Cauchy net in X is o-convergent in X^u*;

(4) *every uo-null net in X is o-null in X^u*;

(5) *every uo-null net in X is eventually order bounded in X^u*;

(6) *every uo-convergent net in X is eventually order bounded in X^u*;

(7) *every uo-convergent net in X is eventually order bounded in X*;

(8) *every uo-convergent net in X o-converges in X^u to the same limit*;

(9) *every uo-convergent net in X^u o-converges in X^u to the same limit.*

Before proving the theorem, we include the following modification of [13, Example 2.6]. Given a nonempty subset $A \subset X$, pr_A stands for the band projection in X^u onto the band in X^u generated by A.

Example 1. In any infinite-dimensional Archimedean vector lattice X there exists a *uo*-null net which is not eventually order bounded in X^u.

As $\dim(X) = \infty$, there is a sequence e_n of pairwise disjoint positive nonzero elements of X. Let $\mathbb{N}^2$ be the coordinatewise directed set of pairs of naturals. A net in X is defined via $x_{(n,m)} = (n \vee m) \cdot e_{n\wedge m}$. Since $\{x_{(n,m)} : (n,m) \in \mathbb{N}^2\} \subseteq B_{\{e_k : k\in\mathbb{N}\}}$ and

$$\lim_{(n,m)\to\infty} pr_{\{e_k\}}(x_{(n,m)}) = \lim_{(n,m)\to\infty} (n \vee m) pr_{\{e_k\}}(e_{n\wedge m}) = 0 \quad (\forall\, k \in \mathbb{N}),$$

then $x_{(n,m)} \xrightarrow{uo} 0$ as $(n,m) \to \infty$ (e. g., it can be seen by use of [3, Corollary 3.5.] for a weak unit u in X^u s.t. $u \wedge e_k = e_k$ for all k). If $x_{(n,m)}$ is eventually order bounded by some $y \in X^u$, then for some $(n_0, m_0) \in \mathbb{N}^2$ we have $y \geqslant x_{(n,m)} \quad (\forall\,(n,m) \geqslant (n_0, m_0))$. Since $n \wedge m_0 = m_0$ and $(n, m_0) \geqslant (n_0, m_0)$ for $n \geqslant n_0 \vee m_0$, then

$$y \geqslant x_{(n,m_0)} = (n \vee m_0) \cdot e_{n \wedge m_0} = (n \vee m_0) \cdot e_{m_0} = n \cdot e_{m_0} > 0 \quad (\forall\, n \geqslant n_0 \vee m_0)$$

which is impossible. Therefore, the net $x_{(n,m)}$ is not eventually order bounded in X^u.

$\lhd$ Proof of Theorem 4. (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) $\Leftrightarrow$ (6), and (7) $\Rightarrow$ (6) are trivial.

(2) $\Rightarrow$ (3): Suppose x_α is *uo*-Cauchy in X. Then x_α is *uo*-Cauchy in X^u by [3, Theorem 3.2], because X is regular in X^u. It follows from [15, Theorem 17] that $x_\alpha \xrightarrow{uo} y$ for some $y \in X^u$. Since x_α is eventually order bounded in X^u by the assumption, then $x_\alpha \xrightarrow{o} y$.

(3) $\Rightarrow$ (4) follows since every *uo*-null net is *uo*-Cauchy, *o*-convergent implies *uo*-convergent, and the *uo*-limit of any *uo*-convergent net is unique.

(5) $\Rightarrow$ (1) is Example 1.

(6) $\Rightarrow$ (7) follows from the equivalence (6) $\Leftrightarrow$ (1) because (1) $\Rightarrow$ (7) is obvious.

(1) $\Leftrightarrow$ (8) follows from the equivalence (1) $\Leftrightarrow$ (4), since (8) is equivalent to the fact that every *uo*-null net in X is *o*-null in X^u.

(1) $\Leftrightarrow$ (9) follows from (1) $\Leftrightarrow$ (8) since $(X^u)^u = X^u$ and $\dim(X) < \infty$ iff $\dim(X^u) < \infty$. $\rhd$

While preparing this paper, we became aware of the still unpublished work [18] by Taylor which provides the construction [18, Proposition 15.2] similar to Example 1. The equivalence (1) $\Leftrightarrow$ (8) of Theorem 4 is also contained in [18, Corollary 15.3].

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Received 4 Jule, 2019

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Vladikavkaz Mathematical Journal
2019, Volume 21, Issue 4, P. 56–62

НЕОГРАНИЧЕННАЯ ПОРЯДКОВАЯ СХОДИМОСТЬ И ТЕОРЕМА ГОРДОНА

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Аннотация. Знаменитая теорема Гордона является естественным инструментом для построения универсального пополнения архимедовой векторной решетки. Она позволяет нам уточнить некоторые недавние результаты о неограниченной порядковой сходимости. Применяя теорему Гордона, мы демонстрируем несколько фактов о сходимость последовательностей. В частности, приводится элементарное доказательство теоремы Гао — Гроблера — Троицкого — Хантоса о том, что последовательность в архимедовой векторной решетке *uo*-сходится к нулю (соответственно, является *uo*-фундаментальной) тогда и только тогда когда она порядково сходится к нулю (соответственно, является порядково сходящейся) в универсальном пополнении этой решетки. В статье дается простое доказательство известной теоремы о том, что архимедова векторная решетка секвенциально *uo*-полна тогда и только тогда когда она σ-универсально полна. Кроме того в статье дается полное решение недавней проблемы Азози о конечномерности всякой архимедовой векторной решетки в которой любая *uo*-фундаментальная последовательность порядково сходится в универсальном пополнении этой решетки.

Ключевые слова: неограниченная порядковая сходимость, расширенное пространство Канторовича, булевозначный анализ.

Mathematical Subject Classification (2010): 03H05, 46S20, 46A40.

Образец цитирования: *Emelyanov E. Y., Gorokhova S. G. and Kutateladze S. S.* Unbounded Order Convergence and the Gordon Theorem // Владикавк. мат. журн.—2019.—Т. 21, № 4.—С. 56–62 (in English). DOI: 10.23671/VNC.2019.21.44624.

Vladikavkaz Mathematical Journal
2019, Volume 21, Issue 4, P. 63–70

УДК 510.898, 517.98
DOI 10.23671/VNC.2019.21.44626

THE GORDON THEOREM: ORIGINS AND MEANING

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To Evgeny Gordon on occasion
of his 70th birthday

Abstract. Boolean valued analysis, the term coined by Takeuti, signifies a branch of functional analysis which uses a special technique of Boolean valued models of set theory. The fundamental result of Boolean valued analysis is Gordon's Theorem stating that each internal field of reals of a Boolean valued model descends into a universally complete vector lattice. Thus, a remarkable opportunity opens up to expand and enrich the mathematical knowledge by translating information about the reals to the language of other branches of functional analysis. This is a brief overview of the mathematical events around the Gordon Theorem. The relationship between the Kantorovich's heuristic principle and Boolean valued transfer principle is also discussed.

Key words: vector lattice, Kantorovich's principle, Gordon's theorem, Boolean valued analysis.

Mathematics Subject Classification (2010): 03C90, 46A40.

For citation: *Kusraev, A. G. and Kutateladze, S. S.* The Gordon Theorem: Origins and Meaning, *Vladikavkaz Math. J.*, 2019, vol. 21, no. 4, pp. 63–70. DOI: 10.23671/VNC.2019.21.44626.

1. Introduction

In 1977, Evgeny Gordon, a young teacher of Lobachevsky Nizhny Novgorod State University, published the short note [1] which begins with the words:

"*This article establishes that the set whose elements are the objects representing reals in a Boolean valued model of set theory* $\mathbb{V}^{(\mathbb{B})}$ *can be endowed with the structure of a vector space and an order relation so that it becomes an extended K-space** *with base*** $\mathbb{B}$*. It is shown that in some cases this fact can be used to generalize the theorems about reals to extended K-spaces.*"

His note has become the bridge between various areas of mathematics which helps, in particular, to solve many problems of functional analysis in "semiordered vector spaces" [4] by using the techniques of Boolean valued models of set theory [5].

* A *K-space* or a *Kantorovich space* is a Dedekind complete vector lattice. An *extended K-space* is a universally complete vector lattice, cp. [2] and [3].

** The *base* of a vector lattice is the Boolean algebra of all of its bands [3].

In the same year, at the Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis (Durham, July 9–11, 1977), Gaisi Takeuti, a renowned expert in proof theory, observed that if $\mathbb{B}$ is a complete Boolean algebra of orthogonal projections in a Hilbert space H, then the set whose elements represent reals in the Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ can be identified with the vector lattice of selfadjoint operators in H whose spectral resolutions take values in $\mathbb{B}$; see [6].

These two events marked the birth of a new section of functional analysis, which Takeuti designated by the term *Boolean valued analysis*. The history and achievements of Boolean valued analysis are reflected in [7–9].

It should be mentioned that Dana Scott foresaw in 1969 [10] that the new nonstandard models must be of mathematical interest aside from the independence proof, but he was unable to give a really good evidence of this. In fact Takeuti found a narrow path whereas Gordon paved a turnpike to the core of mathematics, which justifies the vision of Scott.

Boolean valued analysis signifies the technique of studying the properties of an arbitrary mathematical object by comparison between its representations in two different Boolean valued models of set theory. As the models, we usually take the *von Neumann universe* $\mathbb{V}$ (the mundane embodiment of the classical Cantorian paradise) and the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ (a specially-trimmed universe whose construction utilizes a complete Boolean algebra $\mathbb{B}$ with a top element $\mathbb{1}$). The principal difference between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$ is the way of verification of statements: There is a natural way of assigning to each statement ϕ about $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$ the *Boolean truth-value* $[\![\phi(x_1, \dots, x_n)]\!] \in \mathbb{B}$. The sentence $\phi(x_1, \dots, x_n)$ is called true in $\mathbb{V}^{(\mathbb{B})}$ if $[\![\phi(x_1, \dots, x_n)]\!] = \mathbb{1}$. All theorems of Zermelo–Fraenkel set theory with the axiom of choice are true in $\mathbb{V}^{(\mathbb{B})}$ for every complete Boolean algebra $\mathbb{B}$. There is a smooth and powerful mathematical technique for revealing interplay between the interpretations of one and the same fact in the two models $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. The relevant *ascending-and-descending machinery* rests on the functors of *canonical embedding* $X \mapsto X^{\wedge}$ and *ascent* $X \mapsto X{\uparrow}$ acting from $\mathbb{V}$ into $\mathbb{V}^{(\mathbb{B})}$ and *descent* $X \mapsto X{\downarrow}$ acting from $\mathbb{V}^{(\mathbb{B})}$ into $\mathbb{V}$; see [7, 8].

Everywhere below $\mathbb{B}$ is a complete Boolean algebra and $\mathbb{V}^{(\mathbb{B})}$ the corresponding Boolean valued model of set theory; see [5, 11, 12]. We let := denote the assignment by definition, while $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ symbolize the naturals, the reals, and the complexes.

2. Kantorovich's Heuristic Principle

The unexplained terms of vector lattice theory can be found in [2, 3, 13, 14]. All vector lattices below are assumed to be Archimedean.

Definition 1. A *vector lattice* or a *Riesz space* is a real vector space X equipped with a partial order $\leqslant$ for which the *join* $x \vee y$ and the *meet* $x \wedge y$ exist for all $x, y \in X$, and such that the *positive cone* $X_+ := \{x \in X : 0 \leqslant x\}$ is closed under addition and multiplication by positive reals and for any $x, y \in X$ the relations $x \leqslant y$ and $0 \leqslant y - x$ are equivalent. A *band* in a vector lattice X is the *disjoint complement* $Y^{\perp}$ of any subset $Y \subset X$ where $Y^{\perp} := \{x \in X : \ (\forall\, y \in Y)\, |x| \wedge |y| = 0\}$. Let $\mathbb{B}(X)$ and $\mathbb{P}(X)$ stand for the inclusion ordered sets of all bands and all band projections in X, respectively.

Definition 2. A subset $U \subset X$ is *order bounded* if U lies in an *order interval* $[a, b] := \{x \in X : \ a \leqslant x \leqslant b\}$ for some $a, b \in X$. A vector lattice X is *Dedekind complete* (respectively, *laterally complete*) if each nonempty order bounded set (respectively, each nonempty set of pairwise disjoint positive vectors) U in X has a least upper bound $\sup(U) \in X$. Note that $\mathbb{B}(X)$ and $\mathbb{P}(X)$ are isomorphic Boolean algebras for such X. The vector lattice that is laterally complete and Dedekind complete simultaneously is referred to as *universally complete*.

Definition 3. An *f-algebra* is a vector lattice X equipped with a distributive multiplication such that if $x, y \in X_+$ then $xy \in X_+$, and if $x \wedge y = 0$ then $(ax) \wedge y = (xa) \wedge y = 0$ for all $a \in X_+$. An f-algebra is *semiprime* provided that $xy = 0$ implies $x \perp y$ for all x and y. A *complex vector lattice* $X_{\mathbb{C}}$ is the complexification $X_{\mathbb{C}} := X \oplus iX$ (with i standing for the imaginary unity) of a real vector lattice X.

Leonid Kantorovich was among the first who studied operators in ordered vector spaces. He distinguished an important instance of ordered vector spaces, a Dedekind complete vector lattice, often called a *Kantorovich space* or a *K-space*. This notion appeared in Kantorovich's first fundamental article [15] on this topic where he wrote:

"In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals."

Here Kantorovich stated an important methodological principle, the *heuristic transfer principle* for K-spaces, claiming that the elements of a K-space can be considered as generalized reals. Essentially, this principle turned out to be one of those profound ideas that, playing an active and leading role in the formation of a new branch of analysis, led eventually to a deep and elegant theory of K-space rich in various applications. At the very beginning of the development of the theory, attempts were made at formalizing the above heuristic argument. In this direction, there appeared the so-called *identity preservation theorems* which claimed that if some proposition involving finitely many relations is proven for the reals then an analogous fact remains valid automatically for the elements of every K-space (see [3, 4, 14]). The depth and universality of Kantorovich's principle were demonstrated within Boolean valued analysis. See more about the Kantorovich's universal heuristics and innate integrity of his methodology in [16]. The contemporary forms of above mentioned relation preservation theorems, basing on Boolean valued models, may be found in Gordon [17–19] and Jech [20, 21].

3. Boolean Valued Reals

Boolean valued analysis stems from the fact that each internal field of reals of a Boolean valued model descends into a universally complete vector lattice. Thus, a remarkable opportunity opens up to expand and enrich the mathematical knowledge by translating information about the reals to the language of other branches of functional analysis.

According to the principles of Boolean valued set theory there exists an internal field of reals $\mathcal{R}$ in $\mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. In other words, there exists $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\, \mathcal{R} \text{ is a field of reals} \,]\!] = \mathbb{1}$. Moreover, if $[\![\, \mathcal{R}' \text{ is a field of reals} \,]\!] = \mathbb{1}$ for some $\mathcal{R}' \in \mathbb{V}^{(\mathbb{B})}$ then $[\![\, \text{the ordered fields } \mathcal{R} \text{ and } \mathcal{R}' \text{ are isomorphic} \,]\!] = \mathbb{1}$.

By the same reasons there exists an internal field of complexes $\mathcal{C} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. Moreover, $\mathbb{V}^{(\mathbb{B})} \models \mathcal{C} = \mathcal{R} \oplus i\mathcal{R}$. We call $\mathcal{R}$ and $\mathcal{C}$ is the *internal reals* and *internal complexes* in $\mathbb{V}^{(\mathbb{B})}$.

The fundamental result of Boolean valued analysis is *Gordon's Theorem* [1] which reads as follows: *Each universally complete vector lattice is an interpretation of the reals in an appropriate Boolean valued model.* Formally:

Gordon Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra, $\mathcal{R}$ be a field of reals within $\mathbb{V}^{(\mathbb{B})}$. Endow $\mathbf{R} := \mathcal{R}{\downarrow}$ with the descended operations and order. Then*

(1) *The algebraic structure $\mathbf{R}$ is an universally complete vector lattice.*

(2) *The field $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ can be chosen so that $[\![\, \mathbb{R}^{\wedge} \text{ is a dense subfield of } \mathcal{R} \,]\!] = \mathbb{1}$.*

(3) *There is a Boolean isomorphism* χ *from* $\mathbb{B}$ *onto* $\mathbb{P}(\mathbf{R})$ *such that*

$$\chi(b)x = \chi(b)y \iff b \leqslant [\![x = y]\!],$$
$$\chi(b)x \leqslant \chi(b)y \iff b \leqslant [\![x \leqslant y]\!]$$
$$(x, y \in \mathbf{R};\ b \in \mathbb{B}).$$

The converse is also true: *Each Archimedean vector lattice embeds into an appropriated Boolean valued model, becoming a vector sublattice of the reals* (*viewed as such over some dense subfield of the reals*). More details on the Boolean valued theory of vector lattices and positive operators can be found in [7–9, 22].

Gutman [23] characterized those complete Boolean algebras $\mathbb{B}$ for which the internal fields $\mathbb{R}^{\wedge}$ and $\mathcal{R}$ coincide: $\mathbb{V}^{(\mathbb{B})} \models \mathcal{R} = \mathbb{R}^{\wedge}$ *if and only if* $\mathbb{B}$ *is the vector lattice* σ*-distributive** *if and only if* $\mathcal{R}{\downarrow}$ *is locally one-dimensional***. He also proved that there exist nondiscrete locally one-dimensional Dedekind complete vector lattice. Observe also some additional properties of Boolean valued reals, multiplicative structure and complexification:

Corollary 1. *The universally complete vector lattice* $\mathcal{R}{\downarrow}$ *with the descended multiplication is a semiprime* f*-algebra with the ring unity* $\mathbb{1} := 1^{\wedge}$. *Moreover, for every* $b \in \mathbb{B}$ *the band projection* $\chi(b) \in \mathbb{P}(\mathbf{R})$ *acts as multiplication by* $\chi(b)\mathbb{1}$.

Corollary 2. *Let* $\mathscr{C}$ *be the complexes within* $\mathbb{V}^{(\mathbb{B})}$. *Then the algebraic system* $\mathcal{C}{\downarrow}$ *is a universally complete complex* f*-algebra. Moreover,* $\mathcal{C}{\downarrow}$ *is the complexification of the universally complete real* f*-algebra* $\mathcal{R}{\downarrow}$*; i. e.,* $\mathcal{C}{\downarrow} = \mathcal{R}{\downarrow} \oplus i\mathcal{R}{\downarrow}$.

Example 1. Assume that a measure space (Ω, Σ, μ) is semi-finite; i. e., if $A \in \Sigma$ and $\mu(A) = \infty$ then there exists $B \in \Sigma$ with $B \subset A$ and $0 < \mu(B) < \infty$. The vector lattice $L^0(\mu) := L^0(\Omega, \Sigma, \mu)$ (of cosets) of μ-measurable functions on Ω is universally complete if and only if (Ω, Σ, μ) is *localizable* ($\equiv$ *Maharam*). In this event $L^p(\Omega, \Sigma, \mu)$ is Dedekind complete; see [24, 241G]. Note that $\mathbb{P}(L^0(\Omega, \Sigma, \mu)) \simeq \Sigma/\mu^{-1}(0)$. In [25], Scott observed that in the algebra of random variables of a probability space (as in a Boolean structure) all Boolean truth values of the axioms of the field of reals are $\mathbb{1}$.

Example 2. Given a complete Boolean algebra $\mathbb{B}$ of orthogonal projections in a Hilbert space H, denote by $\langle\mathbb{B}\rangle$ the space of all selfadjoint operators on H whose spectral resolutions are in $\mathbb{B}$; i. e., $A \in \langle\mathbb{B}\rangle$ if and only if $A = \int_{\mathbb{R}} \lambda\, dE_\lambda$ and $E_\lambda \in \mathbb{B}$ for all $\lambda \in \mathbb{R}$. Define the partial order in $\langle\mathbb{B}\rangle$ by putting $A \geqslant B$ whenever $\langle Ax, x\rangle \geqslant \langle Bx, x\rangle$ for all $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$, where $\mathcal{D}(A) \subset H$ stands for the domain of A. Then $\langle\mathbb{B}\rangle$ is a universally complete vector lattice and the Boolean algebras $\mathbb{P}(\langle\mathbb{B}\rangle)$ and $\mathbb{B}$ are isomorphic.

If μ is a Maharam measure and $\mathbb{B}$ in the Gordon Theorem is the algebra of all μ-measurable sets modulo μ-negligible sets, then $\mathscr{R}{\downarrow}$ is lattice isomorphic to $L^0(\mu)$; see Example 1. If $\mathbb{B}$ is a complete Boolean algebra of projections in a Hilbert space H then $\mathscr{R}{\downarrow}$ is isomorphic to $\langle\mathbb{B}\rangle$; see Example 2. The two indicated particular cases of Gordon's Theorem were intensively and fruitfully exploited by Takeuti [6, 26]. The object $\mathscr{R}{\downarrow}$ for the general Boolean algebras was also studied by Jech [20, 21], who in fact rediscovered Gordon's Theorem. The difference is that in [20] a (complex) universally complete vector lattice with unit is defined by another system of axioms and is referred to as a complete *Stone algebra*.

* A Boolean algebra $\mathbb{B}$ is called *σ-distributive* if for every double sequence $(b_{n,m})_{n,m\in\mathbb{N}}$ in $\mathbb{B}$ the following equation holds: $\bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} b_{n,m} = \bigwedge_{\varphi\in\mathbb{N}^{\mathbb{N}}} \bigvee_{n\in\mathbb{N}} b_{n,\varphi(n)}$.

** A universally complete vector lattice G is called *locally one-dimensional* if all positive elements of G are *locally constants* with respect to and arbitrary order unit $\mathbb{1}$, that is, every $x \in G_+$ is reprentable as $e = \sup_{\xi\in\Xi} \lambda_\xi \pi_\xi \mathbb{1}$ for some numeric family $(\lambda_\xi)_{\xi\in\Xi}$ and a family $(\pi_\xi)_{\xi\in\Xi}$ of pairwise disjoint band projections.

4. Concluding Remarks

1. In 1963 Cohen discovered his *method of forcing* and also proved the independence of the Continuum Hypothesis. A comprehensive presentation of the Cohen forcing method gave rise to the *Boolean valued models of set theory*, which were first introduced by Scott and Solovay (see Scott [10]*) and Vopěnka [27]. In an extremely interesting and illuminating foreword to Bell's book [5] written by Scott, the development following Cohen's discovery is characterized as follows:

"*It was in 1963 that we were hit by a real bomb, however, when Paul J. Cohen discovered his method of 'forcing', which started a long chain reaction of independence results stemming from his initial proof of the independence of the Continuum Hypothesis. Set theory could never be the same after Cohen, and there is simply no comparison whatsoever in the sophistication of our knowledge about models for set theory today as contrasted to the pre-Cohen era.*"

Many delicate properties of the objects within $\mathbb{V}^{(\mathbb{B})}$ depend essentially on the structure of the initial complete Boolean algebra $\mathbb{B}$. The diversity of opportunities together with a great stock of information on particular Boolean algebras ranks Boolean valued models among the most powerful tools of foundational studies. A systematic account of the theory of Boolean valued models and its applications to independence proofs can be found in [5, 11, 12, 28].

2. Recal that ZF is *Zermelo–Fraenkel set theory*, AC and DC stand for the *Axiom of Choice* and the *Principle of Dependent Choice*, respectively, ZFC=ZF+AC, and LM denotes the sentence "Every set of reals is Lebesgue measurable." Solovay, in his celebrated work [29], proved the following result by constructing a model for ZF+DC+LM.

Theorem 1. *If the existence of an inaccessible cardinal is consistent with* ZFC*, then*

(1) *The statement "Every subset of* $\mathbb{R}$ *definable by a countable sequence of ordinals is Lebesgue measurable" is consistent with* ZF.

(2) LM *is consistent with* ZF+DC.

Solovay then posed the famous problem: *Does Theorem 1 remains true without assumption of consistency of the existence of an inaccessible cardinal?*

Solovay's model have many interesting properties. For example, the Hahn-Banach theorem fails in the model of Theorem 1, while it follows readily from DC for separable Banach spaces [29, p. 3]. Moreover, in Solovay's model each linear operator on a Hilbert space is a bounded linear operator; see [30, Theorem 6].

3. Gordon came to his theorem, while trying to attack Solovay's problem. He failed to solve the problem but proved the following weaker statement; see [1, Theorem 7] and [31].

Theorem 2. *The statement "The Lebesgue measure on* $\mathbb{R}$ *can be extended to a* σ*-additive invariant measure on the* σ*-algebra of sets definable by a countable sequence of ordinals" is consistent with* ZFC.

In order to prove Theorem 2, he needed to consider the elements $\mathcal{B}, \mu \in \mathbb{V}^{(\mathbb{B})}$, where $\mathbb{B}$ is a complete Boolean algebra with measure and

$$[\![(\mathcal{B}, \mu) \text{ is a complete Boolean algebra with measure}]\!] = \mathbb{1},$$

and identify in $\mathbb{V}$ the descent $\mu{\downarrow} : \mathcal{B}{\downarrow} \to \mathcal{R}{\downarrow}$ of μ as a vector measure on the complete Boolean algebra $\mathcal{B}{\downarrow}$ with values in $\mathcal{R}{\downarrow}$. The fact that $\mathcal{B}{\downarrow}$ is a complete Boolean algebra that contains $\mathbb{B}$ as a complete subalgebra was known from the paper [28] about the iterated forcing.

* There are many references in the literature to the Scott–Solovay unpublished paper *"Boolean valued models of set theory."* The reasons for this are discussed in the preface to the book [5].

So, he considered the algebraic structure $\mathcal{R}{\downarrow}$ and proved that it is the extended K-space with the base isomorphic to $\mathbb{B}$. He learned about K-spaces from the book [32]. Now, since $\mathbb{B}$ is an algebra with measure, a real-valued measure on $\mathcal{B}{\downarrow}$ can be produced by integration of elements $\mu{\downarrow}(b) \in \mathcal{R}{\downarrow}$ for all $b \in \mathcal{B}{\downarrow}$.

4. The Solovay problem was settled by Shelah [33], who showed that the assumption about inaccessible cardinal cannot be removed from Theorem 1. More precisely, he proved that ZF+DC+LM implies that ω_1 is inaccessible in L, the universe of Gödel constructible sets. It is also worth mentioning that Sacks [34] obtained the following result without assuming the existence of an inaccessible cardinal.

Theorem 3. *The statement "The Lebesgue measure on* $\mathbb{R}$ *can be extended to the* σ*-additive invariant measure defined on all subsets of* $\mathbb{R}$*" is consistent with* ZF+DC.

In particular, Shelah's result brings to light the importance of Theorems 2 and 3.

5. Two more remarkable independence results are worth mentioning here. We first recall the following abbreviations:

SH (*Souslin's Hypothesis*): Every order complete order dense linearly ordered set having neither bottom nor top element is order isomorphic to the ordered set of the reals $\mathbb{R}$, provided that every collection of mutually disjoint nonempty open intervals in it is countable.

NDH (*No Discontinuous Homomorphisms*): For each compact space X, each homomorphism from $C(X, \mathbb{C})$, the Banach algebra of all continuous complex-valued functions on X, into arbitrary complex Banach algebra is continuous. NDH is equivalent to saying that every algebra norm on $C(X, \mathbb{C})$ is equivalent to the uniform norm.

The problem whether or not SH is true was posed by Souslin in 1920. The corresponding problem for NDH dates back to the Kaplansky article of 1948.

Theorem 4. *Both statements SH and NDH are independent of ZFC.*

Tennenbaum [35] and Jech [36] both gave models in which SH is false. Solovay and Tennenbaum [28] extended Cohen's method to define models in which SH holds. The consistency of $\neg$NDH is due to Dales and Esterly, while the consistency of NDH was proved by Solovay and Woodin; see [37] for details. Thus, like the Continuum Hypothesis, SH and NDH are undecidable on using the contemporary axioms of set theory.

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Received July 2, 2019

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Владикавказский математический журнал
2019, Том 21, Выпуск 4, С. 63–70

ТЕОРЕМА ГОРДОНА: ИСТОКИ И СМЫСЛ

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Аннотация. Термин булевозначный анализ, введенный Такеути, обозначает раздел функционального анализа, в котором используется специальная техника булевозначных моделей теории множеств. Фундаментальным результатом булевозначного анализа является теорема Гордона о том, что каждое внутреннее поле вещественных чисел булевозначной модели спускается в универсально полную векторную решетку. Таким образом, открывается замечательная возможность расширить и обогатить математические знания, переводя информацию о вещественных числах на язык других разделов функционального анализа. Настоящая работа — краткий обзор математических событий вокруг теоремы Гордона. Обсуждается также связь между эвристическим принципом Канторовича и принципом булевозначного переноса.

Ключевые слова: векторная решетка, принцип Канторовича, теорема Гордона, булевозначный анализ.

Mathematical Subject Classification (2010): 03C90, 46A40.

Образец цитирования: *Kusraev, A. G. and Kutateladze, S. S.* The Gordon Theorem: Origins and Meaning // Владикавк. мат. журн.—2019.—Т. 21, № 4.—С. 63–70 (in English). DOI: 10.23671/VNC.2019.21.44626.

Vladikavkaz Mathematical Journal
2019, Volume 21, Issue 4, P. 71–89

УДК 510.898, 517,98, 519.866
DOI 10.23671/VNC.2019.21.44629

A BOOLEAN VALUED ANALYSIS APPROACH TO CONDITIONAL RISK[♯]

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To Evgeny Israilevich Gordon
with respect and admiration
on the occasion of his anniversary

Abstract. By means of the techniques of Boolean valued analysis, we provide a transfer principle between duality theory of classical convex risk measures and duality theory of conditional risk measures. Namely, a conditional risk measure can be interpreted as a classical convex risk measure within a suitable set-theoretic model. As a consequence, many properties of a conditional risk measure can be interpreted as basic properties of convex risk measures. This amounts to a method to interpret a theorem of dual representation of convex risk measures as a new theorem of dual representation of conditional risk measures. As an instance of application, we establish a general robust representation theorem for conditional risk measures and study different particular cases of it.

Key words: Boolean valued analysis, conditional risk measures, duality theory, transfer principle.

Mathematical Subject Classification (2010): 03C90, 46H25, 91B30.

For citation: *Zapata, J. M.* A Boolean Valued Analysis Approach to Conditional Risk, *Vladikavkaz Math. J.*, 2019, vol. 21, no. 4, pp. 71–89. DOI: 10.23671/VNC.2019.21.44629.

1. Introduction

The present paper contributes to mathematical finance by means of the tools of Boolean valued analysis, a branch of functional analysis that applies special model-theoretic techniques to analysis.

Let us start by explaining the mathematical finance problem that we are interested in. Over the past two decades and having its origins in the seminal paper [1], duality theory of risk measures has been an active and prolific area of research, see e. g. [2–12] and references therein. The simplest situation is the case in which only two instants of time matter: today 0 and tomorrow $T > 0$. In this case, the market information that will be observable at time T is described by a probability space $(\Omega, \mathcal{E}, \mathbb{P})$. A *risk measure* is a function that assigns to any $\mathcal{E}$-measurable random variable x, which models a final payoff, a real number $\rho(x)$, which quantifies the riskiness of x. Generally speaking, duality theory of risk measures studies what

[♯] The author was partially supported by the grants MINECO MTM2014-57838-C2-1-P and Fundación Séneca 20903/PD/18.

are the desirable economic properties that should have a risk measure and which is the dual representation of a risk measure with these properties.

A more intricate situation is when we have a dynamic configuration of time, in which the arrival of new information at an intermediate date $0 < t < T$ is taken into account. Suppose that the available information at time t is encoded in a sub-σ-algebra $\mathcal{F}$ of the σ-algebra $\mathcal{E}$ of general information. In that case, the riskiness at time t of any final payoff is contingent on the information contained in $\mathcal{F}$. Then a *conditional risk measure* is a mapping (fulfilling some desirable economic conditions) that assigns to any final payoff, modeled by an $\mathcal{E}$-measurable random variable x, an $\mathcal{F}$-measurable random variable $\rho(x)$, which quantifies the risk arisen from x. A problem that has drawn the attention for a long time is the dual representation of a risk measure in a multi-period setup, see for instance [13–22] and references therein.

As explained in [23], whereas classical convex analysis perfectly applies to the one-period case, it has a rather delicate application to the multi-period model: consider the properties of the conditional risk measure ρ such as convexity, continuity, differentiability and so on. These properties have to be satisfied by the function $x \mapsto \rho(x)(\omega)$ for each $\omega \in \Omega$, but they should be fulfilled under a measurable dependence on ω, in order to enable a recursive multi-period scheme. This approach would require heavy measurable selection criteria. These difficulties have motivated some new developments in functional analysis. For instance, Filipovic et al. [23] proposed to consider modules over $L^0(\mathcal{F})$, the space of (equivalence classes of) $\mathcal{F}$-measurable random variables (see also [19, 24, 25]). More sophisticated machinery is provided in [26], where the so-called *conditional set theory* is introduced and developed. Other related approaches are introduced in [27–29].

A step forward is given in [30], where it is established a method to interpret any theorem of convex analysis as a theorem of L^0-convex analysis. The machinery is taken from Boolean valued analysis, a branch of functional analysis that consists in studying the properties of a mathematical object by interpreting it as a simpler object in a different set-theoretic model whose construction utilizes a Boolean algebra. Boolean valued analysis stems from the method of forcing that Paul Cohen created to prove the independence of the continuum hypothesis from the system of axioms of the Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC) [31]. The main tool of Boolean valued analysis are Boolean valued models of set theory, which were developed by Scott, Solovay, and Vopěnka as a way to simplify the Cohen's method of forcing. Boolean valued analysis started with Gordon [32] and Takeuti [33]*, and has undergone a fruitful and deep development due to Kusraev and Kutateladze. For a thorough account, we refer the reader to [34] and its extensive list of references.

The present paper is aimed to extend and exploit the connections provided in [30], to establish a general transfer method between duality theory of one-period risk measures and duality theory of conditional risk measures, putting at the disposal of mathematical finance a powerful tool to obtain different duality representation results. Namely, we show that if $\rho : \mathscr{X} \to L^0(\mathcal{F})$ is a conditional risk measure, then we can interpret ρ as a one-period risk measure $\rho\!\uparrow$ defined on a space of (classes of equivalence of) random variables $\mathscr{X}\!\uparrow$ within a suitable set-theoretic model. Then, inside of this model, any available theorem about the dual representation of the one-period risk measure $\rho\!\uparrow$ has a counterpart that is satisfied by the conditional risk measure ρ. This means that any theorem of duality theory of one-period risk measures gives rise to a new theorem of duality theory of conditional risk measures.

The paper is structured as follows: In Section 1, we give some preliminaries and review duality theory of risk measures both in the one-period and multi-period setups. In Section 2, we

* Actually the term Boolean valued analysis was coined by Takeuti [33].

recall the basics of Boolean valued models. In Section 3, we establish a Boolean valued transfer principle between duality theory of convex risk measures and duality theory of conditional risk measures. By applying this transfer principle we derive a general robust representation theorem of conditional risk measures and study different particular cases. Finally, in Section 4, due to limited space, we sketch the proof of the transfer method.

1. Preliminaries on Duality Theory of Risk Measures

In this section, we review the main elements of duality theory of risk measures. We start by the one-period setup, recalling the notion of convex risk measure and different properties that matter in the dual representation of a convex risk measure. After this, we move on to the multi-period setup. We recall the notion of conditional risk measure and introduce conditional analogues of the different elements of the one-period case.

1.1. One-period setup: convex risk measures. Let us recall some basics of duality theory of risk measures. For an introduction to this topic, we refer the reader to [35, Chapter 4]. Let $(\Omega, \mathcal{E}, \mathbb{P})$ be a probability space. We denote by $L^0(\mathcal{E})$ the space of $\mathcal{E}$-measurable real-valued random variables on Ω identified whenever their difference is $\mathbb{P}$-negligible. Given $x, y \in L^0(\mathcal{E})$ we understand $x \leqslant y$ and $x < y$ in the almost surely sense. Endowed with the order $\leqslant$, $L^0(\mathcal{E})$ is a Dedekind complete lattice ring. We say that $\lim_n x_n = x$ a.s. in $L^0(\mathcal{E})$ whenever (x_n) converges almost surely to $x \in L^0(\mathcal{E})$ (or equivalently, x_n order converges to x).

Suppose that our probability space $(\Omega, \mathcal{E}, \mathbb{P})$ models the market information at some time horizon $T > 0$. The final payoff of each financial position is going to be modeled by a subspace $\mathscr{X}$ of $L^0(\mathcal{E})$ with the following properties:

- $\mathcal{X}$ is *solid*, that is, $y \in \mathcal{X}$ and $|x| \leqslant |y|$ imply that $x \in \mathcal{X}$;*
- $\mathbb{E}_{\mathbb{P}}[|x|] < \infty$ for any $x \in \mathcal{X}$;
- the classes of equivalence of constant functions are contained in $\mathcal{X}$.

Example 1.1. The following subspaces of $L^0(\mathcal{E})$ satisfy the properties above:

1. L^p spaces: $L^p(\mathcal{E}) := L^p(\Omega, \mathcal{E}, \mathbb{P})$ with $1 \leqslant p \leqslant \infty$.

2. Orlicz spaces: Let $\phi : [0, \infty) \to [0, \infty]$ be a *Young function*, that is, an increasing left-continuous convex function finite on a neighborhood of 0 with $\varphi(0) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$. The associated *Orlicz space* is

$$L^\phi(\mathcal{E}) = \big\{x \in L^0(\mathcal{E})\colon (\exists\, r \in (0, \infty))\ \ \mathbb{E}_{\mathbb{P}}[\phi(r|x|)] < \infty\big\}.$$

3. Orlicz-heart spaces: If ϕ is a Young function, the associated *Orlicz-heart space* is

$$H^\phi(\mathcal{E}) = \big\{x \in L^0(\mathcal{E})\colon (\forall\, r \in (0, \infty))\ \ \mathbb{E}_{\mathbb{P}}[\phi(r|x|)] < \infty\big\}.$$

The riskiness of any final payoff $x \in \mathcal{X}$ is quantified by a function $\rho : \mathcal{X} \to \mathbb{R}$ satisfying for all $x, y \in \mathcal{X}$:

1. *Convexity*: i. e. $\rho(rx + (1 - r)y) \leqslant r\rho(x) + (1 - r)\rho(y)$ for all $r \in \mathbb{R}$ with $0 \leqslant r \leqslant 1$;
2. *Monotonicity*: i. e. $x \leqslant y$ implies $\rho(y) \leqslant \rho(x)$;
3. *Cash-invariance*: i. e. $\rho(x + r) = \rho(x) - r$ for all $r \in \mathbb{R}$.

Such a function ρ is called a *convex risk measure*. The notion of convex risk measure was independently introduced in [36] and [37] as a generalization of the notion of *coherent risk measure* introduced in [1].

* Solid subspaces are also called *order ideals*.

Associated to the model space $\mathcal{X}$, we can consider a dual pair. Namely, the Köthe dual space of $\mathcal{X}$ is defined by

$$\mathcal{X}^{\#} := \left\{ y \in L^0(\mathcal{E}) \colon xy \in L^1(\mathcal{E}) \text{ for all } x \in \mathcal{X} \right\}.$$

Then $\mathcal{X}^{\#}$ is also a solid subspace of $L^1(\mathcal{E})$ with $\mathbb{R} \subset \mathcal{X}^{\#}$. This gives rise to the dual pair $\langle \mathcal{X}, \mathcal{X}^{\#} \rangle$ associated to the bilinear form $(x, y) \mapsto \mathbb{E}_{\mathbb{P}}[xy]$ and the weak topologies $\sigma(\mathcal{X}, \mathcal{X}^{\#})$ and $\sigma(\mathcal{X}^{\#}, \mathcal{X})$.

If $\mathcal{X} = L^p(\mathcal{E})$ with $1 \leqslant p \leqslant \infty$, it is known that $\mathcal{X}^{\#} = L^q(\mathcal{E})$, where $q := (1 - 1/p)^{-1}$ is the Hölder conjugate of p, see e.g. [38, Example 29.4]. Suppose that ϕ is a Young function and let $\psi(r) := \sup_{s \geqslant 0} \{rs - \phi(s)\}$ be the conjugate Young function of ϕ. If $\mathcal{X} = L^{\phi}(\mathcal{E})$, then one has that $\mathcal{X}^{\#} = L^{\psi}(\mathcal{E})$, see e.g. [39, 40]. If $\mathcal{X} = H^{\phi}(\mathcal{E})$ and ϕ is finite-valued (otherwise $H^{\phi}(\mathcal{E}) = \{0\}$), then $\mathcal{X}^{\#} = L^{\psi}(\mathcal{E})$, see e.g. [41].

The *Fenchel transform* of a convex risk measure ρ is defined to be

$$\rho^{\#}(y) := \sup \left\{ \mathbb{E}_{\mathbb{P}}[xy] - \rho(x) \colon x \in \mathcal{X} \right\}.$$

Duality theory of convex risk measures is aimed to study when the Fenchel transform is involutive. More precisely, given a convex risk measure ρ say that:

- ρ is *representable* if it admits the following dual representation:

$$\rho(x) = \sup \left\{ \mathbb{E}_{\mathbb{P}}[xy] - \rho^{\#}(y) \colon y \in \mathcal{X}^{\#} \right\} \quad \text{for all } x \in \mathcal{X}.$$

- ρ attains its representation whenever for any $x \in \mathcal{X}$ there exists a $y \in \mathcal{X}^{\#}$ such that

$$\rho(x) = \mathbb{E}_{\mathbb{P}}[xy] - \rho^{\#}(y).$$

Remark 1.1. We have that $\rho^{\#}(y) < \infty$ only if $y \leqslant 0$ and $\mathbb{E}_{\mathbb{P}}[y] = -1$.* Thus ρ is representable if and only if

$$\rho(x) = \sup \left\{ \mathbb{E}_{\mathbb{P}}[xy] - \rho^{\#}(y) \colon y \in \mathcal{X}^{\#},\ y \leqslant 0,\ \mathbb{E}_{\mathbb{P}}[y] = -1 \right\} \quad \text{for all } x \in \mathcal{X}. \tag{1}$$

Notice that an element $y \in L^1(\mathcal{E})$ with $y \leqslant 0$ and $\mathbb{E}_{\mathbb{P}}[y] = -1$ can be identified with a probability measure $Q_y \ll \mathbb{P}$ via the Radon-Nikodym derivative $y = -\frac{dQ_y}{d\mathbb{P}}$. The economic interpretation of the representation (1) is that a convex risk measure can be seen as a stress test of the financial position x among the different market models given by the probabilities Q_y and the penalty function $\rho^{\#}$.

Next, we recall some properties that matter in duality theory of convex risk measures:

A function f from $\mathcal{X}$ to the extended real numbers $\overline{\mathbb{R}}$ is said to be *proper* if $f > -\infty$ and $f(x) < \infty$ for at least one $x \in \mathcal{X}$. Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be proper. For any $r \in \mathbb{R}$, we define the sublevel set

$$V_r(f) := \{x \in \mathcal{X} \colon f(x) \leqslant r\}.$$

Say that:

- f has the *Fatou property* if

$$\lim_n x_n = x \text{ a.s., } y \in \mathcal{X},\ |x_n| \leqslant y \text{ for all } n \in \mathbb{N} \text{ implies } \liminf_n f(x_n) \geqslant f(x);$$

* Indeed, suppose that $y \in X^{\#}$ and fix $n \in \mathbb{N}$. Then $\rho^{\#}(y) \geqslant \mathbb{E}_{\mathbb{P}}[n 1_{\{y \geqslant 0\}} y] - \rho(n 1_{\{y \geqslant 0\}}) \geqslant n \mathbb{E}_{\mathbb{P}}[y^+] - \rho(0)$. Since n is arbitrary, $\rho^{\#}(y) < \infty$ only if $y^+ = 0$. We also have that $\rho^{\#}(y) \geqslant \mathbb{E}_{\mathbb{P}}[ny] - \rho(n) \geqslant n(\mathbb{E}_{\mathbb{P}}[y] + 1) - \rho(0)$. Being n arbitrary, we conclude that $\rho^{\#}(y) < \infty$ only if $\mathbb{E}_{\mathbb{P}}[y] = -1$.

• f has the *Lebesgue property* if

$$\lim_n x_n = x \text{ a.s.}, \ y \in \mathcal{X}, \ |x_n| \leqslant y \text{ for all } n \in \mathbb{N} \text{ implies } \lim_n f(x_n) = f(x);$$

• f is *law invariant* if $f(x) = f(y)$ whenever x and y have the same law (i. e. $\mathbb{P}(x \leqslant r) = \mathbb{P}(y \leqslant r)$ for each $r \in \mathbb{R}$);

• f is *lower semicontinuous* w.r.t. $\sigma(\mathcal{X}, \mathcal{X}^{\#})$, if $V_r(f)$ is closed w.r.t. $\sigma(\mathcal{X}, \mathcal{X}^{\#})$ for each $r \in \mathbb{R}$;

• f is *inf-compact* w.r.t. $\sigma(\mathcal{X}, \mathcal{X}^{\#})$, if $V_r(f)$ is compact w.r.t. $\sigma(\mathcal{X}, \mathcal{X}^{\#})$ for each $r \in \mathbb{R}$.

1.2. Multi-Period Setup: Conditional Risk Measures. The notion of conditional risk measure was independently introduced by [17] and [18]. Next, we recall the main elements of duality theory of conditional risk measures. Namely, we adopt the module-based approach introduced in [19, Section 3].

Now, we suppose that $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{E}$, which models the available market information at some future date $t \in (0, T)$. Let us introduce some notation. We denote by $L^0_+(\mathcal{F})$, $L^0_{++}(\mathcal{F})$, and $\overline{L^0}(\mathcal{F})$ the spaces of (classes of equivalence of) $\mathcal{F}$-measurable random variables with values in the intervals $[0, \infty)$, $(0, \infty)$, and $[-\infty, \infty]$, respectively.

Let $\bar{\mathcal{F}}$ be the probability algebra associated to $(\Omega, \mathcal{F}, \mathbb{P})$, where $\bar{\mathcal{F}}$ is defined by identifying events modulo null sets. It is well-known that $\bar{\mathcal{F}}$ is a complete Boolean algebra which satisfies the countable chain condition (ccc), i. e. every family of positive pairwise disjoint elements in $\bar{\mathcal{F}}$ is at most countable. The 0 of $\bar{\mathcal{F}}$ is represented by the empty set $\varnothing$ and the unity I of $\bar{\mathcal{F}}$ is represented by Ω. We denote by $p(I)$ the set of all partitions of I to $\bar{\mathcal{F}}$. Given $a \in \bar{\mathcal{F}}$, we write 1_a for the class in $L^0(\mathcal{F})$ of the characteristic function 1_A of some representative $A \in \mathcal{F}$ of a. Given a partition $(a_k)_{k\in\mathbb{N}} \in p(I)$ and a sequence $(x_k)_{k\in\mathbb{N}}$, we define $\sum 1_{a_k} x_k := \lim_k \sum_{i=1}^k 1_{a_i} x_i$ a.s.

Classically, the conditional expectation $\mathbb{E}_{\mathbb{P}}[\cdot|\mathcal{F}]$ is defined for elements with finite expectation. We consider the extended conditional expectation. Namely, suppose that $x \in L^0(\mathcal{E})$ satisfies that at least one of $\lim_n \mathbb{E}_{\mathbb{P}}[x^+ \wedge n|\mathcal{F}]$ and $\lim_n \mathbb{E}_{\mathbb{P}}[x^- \wedge n|\mathcal{F}]$ (a.s.) is finite, then we define the *extended conditional expectation* of x to be

$$\mathbb{E}_{\mathbb{P}}[x|\mathcal{F}] := \lim_n \mathbb{E}_{\mathbb{P}}[x^+ \wedge n|\mathcal{F}] - \lim_n \mathbb{E}_{\mathbb{P}}[x^- \wedge n|\mathcal{F}] \in \overline{L^0}(\mathcal{F}).$$

Now, our model space is an $L^0(\mathcal{F})$-submodule $\mathscr{X}$ of $L^0(\mathcal{E})$ which satisfies the following properties:

• $\mathscr{X}$ is solid;

• $\mathbb{E}_{\mathbb{P}}[|x||\mathcal{F}] < \infty$ for all $x \in \mathscr{X}$;

• $L^0(\mathcal{F}) \subset \mathscr{X}$;

• $\mathscr{X}$ is *stable*, that is, $\sum 1_{a_k} x_k \in \mathscr{X}$ whenever $(a_k) \in p(I)$ and $(x_k) \subset \mathscr{X}$.

Example 1.2. The following $L^0(\mathcal{F})$-submodules of $L^0(\mathcal{E})$ satisfy the properties above:

1. *L^p type modules* (see [23]): We define

$$L^\infty_{\mathcal{F}}(\mathcal{E}) := \left\{x \in L^0(\mathcal{E}) \colon |x| \leqslant \eta, \text{ for some } \eta \in L^0(\mathcal{F})\right\},$$

if $1 \leqslant p < \infty$, let

$$L^p_{\mathcal{F}}(\mathcal{E}) := \left\{x \in L^0(\mathcal{E}) \colon \mathbb{E}_{\mathbb{P}}\left[|x|^p \,|\mathcal{F}\right] < \infty\right\}.$$

2. *Orlicz type modules* (see [42]) and *Orlicz-heart type modules*: Let $\phi : [0, \infty) \to [0, \infty]$ be a Young function and let

$$L^\phi_{\mathcal{F}}(\mathcal{E}) := \left\{x \in L^0(\mathcal{E}) : (\exists\, \eta \in L^0_{++}(\mathcal{F}))\ \mathbb{E}_{\mathbb{P}}[\phi(\eta|x|)|\mathcal{F}] \in L^0(\mathcal{F})\right\} \quad \text{(Orlicz type module)},$$

$H^{\phi}_{\mathcal{F}}(\mathcal{E}) := \{x \in L^0(\mathcal{E}) : (\forall \eta \in L^0_{++}(\mathcal{F}))\ \mathbb{E}_{\mathbb{P}}[\phi(\eta|x|)|\mathcal{F}] \in L^0(\mathcal{F})\}$ (Orlicz-heart type module).

Our model module $\mathscr{X}$ is going to describe all possible final payoffs of the positions at T.

The riskiness at time t of any financial position $x \in \mathscr{X}$ is uncertain and contingent to the information encoded in $\mathcal{F}$. Thus the riskiness is quantified by a function $\rho : \mathscr{X} \to L^0(\mathcal{F})$ which satisfies:

1. $L^0(\mathcal{F})$-*convexity*: $\rho(\eta x + (1-\eta)y) \leqslant \eta\rho(x) + (1-\eta)\rho(y)$ whenever $\eta \in L^0(\mathcal{F})$ with $0 \leqslant \eta \leqslant 1$ and $x, y \in \mathscr{X}$;
2. *Monotonicity*: if $x \leqslant y$ in $\mathscr{X}$, then $\rho(y) \leqslant \rho(x)$;
3. $L^0(\mathcal{F})$-*cash invariance*: $\rho(x+\eta) = \rho(x) - \eta$ whenever $\eta \in L^0(\mathcal{F})$, $x \in \mathscr{X}$.

Such a function is called a *conditional risk measure*.

Dual systems of modules were introduced and studied in [43]. Associated to the model space $\mathscr{X}$, we can consider a dual system of $L^0(\mathcal{F})$-modules. Namely, we define the *Köthe dual* $L^0(\mathcal{F})$-*module of* $\mathscr{X}$ to be

$$\mathscr{X}^{\#} := \{y \in L^0(\mathcal{E}) \colon xy \in L^1_{\mathcal{F}}(\mathcal{E}) \text{ for all } x \in \mathscr{X}\}.$$

It is simple to verify that $\mathscr{X}^{\#}$ enjoys the same properties as $\mathscr{X}$; namely, $\mathscr{X}^{\#}$ is a solid and stable $L^0(\mathcal{F})$-submodule with $L^0(\mathcal{F}) \subset \mathscr{X}^{\#} \subset L^1_{\mathcal{F}}(\mathcal{E})$.

The dual system $\langle \mathscr{X}, \mathscr{X}^{\#}\rangle$ allows for the definition of the following module analogue of the Fenchel transform:

$$\rho^{\#}(y) := \sup\{\mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}] - \rho(x) \colon x \in \mathscr{X}\} \quad \text{for } y \in \mathscr{X}^{\#}.$$

Again, we are interested in the involutivity of the Fenchel transform. Thus we introduce the following nomenclature: Given a conditional risk measure $\rho : \mathscr{X} \to L^0(\mathcal{F})$, say that:

- ρ is *representable* if

$$\rho(x) = \sup\{\mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}] - \rho^{\#}(y) \colon y \in \mathscr{X}^{\#}\} \quad \text{for all } x \in \mathscr{X}.$$

- ρ attains its representation if for any $x \in \mathscr{X}$ there exists $y \in \mathscr{X}^{\#}$ such that

$$\rho(x) = \mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}] - \rho^{\#}(y).$$

REMARK 1.2. Due to [19, Corollary 3.14], a conditional risk measure ρ is representable if and only if

$$\rho(x) = \sup\{\mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}] - \rho^{\#}(y) : y \in \mathscr{X}^{\#},\ y \leqslant 0,\ \mathbb{E}_{\mathbb{P}}[y|\mathcal{F}] = -1\} \quad \text{for all } x \in \mathscr{X}.$$

Next, we will introduce some notions that are useful in the dual representation of a conditional risk measure.

Given the dual system of $L^0(\mathcal{F})$-modules $\langle \mathscr{X}, \mathscr{X}^{\#}\rangle$, we can define the so-called *stable weak topologies* induced by $\langle \mathscr{X}, \mathscr{X}^{\#}\rangle$. Namely, given a partition $(a_k) \in p(I)$, a family (F_k) of non-empty finite subsets of $\mathscr{X}^{\#}$, and $\varepsilon \in L^0_{++}(\mathcal{F})$, we define

$$U_{(F_k),(a_k),\varepsilon} := \left\{x \in \mathscr{X} \colon \sum 1_{a_k} \sup_{y \in F_k} |\mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}]| < \varepsilon\right\}.$$

The collection of sets

$$\mathscr{B}_{\sigma_s(\mathscr{X},\mathscr{X}^{\#})} := \left\{x + U_{(F_k),(a_k),\varepsilon} \colon x \in \mathscr{X},\ (a_k) \in p(I),\ \varnothing \neq F_k \subset \mathscr{X}^{\#} \text{ finite},\ \varepsilon \in L^0_{++}(\mathcal{F})\right\}$$

is a base for a topology on $\mathscr{X}$, which will be denoted by $\sigma_s(\mathscr{X}, \mathscr{X}^\#)$. Similarly, we define $\sigma_s(\mathscr{X}^\#, \mathscr{X})$.

Stable weak topologies were introduced in [43, 1.1.8] as the topology induced by the multinorm associated to a dual system of modules. Also, they are the transcription in a modular setting of the conditional weak topologies introduced in [26].

A function f from $\mathscr{X}$ to $\overline{L^0}(\mathcal{F})$ is said to be *proper* if $f(x) > -\infty$ for all $x \in \mathscr{X}$ and $f(x_0) \in L^0(\mathcal{F})$ for at least one $x_0 \in \mathscr{X}$. Let $f : \mathscr{X} \to \overline{L^0}(\mathcal{F})$ be proper. For any $\eta \in L^0(\mathcal{F})$, we define the sublevel set

$$V_\eta(f) := \{x \in \mathscr{X} : f(x) \leqslant \eta\}.$$

Say that:

• f has the *Fatou property* if

$$\lim_n x_n = x \text{ a.s., } y \in \mathscr{X},\ |x_n| \leqslant y \text{ for all } n \in \mathbb{N} \text{ implies } \liminf_n f(x_n) \geqslant f(x);$$

• f has the *Lebesgue property* if

$$\lim_n x_n = x \text{ a.s., } y \in \mathscr{X},\ |x_n| \leqslant y \text{ for all } n \in \mathbb{N} \text{ implies } \lim_n f(x_n) = f(x) \text{ a.s.};$$

• f is *conditionally law invariant* is $f(x) = f(y)$ whenever x and y have the same *conditional law* (i. e. $\mathbb{P}(x \leqslant \eta|\mathcal{F}) = \mathbb{P}(y \leqslant \eta|\mathcal{F})$ for each $\eta \in L^0(\mathcal{F})$);

• lower semicontinuous w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^\#)$ if $V_\eta(f)$ is closed w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^\#)$ for every $\eta \in L^0(\mathcal{F})$.

Next, we will recall some notions that we will be needed later.

• A non-empty subset S of $L^0(\mathcal{E})$ is *stable* if $\sum 1_{a_k} x_k \in S$ whenever $(x_k) \subset S$ and $(a_k) \in p(I)$.

• A collection $\mathscr{B}$ of non-empty subsets of $L^0(\mathcal{E})$ is said to be *stable* if for any sequence (S_k) of members of $\mathscr{B}$ and any partition $(a_k) \in p(I)$ one has

$$\sum 1_{a_k} S_k = \left\{ \sum_k 1_{a_k} x_k \colon x_k \in S_k \text{ for all } k \right\} \in \mathscr{B}.$$

• A *stable filter base* is a filter base $\mathscr{B}$ on $L^0(\mathcal{E})$ such that $\mathscr{B}$ is a stable collection consisting of stable subsets of $L^0(\mathcal{E})$.

• A non-empty subset S of $\mathscr{X}$ is *stably compact* with respect to $\sigma_s(\mathscr{X}, \mathscr{X}^\#)$, if S is stable and any stable filter base $\mathscr{B}$ on S has a cluster point $x \in S$ w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^\#)$.

• A proper function $f : \mathscr{X} \to \overline{L^0}(\mathcal{E})$ is said to be *stably inf-compact* w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^\#)$ if $V_\eta(f)$ is stably compact w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^\#)$ for every $\eta \in L^0(\mathcal{F})$ such that $V_\eta(f) \neq \varnothing$.

The notion of stability is crucial in some related frameworks. In Boolean valued analysis it is used the terminology *cyclic* or *universally complete* $\mathcal{A}$-sets (here $\mathcal{A}$ is any complete Boolean algebra, for instance we can take $\mathcal{A} = \bar{\mathcal{F}}$), see [34]. In particular, in the case of dual systems of modules this notion was introduced in [43].

In conditional set theory it is used the terminology stable set and stable collection, see [26]. Actually, the notion of conditional set is a reformulation of that of cyclic $\mathcal{A}$-set. However, it should be mentioned that conditional set theory provides us with an intuitive and useful tool for dealing with $\mathcal{A}$-sets and their Boolean valued representation. In theory of L^0-modules the notion of stability is called the *countable concatenation property*, see [25].

Stable compactness was first time studied by Kusraev [44] under the name of *cyclic compactness*. The notion of stable filter base and stable compactness were defined in [26]. The transcriptions of these notions in L^0-modules are studied in [45].

2. Foundations of Boolean Valued Models

The precise formulation of Boolean valued models requires some familiarity with the basics of set theory and logic, and in particular with first-order logic, ordinals and transfinite induction. For the convenience of the reader, we will give some background of this theory. For a more detailed description we refer the reader to [34].

Let us consider a universe of sets V satisfying the axioms of the Zermelo–Fraenkel set theory with the axiom of choice (ZFC), and a first-order language $\mathcal{L}$, which allows for the formulation of statements about the elements of V. In the universe V we have all possible mathematical objects (real numbers, topological spaces, and so on). The language $\mathcal{L}$ consists of names for the elements of V together with a finite list of symbols for logic symbols ($\forall$, $\wedge$, $\neg$ and parenthesis), variables and the predicates $=$ and $\in$. Though we usually use a much richer language by introducing more and more intricate definitions, in the end any usual mathematical statement can be written using only those mentioned. The elements of the universe V are classified into a transfinite hierarchy: $V_0 \subset V_1 \subset V_2 \subset \cdots V_\omega \subset V_{\omega+1} \subset \cdots$, where $V_0 = \varnothing$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ is the family of all sets whose elements come from V_α, and $V_\beta = \bigcup_{\alpha<\beta} V_\alpha$ for limit ordinal β.

The following constructions and principles work for any complete Boolean algebra $\mathcal{A}$, even if it is not associated to a probability space or even does not have the countable chain condition. However, for the sake of simplicity, we will consider our underlying probability algebra $\mathcal{A} := \bar{\mathcal{F}}$, which encodes the future market information.

We will construct $V^{(\mathcal{A})}$, the *Boolean valued model* of $\mathcal{A}$, whose elements we interpret as objects which we can talk about at the future time t. We proceed by induction over the class Ord of ordinals of the universe V. We start by defining $V_0^{(\mathcal{A})} := \varnothing$. If $\alpha + 1$ is the successor of the ordinal α, we define

$$V_{\alpha+1}^{(\mathcal{A})} := \left\{ u\colon u \text{ is an } \mathcal{A}\text{-valued function with } \operatorname{dom}(u) \subset V_\alpha^{(\mathcal{A})} \right\}.$$

If α is a limit ordinal $V_\alpha^{(\mathcal{A})} := \bigcup\limits_{\xi<\alpha} V_\xi^{(\mathcal{A})}$. Finally, let $V^{(\mathcal{A})} := \bigcup\limits_{\alpha\in\mathrm{Ord}} V_\alpha^{(\mathcal{A})}$.

The idea is that any member v of the class $V^{(\mathcal{A})}$ is a *fuzzy* set in the sense that, for $v \in \operatorname{dom}(u)$, v will become an element of u at the future time t if $u(v)$ happens. Given u in $V^{(\mathcal{A})}$, we define its *rank* as the least ordinal α such that u is in $V_{\alpha+1}^{(\mathcal{A})}$.

We consider a first-order language which allows us to produce statements about $V^{(\mathcal{A})}$. Namely, let $\mathcal{L}^{(\mathcal{A})}$ be the first-order language which is the extension of $\mathcal{L}$ by adding *names* for each element in $V^{(\mathcal{A})}$. Throughout, we will not distinguish between an element in $V^{(\mathcal{A})}$ and its name in $\mathcal{L}^{(\mathcal{A})}$. Thus, hereafter, the members of $V^{(\mathcal{A})}$ will be referred to as names.

Suppose that φ is a formula in set theory, that is, φ is constructed by applying logical symbols to atomic formulas $u = v$ and $u \in v$. If φ does not have any free variable and all the constants in φ are names in $V^{(\mathcal{A})}$, then we define its *Boolean truth value*, say $[\![\varphi]\!]$, which is a member of $\mathcal{A}$ and is constructed by induction in the length of φ by naturally giving Boolean meaning to the predicates $=$ and $\in$, the logical connectives and the quantifiers.

We start by defining the Boolean truth value of the *atomic formulas* $u \in v$ and $u = v$ for u and v in $V^{(\mathcal{A})}$. Namely, proceeding by transfinite recursion we define

$$[\![u \in v]\!] = \bigvee_{t \in \mathrm{dom}(v)} v(t) \wedge [\![t = u]\!],$$

$$[\![u = v]\!] = \bigwedge_{t \in \mathrm{dom}(u)} \big(u(t) \Rightarrow [\![t \in v]\!]\big) \wedge \bigwedge_{t \in \mathrm{dom}(v)} \big(v(t) \Rightarrow [\![t \in u]\!]\big),$$

where, for $a, b \in \mathcal{A}$, we denote $a \Rightarrow b := a^c \vee b$. For non-atomic formulas we have

$$[\![(\exists\, x)\varphi(x)]\!] := \bigvee_{u \in V^{(\mathcal{A})}} [\![\varphi(u)]\!] \quad \text{and} \quad [\![(\forall\, x)\varphi(x)]\!] := \bigwedge_{u \in V^{(\mathcal{A})}} [\![\varphi(u)]\!];$$

$$[\![\varphi \vee \psi]\!] := [\![\varphi]\!] \vee [\![\psi]\!], \quad [\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \wedge [\![\psi]\!], \quad [\![\varphi \Rightarrow \psi]\!] := [\![\varphi]\!]^c \vee [\![\psi]\!], \quad [\![\neg\varphi]\!] := [\![\varphi]\!]^c.$$

We say that a formula φ is satisfied within $V^{(\mathcal{A})}$, and write $V^{(\mathcal{A})} \models \varphi$, whenever it is true with the Boolean truth value, that is, $[\![\varphi]\!] = I$.

We say that two names u, v are equivalent when $[\![u = v]\!] = I$. It is not difficult to verify that the Boolean truth value of a formula is not affected when we change a name by an equivalent one. However, the relation $[\![u = v]\!] = I$ does not mean that the functions u and v (considered as elements of V) coincide. For example, the empty function $u := \varnothing$ and the function $v : \{\varnothing\} \to \mathcal{A}$ $v(\varnothing) := 0$ are different as functions; however, $[\![u = v]\!] = I$. In order to avoid technical difficulties, we will consider the so-called *separated universe*. Namely, let $\overline{V}^{(\mathcal{A})}$ be the subclass of $V^{(\mathcal{A})}$ defined by choosing a representative of the least rank in each class of the equivalence relation $\{(u, v)\colon [\![u = v]\!] = I\}$.

The universe V can be embedded into $V^{(\mathcal{A})}$. Given a set x in V, we define its canonical name $\check{x}$ in $V^{(\mathcal{A})}$ by transfinite induction. Namely, we put $\check{\varnothing} := \varnothing$ and for x in V we define $\check{x}$ to be the unique representative in $\overline{V}^{(\mathcal{A})}$ of the name given by the function

$$\{\check{y}\colon y \in x\} \to \mathcal{A}, \quad \check{y} \mapsto I.$$

Given a name u with $[\![u \neq \varnothing]\!] = I$ we define its *descent* by

$$u{\downarrow} := \big\{v \in \overline{V}^{(\mathcal{A})}\colon [\![v \in u]\!] = I\big\}.$$

$V^{(\mathcal{A})}$ is a model of ZFC. More precisely we have:

Theorem 2.1 (Transfer Principle)**.** *If φ is a theorem of ZFC, then $V^{(\mathcal{A})} \models \varphi$.*

Other two important principles are the following:

Theorem 2.2 (Maximum Principle)**.** *Let $\varphi(x_1, \dots, x_n)$ be a formula with free variables $x_1, \dots, x_n$. Then there exist names $u_1, \dots, u_n$ such that $[\![\varphi(u_1, \dots, u_n)]\!] = [\![(\exists\, x_1) \dots (\exists\, x_n)\varphi(x_1, \dots, x_n)]\!]$.*

Theorem 2.3 (Mixing Principle)**.** *Let $(a_k) \in p(I)$ and let (u_k) be a sequence of names. Then there exists a unique member u of $\overline{V}^{(\mathcal{A})}$ such that $[\![u = u_k]\!] \geqslant a_k$ for all $k \in \mathbb{N}$.*

Given a partition $(a_k) \in p(I)$ and a sequence (u_k) of elements of $V^{(\mathcal{A})}$, we denote by $\sum u_k a_k$, the unique name u in $\overline{V}^{(\mathcal{A})}$ satisfying $[\![u = u_k]\!] \geqslant a_k$ for all $k \in \mathbb{N}$.

The following result is very useful to manipulate Boolean truth values:

Proposition 2.1. *Let $\varphi(x)$ be a formula with a free variable x and v a name with $[\![v \neq \varnothing]\!] = I$. Then:*

$$[\![(\forall\, x \in v)\varphi(x)]\!] = \bigwedge_{u \in v\downarrow} [\![\varphi(u)]\!], \quad [\![(\exists\, x \in v)\varphi(x)]\!] = \bigvee_{u \in v\downarrow} [\![\varphi(u)]\!].$$

Moreover, one has

1. $[\![(\forall\, x \in v)\varphi(x)]\!] = I$ *if and only if* $[\![\varphi(u)]\!] = I$ *for all* $u \in v\downarrow$;
2. $[\![(\exists\, x \in v)\varphi(x)]\!] = I$ *if and only if there exists* $u \in v\downarrow$ *such that* $[\![\varphi(u)]\!] = I$.

In general, in the universe $V^{(\mathcal{A})}$ we have all possible mathematical objects (real numbers, topological spaces, and so on). If u is a name which satisfies $[\![u \text{ is a function}]\!] = I$, that is, u satisfies the definition of function in the language $\mathcal{L}^{(\mathcal{A})}$, then we say that u is a *name for a function*. Of course, this can be done for any mathematical concept. Thus, in the sequel of this article, we will use the terminology *name for a vector space*, *name for a topology*, and so on without further explanations.

DEFINITION 2.1. Suppose that u, v are two names with $[\![(u \neq \varnothing) \wedge (v \neq \varnothing)]\!] = I$. A function $f : u\downarrow \to v\downarrow$ such that

$$[\![w = t]\!] \leqslant [\![f(w) = f(t)]\!] \quad \text{for all } w, t \in u\downarrow$$

is called *extensional*.

Extensional functions allows for the definition of names for functions. More precisely, we have the following:

Proposition 2.2. *Let u, v be names with $[\![(u \neq \varnothing) \wedge (v \neq \varnothing)]\!] = I$ and suppose that $f : u\downarrow \to v\downarrow$ is an extensional function. Then there exists a name $f\uparrow$ for a function from u to v, such that $[\![f\uparrow(t) = f(t)]\!] = I$ for all $t \in u\downarrow$.*

3. A Transfer Principle Between Duality Theory of Convex Risk Measure and Duality Theory of Conditional Risk Measures

Let us go back to our model probability space $(\Omega, \mathcal{E}, \mathbb{P})$ with $\mathcal{F} \subset \mathcal{E}$. Next, we state the main result of the present paper, which allows for the interpretation of a conditional risk measure $\rho : \mathscr{X} \to L^0(\mathcal{F})$ as a name for a convex risk measure, let us say $\rho\uparrow$, defined on some space of random variables, and relates the properties of ρ with the properties of $\rho\uparrow$ in the set-theoretic model $V^{(\mathcal{A})}$. In other words, this result establishes a transfer principle between duality theory of convex risk measures and duality theory of conditional risk measures.

Theorem 3.1. *Let $\rho : \mathscr{X} \to L^0(\mathcal{F})$ be a conditional risk measure. Then there exist members $\rho\uparrow$ and $\mathscr{X}\uparrow$ of $V^{(\mathcal{A})}$ such that*

$$\begin{aligned} V^{(\mathcal{A})} \models\ & \textit{there exists a probability space } (X, \Sigma, Q) \textit{ such that,} \\ & \mathscr{X}\uparrow \textit{ is a solid subspace of } L^1(\Sigma) \textit{ with } \mathbb{R} \subset \mathscr{X}\uparrow, \\ & \textit{and } \rho\uparrow : \mathscr{X}\uparrow \to \mathbb{R} \textit{ is a convex risk measure,} \end{aligned}$$

and so that the names $\rho\uparrow$ and $\mathscr{X}\uparrow$ satisfy the following:

1. ρ *is representable iff* $[\![\rho\uparrow \text{ is representable}]\!] = I$.
2. ρ *attains its representation iff* $[\![\rho\uparrow \text{ attains its representation}]\!] = I$.
3. ρ *has the Fatou property iff* $[\![\rho\uparrow \text{ has the Fatou property}]\!] = I$.

4. *ρ has the Lebesgue property iff* $[\![\rho{\uparrow}$ has the Lebesgue property$]\!] = I$.

5. *ρ is conditionally law invariant iff* $[\![\rho{\uparrow}$ is law invariant$]\!] = I$.

6. *ρ is lower semicontinuous w.r.t $\sigma_s(\mathscr{X}, \mathscr{X}^\#)$ iff*

$$[\![\rho{\uparrow} \text{ is lower semicontinuous w.r.t. } \sigma(\mathscr{X}{\uparrow}, \mathscr{X}{\uparrow}^\#)]\!] = I.$$

7. *$\rho^\#$ is stably inf-compact w.r.t $\sigma_s(\mathscr{X}^\#, \mathscr{X})$ iff*

$$[\![\rho{\uparrow}^\# \text{ is inf-compact w.r.t. } \sigma(\mathscr{X}{\uparrow}^\#, \mathscr{X}{\uparrow})]\!] = I.$$

8. *If $\mathscr{X} = L^p_{\mathcal{F}}(\mathcal{E})$ with $1 \leqslant p \leqslant \infty$, then $[\![\mathscr{X}{\uparrow} = L^p(\Sigma)]\!] = I$. In that case, $\mathscr{X}^\# = L^q_{\mathcal{F}}(\mathcal{E})$ where q is the Hölder conjugate of p.*

9. *If $\mathscr{X} = L^\phi_{\mathcal{F}}(\mathcal{E})$ with ϕ a Young function, then there is a name $\tilde{\phi}$ for a Young function such that $[\![\mathscr{X}{\uparrow} = L^{\tilde{\phi}}(\Sigma)]\!] = I$. In that case, $\mathscr{X}^\# = L^\psi_{\mathcal{F}}(\mathcal{E})$ where ψ is the conjugate Young function of ϕ.*

10. *If $\mathscr{X} = H^\phi_{\mathcal{F}}(\mathcal{E})$ with ϕ a finite-valued Young function, then there is a name $\tilde{\phi}$ for a finite-valued Young function such that $[\![\mathscr{X}{\uparrow} = H^{\tilde{\phi}}(\Sigma)]\!] = I$. In that case, $\mathscr{X}^\# = L^\psi_{\mathcal{F}}(\mathcal{E})$ where ψ is the conjugate Young function of ϕ.*

The proof of the theorem above is postponed to next section. Instead, we focus first on some instances of application.

Theorem 3.1 together with the transfer principle of Boolean-valued models allow for the interpretation of well-known results about the dual representation of convex risk measures as new theorems about the dual representation of conditional risk measures.

For example, suppose that $\rho : \mathcal{X} \to \mathbb{R}$ is a convex risk measure. As a consequence of the Fenchel-Moreau theorem (see [11, Theorem 2.1]) applied to the weak topology $\sigma(\mathcal{X}, \mathcal{X}^\#)$ we have that ρ is representable if and only if ρ is lower semicontinuous w.r.t. $\sigma(\mathcal{X}, \mathcal{X}^\#)$.

Moreover, we have the following dual representation result:

Theorem 3.2 [41, Theorem 1.1]. *Let $\rho : \mathcal{X} \to \mathbb{R}$ be a convex risk measure. Then ρ is representable if and only if ρ is lower semicontinuous w.r.t. $\sigma(\mathcal{X}, \mathcal{X}^\#)$. In that case, the following statements are equivalent:*

1. *ρ attains its representation;*
2. *ρ has the Lebesgue property;*
3. *$\rho^\#$ is inf-compact w.r.t. $\sigma(\mathcal{X}^\#, \mathcal{X})$.*

Let φ denote the theorem above. Due to the transfer principle, it is satisfied that $[\![\varphi]\!] = I$. In view of Theorem 3.1, we have that the statement below is just a reformulation of $[\![\varphi]\!] = I$, so no proof is needed.

Theorem 3.3. *Let $\rho : \mathscr{X} \to L^0(\mathcal{F})$ be a conditional risk measure. Then ρ is representable, i. e.*

$$\rho(x) = \sup\left\{\mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}] - \rho^\#(y) : y \in \mathscr{X}^\#,\ y \leqslant 0,\ \mathbb{E}_{\mathbb{P}}[y|\mathcal{F}] = -1\right\} \quad \text{for all } x \in \mathscr{X}$$

if and only if ρ is lower semicontinuous w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^\#)$.

In that case, the following are equivalent:

1. *ρ attains its representation, i. e. for every $x \in \mathscr{X}$ there exists $y \in \mathscr{X}^\#$ with $y \leqslant 0$ and $\mathbb{E}_{\mathbb{P}}[y|\mathcal{F}] = -1$ such that $\rho(x) = \mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}] - \rho^\#(y)$;*
2. *ρ has the Lebesgue property;*
3. *$\rho^\#$ is stably inf-compact w.r.t. $\sigma_s(\mathscr{X}^\#, \mathscr{X})$.*

Suppose that $\mathcal{X} = L^\infty(\mathcal{E})$. Then, the so-called Jouini–Schachermayer–Touzi theorem (see [46, Theorem 2] and for its original form see [10]) asserts that in Theorem 3.2 we can replace the lower semicontinuity by the Fatou property.* Thus, the transfer principle together

* Actually, the Fatou property is automatically satisfied when ρ is law invariant, see [10].

with Theorem 3.1 yields the following:

Theorem 3.4. *Let $\rho : L^\infty_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ be a conditional risk measure. Then ρ has the Fatou property if and only if it admits a representation*

$$\rho(x) = \sup\left\{\mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}] - \rho^{\#}(y) : \, y \in L^1_{\mathcal{F}}(\mathcal{E}), \, y \leqslant 0, \, \mathbb{E}_{\mathbb{P}}[y|\mathcal{F}] = -1\right\} \quad \text{for all } x \in \mathscr{X}.$$

In this case, the following are equivalent:

1. *ρ attains its representation;*
2. *ρ has the Lebesgue property;*
3. *$\rho^{\#}$ is stably inf-compact w.r.t. $\sigma_s(L^1_{\mathcal{F}}(\mathcal{E}), L^\infty_{\mathcal{F}}(\mathcal{E}))$.*

Suppose that $\mathcal{X} = L^p(\mathcal{E})$ with $1 \leqslant p < \infty$. In this case, every convex risk measure has the Lebesgue property, is representable and the representation is attained for every $x \in L^p(\mathcal{E})$ (see eg [11, Theorem 2.11]). Thus, we have:

Theorem 3.5. *Suppose that (p, q) are Hölder conjugates with $1 \leqslant p < \infty$. If $\rho : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is a conditional risk measure, then ρ has the Lebesgue property, and for every $x \in L^p_{\mathcal{F}}(\mathcal{E})$ there exists $y \in L^q_{\mathcal{F}}(\mathcal{E})$ with $y \leqslant 0$ and $\mathbb{E}_{\mathbb{P}}[y|\mathcal{F}] = -1$ such that $\rho(x) = \mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}] - \rho^{\#}(y)$.*

If $\mathcal{X} := L^\phi(\mathcal{E})$ with ϕ a Young function, due Theorem 3.1 we have that $\mathcal{X}^{\#} := L^\phi(\mathcal{E})$ and applying Theorem 3.3 we have the following:

Theorem 3.6. *Let (ϕ, ψ) be Young conjugate functions and $\rho : L^\phi_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ a conditional risk measure. Then ρ is representable, i. e.*

$$\rho(x) = \sup\{\mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}] - \rho^{\#}(y) \colon y \in L^\psi_{\mathcal{F}}(\mathcal{E}), \, y \leqslant 0, \, \mathbb{E}_{\mathbb{P}}[y|\mathcal{F}] = -1\} \quad \text{for all } x \in \mathscr{X}$$

if and only if ρ is lower semicontinuous w.r.t. $\sigma_s(L^\phi_{\mathcal{F}}(\mathcal{E}), L^\psi_{\mathcal{F}}(\mathcal{E}))$.

In that case, the following are equivalent:

1. *ρ attains its representation;*
2. *ρ has the Lebesgue property;*
3. *$\rho^{\#}$ is stably inf-compact w.r.t. $\sigma_s(L^\psi_{\mathcal{F}}(\mathcal{E}), L^\phi_{\mathcal{F}}(\mathcal{E}))$.*

If $\mathcal{X} := H^\phi(\mathcal{E})$ with ϕ finite-valued, then every convex risk measure on $H^\phi(\mathcal{E})$ has the Lebesgue property, is representable and the representation is attained for every $x \in H^\phi(\mathcal{E})$ (see eg [6, Theorem 4.4]). Thus, we have the following:

Theorem 3.7. *Let (ϕ, ψ) be Young conjugate functions with ϕ finite-valued and $\rho : H^\phi_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ a conditional risk measure. Then, ρ has the Lebesgue property, and for every $x \in H^\phi_{\mathcal{F}}(\mathcal{E})$ there exists $y \in L^\psi_{\mathcal{F}}(\mathcal{E})$ with $y \leqslant 0$ and $\mathbb{E}_{\mathbb{P}}[y|\mathcal{F}] = -1$ such that $\rho(x) = \mathbb{E}_{\mathbb{P}}[xy|\mathcal{F}] - \rho^{\#}(y)$.*

Note that all these theorems are just some examples: we can state a version of any theorem φ on duality theory of convex risk measures and it immediately renders a version for conditional risk measures of the form $[\![\varphi]\!] = I$.

Also, we would like to point out that the relations in Theorem 3.1 can be easily increased. Moreover, it is also possible to cover more general cases: conditional risk measures with values in $L^0(\mathcal{F}, \mathbb{R} \cup \{+\infty\})$, quasi-convex conditional risk measures and so on.

4. Sketch of the Proof of the Main Result

For saving space, we will give only a sketch of the proof of Theorem 3.1. A more detailed exposition can be found in [47, Chapter 4].

The set of real numbers is a definable notion of ZFC. We will denote by $\mathbb{R}_{\mathcal{A}}$ the unique name in $\overline{V}^{(\mathcal{A})}$ that satisfies the definition of real numbers, which exists due to the transfer and maximum principles. Likewise, we will denote by $\mathbb{N}_{\mathcal{A}}$ the unique name in $\overline{V}^{(\mathcal{A})}$ which satisfies the definition of natural numbers.

It is well-known that $L^0(\mathcal{F})$ is a Boolean valued interpretation of the real numbers, see [33, Chapter 2, Section 2]. More precisely, we can state this fact as follows: there is a bijection

$$\begin{array}{rccc} \imath: & L^0(\mathcal{F}) & \longrightarrow & \mathbb{R}_{\mathcal{A}}\!\downarrow \\ & \eta & \longmapsto & \eta^\bullet \\ & u^\circ & \longleftarrow & u \end{array}$$

such that the following is satisfied:

(i) $\imath(L^0(\mathcal{F},\mathbb{N})) = \mathbb{N}_{\mathcal{A}}\!\downarrow$ and $(\sum 1_{a_k} n_k)^\bullet = \sum \check{n}_k a_k$ whenever $(n_k) \subset \mathbb{N}$ and $(a_k) \in p(I)$;*

(ii) $[\![0^\bullet = 0]\!] = I$, $[\![1^\bullet = 1]\!] = I$, $[\![\eta^\bullet + \xi^\bullet = (\eta+\xi)^\bullet]\!] = I$ and $[\![\eta^\bullet \xi^\bullet = (\eta\xi)^\bullet]\!] = I$ for all $\eta, \xi \in L^0(\mathcal{F})$;

(iii) $[\![\eta^\bullet = \xi^\bullet]\!] = \bigvee\{a \in \mathcal{A}\colon 1_a\eta = 1_a\xi\}$ and $[\![\eta^\bullet \leqslant \xi^\bullet]\!] = \bigvee\{a \in \mathcal{A}\colon 1_a\eta \leqslant 1_a\xi\}$ for all $\eta, \xi \in L^0(\mathcal{F})$;

(iv) $(\sum 1_{a_k}\eta_k)^\bullet = \sum \eta_k^\bullet a_k$ for each $(\eta_k) \subset L^0(\mathcal{F})$ and $(a_k) \in p(I)$.

Now, write $\overline{\mathbb{R}}_{\mathcal{A}}$ for the unique name in $\overline{V}^{(\mathcal{A})}$ that satisfies the definition of the extended real numbers. Using the same techniques as in [33] it can be proved that the function $\imath$ extends to a bijection

$$\bar{\imath} : \overline{L^0}(\mathcal{F}) \to \overline{\mathbb{R}}_{\mathcal{A}}\!\downarrow, \quad \eta \mapsto \eta^\bullet$$

such that

$$[\![\eta^\bullet = \xi^\bullet]\!] = \bigvee\{a \in \mathcal{A}\colon 1_a\eta = 1_a\xi\},$$

where, by convention, we take above $0 \cdot (\pm\infty) = 0$.

Suppose that (η_n) is a sequence in $L^0(\mathcal{F})$. Then, we define $(\eta_{\mathfrak{n}})_{\mathfrak{n}\in L^0(\mathcal{F},\mathbb{N})}$ where $\eta_{\mathfrak{n}} = \sum_{k\in\mathbb{N}} 1_{\{\mathfrak{n}=k\}}\eta_k$. Then the function

$$\mathbb{N}_{\mathcal{A}}\!\downarrow \to \mathbb{R}_{\mathcal{A}}\!\downarrow, \quad \mathfrak{n}^\bullet \mapsto \eta_{\mathfrak{n}}^\bullet,$$

is extensional. Due to Proposition 2.2, we can find a name v with $[\![v : \mathbb{N} \to \mathbb{R}]\!] = I$ such that

$$[\![\eta_{\mathfrak{n}}^\bullet = v(\mathfrak{n}^\bullet)]\!] = I,$$

for all $\mathfrak{n}$. Moreover, we have the following:

Proposition 4.1 [48, Proposition 2.2.1]**.** *If (η_n) is a sequence in $L^0(\mathcal{F})$, then*

$$[\![(\liminf_n \eta_n)^\bullet = \liminf_n \eta_{n^\circ}^\bullet]\!] = I \quad \textit{and} \quad [\![(\limsup_n \eta_n)^\bullet = \limsup_n \eta_{n^\circ}^\bullet]\!] = I.$$

In particular, $\lim_n \eta_n = \eta$ *a.s. if and only if* $[\![\lim_n \eta_{n^\circ}^\bullet = \eta^\bullet]\!] = I$.

Suppose that $(X,\Sigma,Q)_{\mathcal{A}}$ is a name for a probability space.** We can consider the name $L^1(\Sigma)_{\mathcal{A}}$ for the space of classes of equivalence of random variables with finite expectation.

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* As usual, $L^0(\mathcal{F},\mathbb{N})$ denotes the set of classes of equivalence of $\mathbb{N}$-valued measurable functions.

** That $(X,\Sigma,Q)_{\mathcal{A}}$ is a name for a probability space means that X is a name for a set, Σ is a name for a σ-algebra on X, Q is a name for a probability measure on Σ, and $(X,\Sigma,Q)_{\mathcal{A}}$ denotes the corresponding ordered triple within $\overline{V}^{(\mathcal{A})}$.

Gordon [49, Theorem 5] proved that the conditional expectation $\mathbb{E}_{\mathbb{P}}[\cdot|\mathcal{F}]$ from $L^1(\mathcal{E})$ to $L^1(\mathcal{F})$ is a Boolean valued interpretation of the name for the expectation $\mathbb{E}_Q[\cdot]$ for some probability measure Q within $V^{(\mathcal{A})}$. We state this fact in the following proposition, whose self-contained proof can be found in [47, Section 4.1].

Proposition 4.2. *There exists a name $(X,\Sigma,Q)_{\mathcal{A}}$ for a probability space and a bijection*

$$\begin{array}{rccc} \jmath\colon & L^1_{\mathcal{F}}(\mathcal{E}) & \longrightarrow & L^1(\Sigma)_{\mathcal{A}}\!\downarrow \\ & x & \longmapsto & x^\bullet \\ & u^\circ & \longleftarrow & u \end{array}$$

such that:

1. $\jmath$ *extends the canonical isomorphism* $\imath$*;*
2. $[\![\mathbb{E}_{\mathbb{P}}[x|\mathcal{F}]^\bullet = \mathbb{E}_Q[x^\bullet]]\!] = I$ *for all* $x \in L^1_{\mathcal{F}}(\mathcal{E})$*;*
3. $[\![x^\bullet = y^\bullet]\!] = \bigvee\{a \in \mathcal{A}\colon 1_a x = 1_a y\}$ *for all* $x, y \in L^1_{\mathcal{F}}(\mathcal{E})$*;*
4. $[\![x^\bullet \leqslant y^\bullet]\!] = \bigvee\{a \in \mathcal{A}\colon 1_a x \leqslant 1_a y\}$ *for all* $x, y \in L^1_{\mathcal{F}}(\mathcal{E})$*;*
5. $[\![x^\bullet + y^\bullet]\!] = [\![(x+y)^\bullet]\!]$ *for all* $x, y \in L^1_{\mathcal{F}}(\mathcal{E})$*;*
6. $(\sum 1_{a_k} x_k)^\bullet = \sum x_k^\bullet a_k$ *for all* $(x_k) \subset L^1_{\mathcal{F}}(\mathcal{E})$ *and* $(a_k) \in p(I)$*.*

For the forthcoming discussion, we will fix a name for a probability space $(X,\Sigma,Q)_{\mathcal{A}}$ as in the theorem above.

Suppose that S is a stable subset of $L^1_{\mathcal{F}}(\mathcal{E})$. Let $S{\uparrow}$ denote the unique representative in $\overline{V}^{(\mathcal{A})}$ of the name given by the function

$$\{x^\bullet\colon x \in S\} \longrightarrow \mathcal{A}, \quad x^\bullet \mapsto I.$$

Using the mixing principle, it is not difficult to prove the following:

Proposition 4.3. *If S is a stable subset of $L^1_{\mathcal{F}}(\mathcal{E})$, then $S{\uparrow}$ is a name for a non-empty subset of $L^1_{\mathcal{F}}(\mathcal{E}){\uparrow}$, and the map $x \mapsto x^\bullet$ is a bijection from S to $S{\uparrow\downarrow}$. In particular, we have that $[\![L^1_{\mathcal{F}}(\mathcal{E}){\uparrow} = L^1(\Sigma)_{\mathcal{A}}]\!] = I$.*

Both $\mathscr{X}$ and $\mathscr{X}^{\#}$ are stable subsets of $L^1_{\mathcal{F}}(\mathcal{E})$. Thus, it makes sense to define $\mathscr{X}{\uparrow}$ and $\mathscr{X}^{\#}{\uparrow}$, which are the names that we refer to in the statement of Theorem 3.1. Indeed, bearing in mind the properties given in Proposition 4.2, a standard manipulation of Boolean truth values proves the following:

Proposition 4.4. *$\mathscr{X}{\uparrow}$ and $\mathscr{X}^{\#}{\uparrow}$ are names for solid subspaces of $L^1_{\mathcal{F}}(\mathcal{E}){\uparrow}$ with $[\![\mathbb{R} \subset \mathscr{X}{\uparrow}]\!] = I$ and $[\![\mathbb{R} \subset \mathscr{X}^{\#}{\uparrow}]\!] = I$. Moreover, $[\![\mathscr{X}^{\#}{\uparrow} = \mathscr{X}{\uparrow}^{\#}]\!] = I$.*

If p is a real number with $1 \leqslant p \leqslant \infty$, we have that its canonical inversion, $\check{p}$ in $\overline{V}^{(\mathcal{A})}$, satisfies that $[\![\check{p} = p]\!] = I$. Then we can consider the corresponding name, say $L^p(\Sigma)_{\mathcal{A}}$, for a L^p space within $V^{(\mathcal{A})}$.

Given a Young function ϕ, consider the function $\eta \mapsto \phi(\eta)\colon L^0(\mathcal{F},[0,\infty)) \to L^0(\mathcal{F},[0,\infty])$. Due to Proposition 2.2, we have a name $\tilde{\phi}$ for a Young function. Then we can consider the corresponding names for an Orlicz space and an Orlicz-heart space within $V^{(\mathcal{A})}$, say $L^{\tilde{\phi}}(\Sigma)_{\mathcal{A}}$ and $H^{\tilde{\phi}}(\Sigma)_{\mathcal{A}}$, respectively.

The following result tells us that L^p, Orlicz and Orlicz-heart type modules can be interpreted as classical L^p, Orlicz and Orlicz-heart spaces within $V^{(\mathcal{A})}$. Actually, general L^p type modules $L^p(\Phi)$ where Φ is a Maharam operator were introduced and their Boolean valued interpretation was provided in [43, 4.2.2]. In fact, a Maharam operator can be viewed as an abstract conditional expectation, see [50, Sections 5.2–5.4]; moreover, if Φ is the conditional expectation, then $L^p(\Phi)$ is precisely $L^p_{\mathcal{F}}(\mathcal{E})$.

By manipulation of Boolean truth values, and bearing in mind the properties given in Proposition 4.2, we can check the following:

Proposition 4.5. *If* $1 \leqslant p \leqslant \infty$, *then* $[\![L^p_{\mathcal{F}}(\mathcal{E})\!\uparrow = L^p(\Sigma)_{\mathcal{A}}]\!] = I$. *If* ϕ *is a Young function, then* $[\![L^{\phi}_{\mathcal{F}}(\mathcal{E})\!\uparrow = L^{\tilde{\phi}}(\Sigma)_{\mathcal{A}}]\!] = I$ *and* $[\![H^{\phi}_{\mathcal{F}}(\mathcal{E})\!\uparrow = H^{\tilde{\phi}}(\Sigma)_{\mathcal{A}}]\!] = I$.

Suppose that (x_n) is a sequence in $L^1_{\mathcal{F}}(\mathcal{E})$. For each $\mathfrak{n} \in L^0(\mathcal{F}, \mathbb{N})$ we define $x_{\mathfrak{n}} := \sum_{k\in\mathbb{N}} 1_{\{\mathfrak{n}=k\}} x_k$. Then the function

$$\mathbb{N}_{\mathcal{A}}\!\downarrow \longrightarrow L^1_{\mathcal{F}}(\mathcal{E})\!\uparrow\downarrow, \quad \mathfrak{n}^\bullet \mapsto x^\bullet_{\mathfrak{n}}$$

is extensional. Due to Proposition 2.2 we can find a name $(x_n)\!\uparrow$ for a sequence in $L^1_{\mathcal{F}}(\mathcal{E})\!\uparrow$. In addition, a standard manipulation of Boolean truth values proves the following:

Proposition 4.6. *Let* (x_n) *be a sequence in* $\mathscr{X}$ *such that* $|x_n| \leqslant y$ *for some* $y \in \mathscr{X}$. *Then* $[\![(\liminf_n x_n)^\bullet = \liminf_n x^\bullet_{n^\circ}]\!] = I$ *and* $[\![(\limsup_n x_n)^\bullet = \limsup_n x^\bullet_{n^\circ}]\!] = I$. *In particular,* $\lim_n x_n = x$ *a.s. in* $\mathscr{X}$ *if and only if* $[\![\lim_n x^\bullet_{n^\circ} = x^\bullet \text{ a.s. in } \mathscr{X}\!\uparrow]\!] = I$.

A function $f : \mathscr{X} \to \overline{L^0}(\mathcal{F})$ is said to have the *local property* if $1_a f(x) = 1_a f(1_a x)$ for all $a \in \mathcal{A}$ and $x \in \mathscr{X}$. It is not difficult to verify that if f has the local property, then the function

$$\mathscr{X}\!\uparrow\downarrow \longrightarrow \overline{\mathbb{R}}_{\mathcal{A}}\!\downarrow, \quad x^\bullet \mapsto f(x)^\bullet,$$

is extensional. Thus, we can find a name $f\!\uparrow$ for a function from $\mathscr{X}\!\uparrow$ to $\overline{\mathbb{R}}_{\mathcal{A}}$ such that $[\![f\!\uparrow(x^\bullet) = f(x)^\bullet]\!] = I$ for all $x \in \mathscr{X}$.

The following is a consequence of Propositions 4.1:

Proposition 4.7. *Let* $f : \mathscr{X} \to \overline{L^0}(\mathcal{F})$ *be a function with the local property. Then*

1. f *has the Fatou property iff* $[\![f\!\uparrow \text{ has the Fatou property}]\!] = I$;
2. f *has the Lebesgue property iff* $[\![f\!\uparrow \text{ has the Lebesgue property}]\!] = I$.

Proposition 2.1 together with the fact that $x \mapsto x^\bullet$ is a bijection from S to $S\!\uparrow\downarrow$ allows us to prove the following:

Proposition 4.8. *Let* $S \subset \mathscr{X}$ *be stable, let* $f : \mathscr{X} \to \overline{L^0}(\mathcal{F})$ *be a function with the local property. Then*

$$\left[\!\!\left[\sup_{u\in S\uparrow} f\!\uparrow(u) = \left(\sup_{x\in S} f(x)\right)^\bullet \right]\!\!\right] = I.$$

Suppose that $\mathscr{B}$ is a stable collection consisting of stable subsets of $L^1_{\mathcal{F}}(\mathcal{E})$.

Let $\mathscr{B}\!\Uparrow$ denote the unique name in $\overline{V}^{(\mathcal{A})}$ equivalent to the name given by the function

$$\{S\!\uparrow : S \in \mathscr{B}\} \longrightarrow \mathcal{A}, \quad S\!\uparrow \mapsto I.$$

By means of a manipulation of Boolean truth values using the mixing principle, one can prove the following:

Proposition 4.9. *Let* $\mathscr{B}$ *be a stable collection consisting of stable subsets of* $L^1_{\mathcal{F}}(\mathcal{E})$. *Then* $\mathscr{B}\!\Uparrow$ *is a name for a non-empty collection of non-empty subsets of* $L^1_{\mathcal{F}}(\mathcal{E})\!\uparrow$, *and the map* $S \mapsto S\!\uparrow$ *is a bijection from* $\mathscr{B}$ *to* $\mathscr{B}\!\Uparrow\downarrow$.

Notice that both $\mathscr{B}_{\langle \mathscr{X}, \mathscr{X}^\# \rangle}$ and $\mathscr{B}_{\langle \mathscr{X}^\#, \mathscr{X} \rangle}$ are stable collections. Then it makes sense to define $\mathscr{B}_{\langle \mathscr{X}, \mathscr{X}^\# \rangle}\!\Uparrow$ and $\mathscr{B}_{\langle \mathscr{X}^\#, \mathscr{X} \rangle}\!\Uparrow$.

Dual systems of modules were introduced and studied in [43]. In addition, their Boolean valued representation can be found in [43, Theorem 3.3.10], which covers the stable weak topologies. Actually, we have the following result, which can be also proved by adapting the proof of [47, Proposition 2.3.20]:

Proposition 4.10. *$\mathscr{B}_{\langle \mathscr{X}, \mathscr{X}^{\#}\rangle}{\Uparrow}$ (resp. $\mathscr{B}_{\langle \mathscr{X}^{\#}, \mathscr{X}\rangle}{\Uparrow}$) is a name for a topological base of the weak topology $\sigma(\mathscr{X}{\uparrow}, \mathscr{X}^{\#}{\uparrow})$ (resp. $\sigma(\mathscr{X}{\uparrow}^{\#}, \mathscr{X}{\uparrow})$) within $V^{(\mathcal{A})}$.*

Next, we deal only with the topology $\sigma_s(\mathscr{X}, \mathscr{X}^{\#})$, but the following results are also valid for the topology $\sigma_s(\mathscr{X}^{\#}, \mathscr{X})$.

The next proposition can be proved by manipulation of the Boolean truth values as in [47, Proposition 2.3.4] and [47, Corollary 2.3.1].

Proposition 4.11. *Let S be a stable subset of $\mathscr{X}$. Then:*

1. *S is open w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^{\#})$ iff* $[\![S{\uparrow}$ is open w.r.t. $\sigma(\mathscr{X}{\uparrow}, \mathscr{X}^{\#}{\uparrow})]\!] = I$;
2. *S is closed w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^{\#})$ iff* $[\![S{\uparrow}$ is closed w.r.t. $\sigma(\mathscr{X}{\uparrow}, \mathscr{X}^{\#}{\uparrow})]\!] = I$;
3. *S is stably compact w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^{\#})$ iff* $[\![S{\uparrow}$ is compact w.r.t. $\sigma(\mathscr{X}{\uparrow}, \mathscr{X}^{\#}{\uparrow})]\!] = I$.

As a consequence of the previous result and by means of a manipulation of the Boolean truth values as in [47, Proposition 2.3.11], we obtain the following:

Proposition 4.12. *Let $f : \mathscr{X} \to \overline{L^0}(\mathcal{F})$ be a function with the local property. Then:*

1. *f is lower semicontinuous w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^{\#})$ iff*
   $[\![f$ is lower semicontinuous w.r.t. $\sigma(\mathscr{X}{\uparrow}, \mathscr{X}^{\#}{\uparrow})]\!] = I$;
2. *f is stably inf-compact w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^{\#})$ iff*
   $[\![f$ is inf-compact w.r.t. $\sigma(\mathscr{X}{\uparrow}, \mathscr{X}^{\#}{\uparrow})]\!] = I$.

At this point, we can already prove Theorem 3.1. Namely, suppose that $\rho : \mathscr{X} \to L^0(\mathcal{F})$ is a conditional risk measure. Since ρ is $L^0(\mathcal{F})$-convex, we know from [23, Theorem 3.2] that ρ has the local property. Then, we have a name $\rho{\uparrow}$ for a function from $\mathscr{X}{\uparrow}$ to $\mathbb{R}_{\mathcal{A}}$ such that $[\![\rho{\uparrow}(x^{\bullet}) = \rho(x)^{\bullet}]\!] = I$ for all $x \in \mathscr{X}$. Moreover, it can be computed that $\rho{\uparrow}$ is a name for a convex risk measure.

We also have that $\rho^{\#}$ has the local property; thus, we can find a name $\rho^{\#}{\uparrow}$ for a proper function from $\mathscr{X}^{\#}$ to $\overline{\mathbb{R}}_{\mathcal{A}}$ so that $[\![\rho^{\#}{\uparrow}(y^{\bullet}) = \rho^{\#}(y)^{\bullet}]\!] = I$ for all $y \in \mathscr{X}^{\#}$. In addition, as a consequence of Proposition 4.8, one has that $[\![\rho^{\#}{\uparrow} = \rho{\uparrow}^{\#}]\!] = I$.

Finally, note that 1 in Theorem 3.1 is a consequence of Proposition 4.8; 2 in Theorem 3.1 is clear from Proposition 4.2; 3–4 in Theorem 3.1 is precisely Proposition 4.7; 5 in Theorem 3.1 is clear from Proposition 4.2; 6–7 in Theorem 3.1 is just Proposition 4.12; and finally we obtain 8–10 in Theorem 3.1 from Proposition 4.5.

Acknowledgment. This paper is based on some parts of the doctoral thesis of the author. He would like to thank his mentor José Orihuela his support and Antonio Avilés López for fruitful discussions. Likewise, he would like to express his gratitude to Anatoly G. Kusraev and Semën S. Kutateladze for taking the time to review his thesis, supporting his work and pointing out some extremely nice references in Boolean valued analysis.

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Received 5 June, 2019

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Владикавказский математический журнал
2019, Том 21, Выпуск 4, С. 71–89

БУЛЕВОЗНАЧНЫЙ ПОДХОД К АНАЛИЗУ УСЛОВНОГО РИСКА

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Аннотация. С помощью методов булевозначного анализа устанавливается принцип переноса между теорией двойственности классической выпуклой меры риска и теорией двойственности меры условного риска. А именно, меру условного риска можно интерпретировать как классическую выпуклую меру риска в подходящей теоретико-множественной модели. Как следствие, многие свойства меры условного риска могут быть получены путем интерпретации свойств выпуклой меры риска. Иными словами, интерпретация теоремы о двойственном представлении выпуклой меры риска приводит к новой теореме о двойственном представлении меры условного риска. В качестве примера приложения устанавливается общую теорема устойчивости представлении меры условного риска и изучаются различные ее частные случаи.

Ключевые слова: булевозначный анализ, мера условного риска, теория двойственности, принцип переноса.

Mathematical Subject Classification (2010): 03C90, 46H25, 91B30.

Образец цитирования: *Zapata J. M.* A Boolean Valued Analysis Approach to Conditional Risk // Владикавк. мат. журн.—2019.—Т. 21, № 4.—С. 71–89 (in English). DOI: 10.23671/VNC.2019.21.44629.

Владикавказский математический журнал
2019, Том 21, Выпуск 4, С. 90–98

MATHEMATICAL LIFE

I. I. GORDON WHO WAS AN ADRESSEE OF L. S. PONTRYAGIN* (INTRODUCTORY NOTES)

E. I. Gordon#

I put before the readers the letters of the eminent mathematician academician Lev Semenovich Pontryagin to my father Izrail Isaakovich Gordon. These letters were written between 1937 and 1969, and contain many interesting facts, pertaining not only to the history of mathematics but also to Russian life during that period. That is why these letters are—I think—of interest not only from the point of view of the history of mathematics, but also as a true-to-life account of the relevant period.

It is well known that L.S. lost his sight at age 13. He did not use the alphabet for the blind and typed his papers on an ordinary typewriter. Hence there are many grammatical mistakes in the original letters. Of course, these mistakes have been corrected in the printed versions of the letters.

The purpose of my introductory remarks is to tell about the recipient of Pontryagin's letters and about other people and events mentioned in them. Since the letters cover a very long period, it is not surprising that, in time, the relation of L.S. to various people mentioned in the letters changed. Sometimes he makes very harsh statements about some people. As a rule, such statements reflect momentary states of mind rather than his considered view of the people in question and characterize his way of speaking. The letters are uncut printed versions of the originals.

I.I. was the first graduate student of L.S. He entered graduate school in 1932 and graduated in 1935. The small age difference between student and teacher (two years) and their youthfulness helped them to become close friends. (It is well known that at 24 L.S. was already a world-famous mathematician.) Their friendship continued until 1969. During all this time they wrote letters to one another. Only three of Pontrygin's letters from the period before WWII have survived. Apparently, this is due to the fact that I.I. and his family were evacuated from Voronezh to Kazakhstan and their house in Voronezh was completely destroyed during the war. I have a copy of a single letter of I.I. to L.S. (This letter contained a question on the proof of the well-known Andronov–Pontryagin theorem, which arose when he was writing the book [12]. According to I.I. quick answer by L.S. allowed to overcome his difficulties).

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* The letters of L. S. Pontryagin to I. I. Gordon, Ibid, p. 27–208.

Gordon E. I. (2005) I. I. Gordon who was an addressee of L. S. Pontryagin (Introductory notes). Istorico Matematicheskie issledovanija (Investigations in the history of Mathematics). Russian Academy of Science, Institute of the History of Science and Technology, second series, issue 9 (44), 2005, pp. 14–26. Translated from Russian by Abe Shenitzer.

I.I. was born on June 16, 1910 in Grodno into the family of the engineer Isaak Israilevich Gordon. His father was a graduate of the famous German Polytechnic in Karlsruhe and was therefore permitted to live outside the Jewish Pale of settlement. For a while the family lived in Petersburg, where I.I.'s mother died in 1913. Then the family moved to Kharkov, for I.I. Senior, a committed bolshevik and member of the VKP(b)*, occupied an important post in the Ukranian Narkompros**. To run ahead, I mention that his intercession for a repressed friend resulted in his exclusion from the party. He was saved by some miracle, and to the end of his life (he died in 1972 at the age of 94) remained a committed Communist.

In 1927, I.I. graduated from a technical school in Kharkov and went to Moscow to study mathematics at Moscow State University. At that time he was a committed member of the Komsomol, a very natural thing for a 17-year-old Jewish youth. But during his very first year at MSU, he was expelled from the Komsomol as a trotskyite. All soviet students were taught in a course of the history of the CPSU that "on the occasion of the celebration of the tenth anniversary of the Great October Revolution a brazen trotskyits sortie was organized: the trotskyites demonstrated and used trotskyite, rather than bolshevik, slogans." During a subsequent meeting of the Komsomol they were all expelled from the Komsomol. Of course, as a committed bolshevik, I.I. condemned the trotskyites and shared the view that they should be expelled from the Komsomol. But when the question was posed at the meeting whether the trotskyites should be allowed to explain their position, I.I. thought it obvious that they should be allowed to do so. When a vote was taken, it turned out that I.I. was the only one to vote yes. Then the secretary of the Komsomol organization Dimitrii Abramovich Raikov, who later became a famous mathematician, expelled him from the Komsomol. I.I. told me that in those years, Raikov was such a fanatical Communist that he was compared with members of the Committee for Public Safety of the French Revolution. Later, in the 1930's, Raikov was himself expelled from the party and sent to Voronezh. For two years he taught at Voronezh University but was later acquitted and readmitted to the party. Then he returned to Moscow.

In 1927, I.I. was readmitted to the Komsomol by some very high instance. After completing a year's work at MSU he went to Leningrad, for he could not find an apartment or even a room for living in Moscow.

In Leningrad, I.I. studied together with, and became a close friend of, Georgii Rudolfovich Lorentz. Later Lorentz became a famous expert in approximation theory. He emigrated from the USSR during the war and worked for many years at the University of Texas in Austin. In his recently published recollections [1], Lorentz mentions the fact that I. M. Vinogradov taught a course that dealt with his research, and that he and I. I. Gordon were the only listeners. The lectures took place in Vinogradov's home. After completion of his university studies, I.I. worked for a year as an assistant at LSU (Leningrad State University)—he ran practice sessions for G. M. Fikhtengolts—and in 1932 became a graduate student at MSU (by then he had no trouble earning a living). Initially he wanted to study number theory, but his examiner at the entrance examination was L. S. Pontryagin, and the encounter with Pontryagin directed his interest to topology. L.S. became his supervisor.

I.I. obtained his first result [2] in the joint seminar of Lyusternik, Pontryagin, and Shnirelman. In it he proved that on any n-dimensional manifold there is a function with $n+1$ critical points. This showed that the lower bound on the number of critical points of a smooth function on a manifold, obtained earlier by Lyusternik and Shnirelman, was exact. For this paper I.I. won the first prize in a competition of papers by graduate students.

*The first abbreviation of CPSU—Soviet communist party.

**State Department of Education

In 1935, I.I. defended a candidate dissertation*. He was one of the first to defend a dissertation after the introduction of scholarly degrees in the USSR. The dissertation was later published in the Annals of Mathematics [3]. In it, simultaneously with Kolmogorov and Aleksandrov and independently of them, I.I. introduced a construction of a cohomological ring. All three lectured on this topic at the famous international topology conference in Moscow in 1935 (J. Alexander. On rings of complexes and combinatorial topology of integration theory; A. N. Kolmogorov. On homology rings in closed sets; I. I. Gordon. On invariants of the intersection of a polyhedron and its complementary space. Ed.) In this connection, the famous Swiss topologist H. Hopf wrote: “For many reasons, the year 1935 turned out to be an especially important landmark in the evolution of topology. In September of that year there took place in Moscow “The First International Conference on Topology.” The independent lectures of J. Alexander, I. Gordon, and A. N. Kolmogorov initiated the theory of cohomology. (The theory goes back to S. Lefschetz, who introduced the notion of a pseudocycle in 1930.)

What impressed me, and, of course, other topologists, most was not the emergence of cohomology groups—after all, they are just groups of characters of ordinary homology groups—but the possibility of defining multiplication of arbitrary complexes and more general spaces, that is, the emergence of *cohomology rings*, which are generalizations of the ring of intersections in the case of manifolds. Before this development we thought that such a situation could arise only because of the local “Euclideanness of manifolds” [4, p. 11].

Gordon’s constructions of multiplication of cohomologies differs of those of Alexander and Kolmogorov. Their constructions are identical. Later, H. Freudenthal [5] proved that the isomorphism of the Gordon and Alaxander–Kolmogorov rings (in this connection see the paper of L.S. of 1 April 1937).

In spite of the fact that I.I.’s dissertation was at the time a rather remarkable event in topology, it was approved by VAK** only in 1938. The delay was connected with political problems. As noted, I.I. was expelled from the Komsomol as a trotskyite in 1927 by a primary instance but reinstated by some higher instance. On 1 December 1934 one of the Soviet communist leaders, S. M. Kirov was killed***. His murder was blamed on trotskyites and zinovevites who were subjected to extensive and intensive repressions. The first to be shot to death without trial were scores of prisoners serving sentences resulting from their being accused of counter-revolutionary activities. Lists of the executed were printed in Pravda under the heading: “In response to the murder of Comrade Kirov, the following enemies of the people were shot to death...” Then there was a purge in the party, followed by a purge in the Komsomol. In the case of the Komsomol, all those who were ever penalized were automatically expelled. One of the expelled was I. I. Gordon. After that, the accusation of being a trotskyite pursued him practically until the outbreak of the war. After the war, when filling out questionnaires, he did not mention that he was expelled from the Komsomol. He thought that he got away with this because the relevant archives were lost during the war. By some miracle he survived and evaded the gulag.

I wish to note that in spite of his having been expelled from the Komsomol as a result of the charge of trotskyism, I.I. could, and did, rely at the time on constant assistance of both Pontryagin and P. S. Aleksandrov. It should be pointed out that, “assistance” has two

*The candidate degree is equivalent to our Ph.D degree, Trans.

**Higher Certifying Commission of the state department of education. All Candidate Science and Doctor Science Degrees granted by Universities in the Soviet Union had to be approved by VAK. The same procedure is used in Russia now.

***It is well known that the assassination of Kirov was organized by Stalin, who used it for extermination of his political opponents.

meanings, "ordinary" and "formal". Thus L.S. and P.S. helped I.I. to find employment, wrote excellent letters of recommendation for him and in addition, tried hard to ensure approval of his candidate dissertation by VAK, in spite of his being a "politically questionable person." L.S. and P.S. submitted to VAK a very positive testimonial relating to I.I.'s candidate dissertation. Had I.I. been arrested, this could have had dismal consequences for both of them. What follows is the text of one of their references.

Reference

I.I. Gordon completed successfully his training for research work at the Mathematical Institute of Moscow University (beginning in 1921, this was the scientific and research institute of mathematics and mechanics; beginning in 1935, this was the scientific and research institute of mathematics (headed by A. N. Kolmogorov), Ed.) and defended a very interesting dissertation on homological properties of complements of polyhedrons in n-dimensional space for which he was awarded the degree of a candidate of the mathematical sciences. Before that, while still a graduate student, I. I. Gordon completed a paper which won the first prize in a competition of papers by graduate students. This paper was published in the Proceedings of the second all-Soviet mathematical conference to which it was submitted by the author. I. I. Gordon's dissertation was published in the American journal Annals of Mathematics at the invitation of its editors.

I. I. Gordon's papers deals with difficult current questions of topology and its applications and show that their author has a creative mathematical talent. They show that I. I. Gordon is a very substantial mathematical researcher.

I. I. Gordon gave a lecture on his investigations at the First International Topology Conference, which took place in Moscow in September 1935.

In addition to being a gifted young scholar, who has already embarked in a fully creative manner on the road to independent scientific research, I.I. Gordon is also a university teacher with high scientific culture and good pedagogical qualities. We can attest that his teaching work at Moscow University was very successful.

In summary, the undersigned regard I. I. Gordon as a talented young mathematician who has already made a valuable contribution to science and one who provides solid reasons for expecting him to achieve further solid successes. Also, he is undoubtedly a valuable university worker who has the essential qualities to give competent lectures in many advanced areas of mathematics which require the lecturer to have very high mathematical qualifications.

Corresponding member of the USSR Academy of Science and doctor of mathematical sciences

P. [S.] Aleksandrov

Professor of Moscow State University and doctor of mathematical sciences

[L. S.] Pontryagin

Crimea, Bati-Liman, 17 September 1936

(The reference was written by P. S. Aleksandrov by hand. The signatures of P. S. Aleksandrov and L. S. Pontryagin were verified by the learned secretary of the Mathematical Institute for Scientific Research on 4 November 1936. The document is kept in the personal archive of I. I. Gordon.)

Were P.S. and L.S. aware that supporting a "trotskyite" spelled danger for them? Hard to say. It seems to me that they never understood this or, simply, never gave it a thought.

I once asked I.I. whether he realized that there was the threat that he might be arrested at any moment. He replied that he never thought of this, although after 1934 he changed his views, and to the end of his life his attitude vis-a-vis the Soviet authorities was one of total enmity. He told me: "At that time I gave it no thought, just as you don't think of death every day." It is possible that such a defensive reaction of the organism is the basis of all courage, and throughout his life, I.I. was a remarkably fearless person. At that time there were people in I.I.'s milieu who reacted altogether differently. For example, in 1938 I.I. moved to Voronezh and got to know his future wife who, at the time, also had problems with the NKVD*. At that time, Nikolai Vladimirovich Efimov. I.I.'s friend from the time of their graduate days, and his wife Roza Yakovlevna Berri, tried to talk I.I. out of associating with her. They told him that both of them had damaged reputations and it was possible to attribute to them the creation of a counter-revolutionary organization.

After completing his graduate studentship, I.I. began work at the university of Saratov. He was offered employment by Gavriil Kirillovich Khvorostin, the rector** of the university between 1935 and 1937. He said that he wanted to turn his university into a "Göthingen on the Volga." That is why he offered positions to such famous mathematicians of the older generation as I. G. Petrovskii and A. Ya. Khinchin, and to "young hopefuls" such as Vietor Vladimirovich Wagner and I. I. Gordon. V. V. Wagner was born in Saratov and worked all his life at Saratov State University. He became a famous algebraist and geometer. Wagner belonged to the large group of graduate students who defended their dissertations but was one of only two who were granted the degree of doctor of mathematical sciences on the basis of their defense. This was in 1935. The friendship of V. V. Wagner and I. I. Gordon continued throughout their lives. I.I.'s archive contains many letters of this remarkably interesting man who knew at least ten European languages and was a connoisseur of history and literature.

I.I. Gordon lived in Saratov between 1936 and 1937. He told me a great deal about this period. I rely on my memory for the most interesting fragments of his account.

The windows of I.I.'s room faced the famous Saratov jail. The street before the jail was full of women who wanted to catch a glimpse of their husbands in the windows of the jail. On a "beautiful" day all this came to an end—at night the jail windows were covered by shutters known as muzzles—inclined shutters that let light penetrate into the cells but made it impossible for their occupants to see the street.

In the streets, one encountered many exiled Leningradians. They stood out in the crowd due to their intelligent and aristocratic "capital" appearance. I.I. mentioned a certain old bolshevik, a red professor (a member of N. I. Bukharin's Institute of the red professoriat) deported from Moscow, who also quietly vanished from sight. The red professor invariably walked alone and always had his red emblem medal pinned on his chest.

I.I. lived in a house with 28 apartments. In the 26 of them at least one of a family members was arrested. A guard sat in the entrance. Every day when I.I. passed the guard, the latter whispered: "today they took away so-and-so." One day, when I.I. was leaving the house, he noticed a new guard. The new guard told him that the old guard was arrested.

It seems that the fact that the university management hired scholars from other cities was detrimental to the allocation of fiats to local workers. This angered some of them and resulted in their starting various intrigues (it was not only "Moscovites" who "turned ugly over flat shortages" as it was mentioned by Voland in the popular novel "Master and Margarita" by M. Bulgakov). I.I. remembered that one day, after a meeting of the learned council, Rector

*the abbreviation for the State Security Department later famous as KGB.

**the Russian analog for the university president.

Khvorostin quoted Yesenin's "You can't manage a brute with a dry branch."* I.I. was the first victim of these intrigues—his "trotskyism" came to light and the rector was forced to dismiss him. He went to Moscow to obtain a reinstatement in Moscow. At that time the Moscow metro had just been opened, and I.I. remembered the depressing effect on him of a sign on the metro door which read: "No outlet". Nevertheless, his dismissal was judged unlawful. Not only was he reinstated but he was paid for half a year of forced unemployment. I remind the reader that this was 1937.

But I.I. did not work very long. At the end of 1937, rector G. K. Khvorostin was arrested and subsequently died in prison. After his arrest, I.I. was immediately dismissed. The dismissal had the following wording: "to be dismissed as a former trotskyite, offered work by an enemy of the people." This time there was no hope for reinstatement. What saved the day was the fact that the number of unemployed who lost their positions for similar reasons was staggering. It seems that a decision was taken on a very high level of government that as long as these people were free (the free people were probably a minority) they should work. M. E. Koltsov** wrote an article in "Pravda" about the "overcautious persons" (this fact was described by L. K. Chukovska in her famous novel "Sofya Petrovna"). Then I.I was called to the Narcompros and asked to choose a place of employment. He wanted to return to the University of Saratov (which shows that he did not fully understand what was happening and failed to understand the danger that threatened him). While his interlocutors did not rule out his request, they amicably advised him to give it up. Then he chose Voronezh university, where N. V. Efimov was in charge of the division of geometry. According to I.I., N.V. was terrified by the fact that his division would employ so politically compromised a person but he could not do anything, for he remembered his friendship with I.I. which dated back to the days of their graduate studentship. Later Wagner told I.I. that members of the NKVD asked him about I.I. But at the time the NKVD did not look for people in other towns. Why mess about elsewhere if one could always find a victim on the spot? It seems that this saved I.I. from being arrested. Had he stayed in Saratov, his arrest would have been unavoidable.

Shortly after I.I.'s arrival in Voronezh was an event of greatest significance for me, the author of these lines—Izrail Isaakovich got to know his future wife, my mother, Nina Aleksandrovna Gubaŕ. They got married in the summer of 1938.

N. A. Gubaŕ was born in Petrograd into a family of doctors on July 29, 1915. After the revolution and the civil war the family ended up in Voronezh. N.A.'s father, an excellent therapeutic doctor, died at the age of 42. N.A.'s older brother, Mikhail Aleksandrovich, was arrested in 1933 and sentenced to 5 years in a prison camp for being a member of a theosophical circle. The judge explained to him that "at this time your circle is not counter-revolutionary, but we can't wait until it is transformed." Just as my parents, so too, M. A. Gubaŕ was lucky. After serving his sentence, he was freed in 1938, readmitted to the medical institute from which he graduated before the war, served during the war as a sanitary inspector, and then, for the rest of his life (he died in 1969), he did research in military hygiene, was a colonel in the medical service, and a doctor of sciences. The luck I write about consisted in the fact that he was not arrested a second time, as were most of political convicted prisoners that were released at that time.

N. A. Gubaŕ was a graduate of the department of mathematics of Voronezh University. Then Voronezh was a rather impressive mathematical center. As mentioned earlier, D. A. Raikov taught in Voronezh between 1933 and 1935. After completing his graduate studentship and defending his dissertation, N. V. Efimov was appointed chair of geometry.

*In Russian "dry branch" sound similar to his last name—"khvorostina".

**A popular Soviet journalist, who was arrested and shot to death a couple of years later.

Boris Abramovich Fuks was chair of functions of a complex variable, and Maksimillian Mikhailovich was chair of analysis. Victoria Semenovna (Vitya) Rabinovich was acquainted with L. S. Pontryagin back in Moscow. He mentioned her in his autobiography [6]. L.S., as well as L. A. Lyusternik, delivered lectures in Voronezh on a number of occasions.

At the time all of these people were very young and students and teachers formed a single group. N. A. Gubaŕ recalled that when N. V. Efimov was supposed lecture and was not, in class, students ran to his apartment at the university to wake him up. Fellow students of N. A. Gubaŕ were: Roza Yakovlevna Berri, Anna Aleksandrovna Gurevich, who later married Vladimir Abramovich Rokhlin, and Aleksandra Ivanovna Tsvetkova, who later married Aleksander Grigofevich Sigalov. All of them maintained friendly relations with L. S. Pontryagin and are frequently mentioned in his letters.

One other student belonging to this group was Vladimir Ivanovich Sobolev, later a well known expert on functional analysis. Sobolev and L. A. Lyusternik were the joint authors of the first Russian textbook on functional analysis (L. A. Lyusternik, V. I. Sobolev, Elements of Functional Analysis, Moscow, 1951 (Ed.)).

I wish to say a few words about yet another student in this group, a man whose life was difficult and interesting. He was Nikolai Aleksevich Zheltukhin. He was arrested when he was a third year student. His arrest was the result of a denunciation by one of the comrades who reported the fact that Zheltukhin was critical of the collectivization and hunger in Ukraine. He told us what had happened to him when he visited us at home in Gorkii, in the 1970s. I tell his story from memory. First he ended up in camps in the Arkhangelsk district where he worked in forest clearing. There he felt that his strength was giving out [2]. Somebody told him that if a prisoner invented something, he could send a description of his invention to a special section of the NKVD, and it could happen that the person in question would be transferred to one of the so-called, “sharashka’s”, special secret research institutes, in which prisoners worked. Living conditions in the “sharashka’s” were incomparably better than in the camps (see A. I. Solzhenitsyn’s “ln the first circle”). When it comes to his scientific interests, N. A. was at the time a pure mathematician, and had written a paper on descriptive set theory. But when he was still a schoolboy he read an article about tractors in the journal, *Technology for the Young* and figured out that a certain mechanism described in the article could be adapted to planes. N.A. didn’t take his invention very seriously but saw no other chance of survival. He described his invention and dropped it in the mailbox on a pine tree. To his amazement, he was transferred to “sharashka” and worked under S. P. Korolev, the future inventor of spacecraft. His invention was evaluated by academician B. S. Stechkin. He was a prisoner at the time but headed a commission which evaluated inventions. Since Stechkin he tried to save as many people as possible, he would often come up with the conclusion, “It makes sense to send for the author” even if the “invention” was worthless (this benevolent activity of B. S. Stechkin is also described in “In the first circle”). N.A. told us that he once ran into Stechkin and had the check to ask him whether he remembered his—Zheltukhin’s—invention and what he thought of it. Stechkin replied: “Of course, I remember. You, my friend, wrote rubbish. What will take place is cooling, not heating.” And went on to explain why. H. A. Zheltukhin’s subsequent scientific lot was very successful. A few years after the end of the war, he was freed, worked for the rest of his life in the Institute of Mechanics in the Novosibirsk Academic Village, and became a corresponding member of the Academy of Science of USSR.

N. A. Gubaŕ was more than once called to the NKVD in connection with Zheltukhin. She refused to give any depositions, behaved in a provocative way, but somehow was not arrested. All that happened was that the NKVD wrote a letter to the university about her improper behavior. She was not even expelled from the university (she was not a member of the Komsomol). All that happened was that she was criticized at a meeting. Nevertheless, many of her friends were later afraid to associate with her, and, until the arrival of I. I. Gordon in Voronezh, she was, as I mentioned, isolated.

I.I. and N.A.'s first son, Aleksandr, was born shortly before the outbreak of the war. When the evacuation of Voronezh University to Siberia was ordered in September 1941, their family, as one with a tiny baby, was among the first to be evacuated. (As a candidate of science, and a person with poor eyesight, I.I. had a "white ticket" and was not a subject for the draft into the army.) When their train stopped in Petropavlovsk (Kazakhstan), they heard a radio announcement about a change in the evacuation order of Voronezh University. It made no sense to travel farther, and, according to the conditions of the war time, it was impossible to return to Voronezh. They were employed as high school teachers of mathematics and physics in a village named Bolshe-Izyum in the Petropavlovsk district. Living conditions in Bolshe-Izyum were extremely difficult, and their family suffered constant hunger. L. S. Pontrygin, who was evacuated to Kazan, was well aware of their difficulties and of the difficulties of many of his other friends, and did all he could to help. The letters show that he tried to obtain for I.I. a position in a university, a matter of utmost difficulty. With the help of his relatives, who had been evacuated to Omsk, I.I. obtained a position at the Omsk Polytechnical Institute. However, he had to have permission to leave his school job, a step firmly objected to by the director of the school. There were just two men in the school, and in the Kazakhstan backwater a white ticket would not have helped I.I. avoid call-up to a reserve regiment (The horror of such regiments is described in the novel by V. P. Astafyev "Damned and killed."). If I.I.were freed from call-up then the director would be called up. L.S. made tremendous efforts to help, and secured the assistance of S. L. Sobolev, head of the Steklov Institute, and of S. V. Kaftanov, the prime minister of the Soviet Goverment at that time. I.I. went with his little son to Omsk and N.A. remained in the Bolshe-Izyum school until the end of the school year. The director of the school was not called up.

I.I. Gordon's family lived in Omsk for a little longer than a year. In the fall of 1944, I.I. obtained a position at the university of Gorkii (now Nizhnii Novgorod). This was due to L.S.'s recommendation of academician Aleksandr Aleksandrovhic Andronov (L.S. was his friend from 1932, Ed.). I.I. and N.A. lived in Gorkii to the end of their lives. After the war, I.I. produced a few more papers on topology [7, 8, 9]. He taught many courses at the mechanical-mathematical and radio-physical Departments. He would sometimes teach six courses a term. Many of his students, who themselves became famous scholars, gratefully recall his lectures and speak of him with warmth and respect.

Nina Aleksandrovna Gubar studied qualitative theory under supervision Evgenya Aleksandrovna Leontovich Andronova, got her candidate of science degree and work until retirement as a docent (associate professor) in the department of mathematics of the Gorkii institute of water engineering.

For a long time life in Gorkii was not easy either. It was only from 1962 that the family enjoyed acceptable living conditions—a small two-bedroom apartment (so-called "khrushchevka") for a family of five. In that same year, I.I. suffered a heavy heart attack, followed by extremely heavy complications. In this case L.S. payed a vital part. He helped I.I. twice to obtain accommodation in a very in a very good academic sanatorium near Moscow. Of course, this helped I.I. to recover after his heart attack and live for over twenty more years.

The therapeutics professor who took care of I.I. after his heart attack asked what preceded the heart attack. When this was told to I.I.'s department head Dmitrii Andreevich Gudkov, a remarkable person and mathematician, he commented that the heart attack was preceded by twenty years of baiting. In fact, for many years the conditions in the department of mathematics of Gorkii University were very difficult. A very active group of second-rate mathematicians with leading party and administrative positions persecuted for various reasons the accomplished and highly qualified members (I wrote about this in my recollections of D. A. Gudkov [10].) I.I. provoked the intense anger of these people because of his independent character and the uprightness and directness of his opinions.

All difficulties notwithstanding, my parents life in Gorkii was happy and interesting owing to their remarkable friends. In the first place, I must mention their friendship with A. A. Andronov, who was not only an eminent scholar but also an extremely interesting, clever, obliging, and grateful person. Unfortunately A.A. died in 1952, age 51. Much has been written about him. Recently, Nizhegorod University published an interesting selection of documents relating to him [11]. After his death, his wife, E. A. Leontovich-Andronova became the head of the mathematical branch of his school. I.I. and E.A. wrote two monograps [12, 13] containing the fundamental results of the Andronov school on the qualitative theory of plane dynamical systems. This is not the place for describing all friends of I.I. and N.A. For me, these people were a splendid example of modern Russian intelligentsia.

I. I. Gordon died on April 22, 1985. N. I. Gubar' lived for another nine years. She died on 13 August 1994.

Contacts between I. I. Gordon and L. S. Pontryagin ended in 1969. The subsequent activities and key position of L.S. are well known and have been described many times by L.S. and many others. Comments on these matters do not belong to the present remarks.

The author is deeply grateful to V. M. Tikhomirov, who inspired him to publish Pontryagin's letters, to G. M. Polotovskii, and to his wife I. N. Gordon for their assistance in the preparation and publication of the letters.

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ПРАВИЛА ДЛЯ АВТОРОВ

Общие положения

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Том 21

Выпуск 4

Зав. редакцией В. В. Бозрова

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информационных технологий и массовых коммуникаций.
Свидетельство о регистрации ПИ № ФС77-70008 от 31 мая 2017 г.

Подписано в печать 20.12.2019. Дата выхода в свет 27.12.2019.
Формат бумаги $60\times84^1/_8$. Гарн. шрифта Computer modern.
Усл. п. л. 11,63. Тираж 100 экз. Цена свободная.

Учредитель:
Федеральное государственное бюджетное учреждение науки
Федеральный научный центр «Владикавказский научный центр
Российской академии наук» (ВНЦ РАН)

Издатель:
Южный математический институт — филиал ФГБУН ФНЦ
«Владикавказский научный центр Российской академии наук»

Адрес издателя:
362027, г. Владикавказ, ул. Маркуса, 22.

Отпечатано ИП Цопановой А. Ю.
362000, г. Владикавказ, пер. Павловский, 3.

Индекс 57380