

New Standards for the Solution of Geometric Calculation Problems by Using Computers

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To pour old wine into new bottles?

Abstract: The availability of computers and of software for the geometric tasks of construction, measurement and calculation, as well as the availability of software for numerical and symbolic computation induce new methods for the solution of geometric calculation problems: computer-aided graphical, numerical and algebraic methods of solution. These computer-aided solutions are explained in relation to suitably chosen tools (Mathematica and Cabri Géomètre) and traditional examples.

Kurzreferat: *Neue Standards für das Lösen geometrischer Berechnungsaufgaben mit Hilfe von Computern.* Die Verfügbarkeit von Computern und von Software für geometrische Aufgaben wie konstruieren, messen und berechnen, sowie für numerische und symbolische Berechnungen induzieren neue Methoden für das Lösen geometrischer Berechnungsaufgaben: computergestützte graphische, numerische und algebraische Lösungsmethoden. Diese computergestützten Methoden werden in bezug auf geeignet ausgewählte Werkzeuge (Mathematica und Cabri Géomètre) und traditionelle Aufgaben erläutert.

ZDM-Classification: G40, R20

1. Introduction

Elementary geometry is a powerful tool to model reality. This becomes apparent, inter alia, in applications to engineering sciences and crafts, where elementary geometric calculation has a large scope. Geometry at school, in particular in German-speaking areas, has a tradition of geometric calculation reaching back to the 19th century. Such geometric calculation is justified as a subject at school by its applicability and by its relevance for practising the solution of mathematical problems. Today this tradition is expressed mostly in geometric problems set for graduation exams at intermediate school levels and in geometric exercises used in preparation for the corresponding written exams. Therefore the treatment of geometric calculation problems in class suffers from a certain stiffness as regards content and method.

To master arithmetic and algebra is, besides corresponding heuristic and geometric knowledge, a prerequisite for the ability to solve geometric calculation problems. So far students have to be practised at these skills. The capacity of suitable computer applications raises the question whether the use of computers can lead to methodical innovations for the solution of (not only) geometric calculation problems. Thus a convivial juxtaposition of new, that is, computer-aided standards and traditional methods to treat geometric calculation problems can keep learning and teaching of geometry alive and attractive.

In the following we explain such new standards to solve geometric calculation problems in relation to suitably cho-

sen computer applications. We distinguish three standards of solution:

- computer-graphical solution
- computer-numerical solution
- computer-algebraic solution.

By a *graphical solution* we mean those possibilities for a solution which arise from the use of school-geometric computer applications which allow interactive variation (drag mode), construction, measurement and calculation (Schumann 1995a). The procedure is essentially described by Diagram 1. Of course, graphical solutions are possible only for so-called special calculation problems. Solutions obtained are approximations to the wanted quantities. Due to yet coarse resolution of pixelation the quality of solutions is low (here we only mention that there is a procedure to overcome the pixelation problem to a large extent).

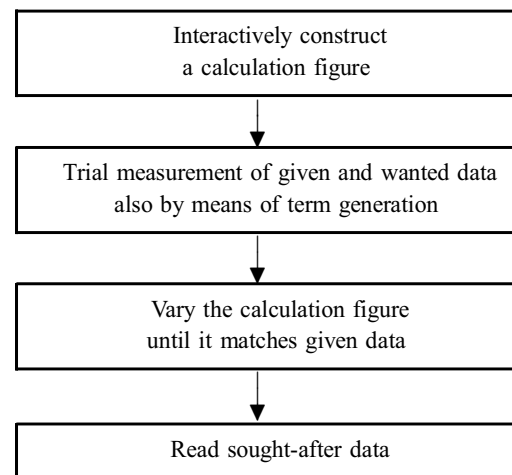


Diagram 1

The advantage of a graphical solution is, that an experimental approach and a vivid geometric penetration of the problem become possible by making a movable on-screen drawing, largely without the hindrance of arithmetical and algebraic barriers.

As a suitable tool for graphical solutions we use Cabri géomètre II by J.-M. Laborde and F. Bellemain (1994), which is currently the probably most developed and ergonomic software as regards geometric content.

For *numerical solutions* we use an approximating algorithm, which allows to solve as a whole those particular geometric problems for which the formulation (*Ansatz*) is an algebraic system of equations. *Algebraic solutions*, which is to say exact solutions of special and general geometric calculation problems, use a corresponding exact algorithm of the computer algebra component of a mathematical package for the solution of the *formulation* of the problem considered. (The exact solvability of such a system of algebraic equations is naturally limited.) A suitable package now with formula editing facilities and the possibility of installing commands as mean options is MATHEMATICA (Schumann 1994, 1995b). Thereby a formulation consists of

- a set of (implicit) algebraic equations which completely describe the relations between given and sought-after quantities and auxiliary quantities as defined by the problem

- the set of wanted quantities
 - a set of auxiliary quantities which are to be eliminated.
- In the syntax of MATHEMATICA, using a modified auto-solver (Gloor/Schumann 1994/95/96):

```
(N)SolveReal[{algebraic equations},
             {wanted quantities},
             {auxiliary quantities}]
```

“NSolveReal” is used for approximate (numerical) real solutions and “SolveReal” for exact (algebraic) real solutions. The action for numerical and algebraic solutions, respectively, is described roughly in Diagram 2.

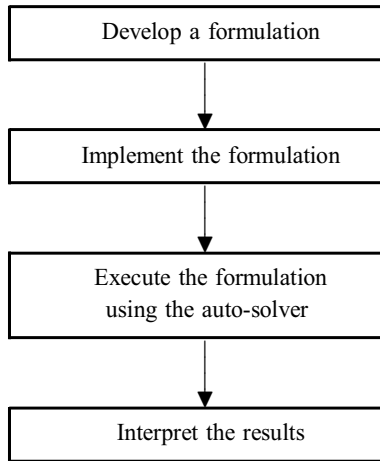
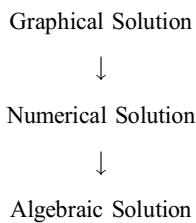


Diagram 2

The advantages of such numerical and algebraic solutions are, above all, the concentration on heuristic development of a formulation as the true original mental effort of the human solver, and the elimination of the need to routinely execute algorithms. Formulation-oriented solutions enhance the aspects of solution planning as well as of solution interpretation and thus improve so-called methodical competence in (computer-aided) solutions of mathematical problems.

The three solution standards are not to be considered separately but induce a computer-aided comprehensive treatment of calculation problems. We propose the following sequence, indicated by increasing abstraction, for a systematic treatment of calculation problems in geometry class:



Deviations from this sequence are possible, for example, in case a numerical solution is to be visualised or checked against a graphical solution.

In the following we give three examples to illustrate the above program. Given the abundance of geometric calculation problems, these examples can not be considered representative. However, they demonstrate computer-aided geometric calculation as it could be implemented today. We restrict ourselves to problems for which the formulation is given by a system of algebraic equations. This

class of geometric calculation problems covers a lot of geometric calculations common in secondary education.

2. Examples of computer-aided solution

Example 1:

Given a trapezium ABCD with AB parallel CD, a (9 cm), b (5 cm), α (90°), β (55°) and the triangle ABE of area A_1 (12 cm^2), E on BC. Sought-after is the area of triangle AED.

Graphical solution:

We construct the trapezium as instructed, with E moveable on BC. We measure the area of ABE and AED using polygonal area measurement and denote the results A_1 , A_2 , respectively (Fig. 1.1).

Now we change the position of E until $A_1 = 12.0\text{ cm}^2$ and read $A_2 = 14.6\text{ cm}^2$ (Fig. 1.2). – We can vary the particular problem by changing the given quantities a, b, β and A_1 and solve graphically (Fig. 1.3).

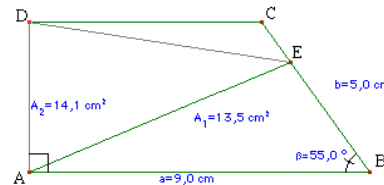


Fig. 1.1

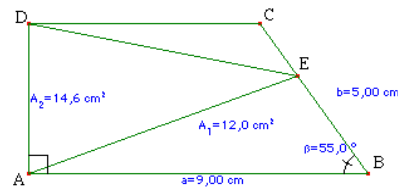


Fig. 1.2

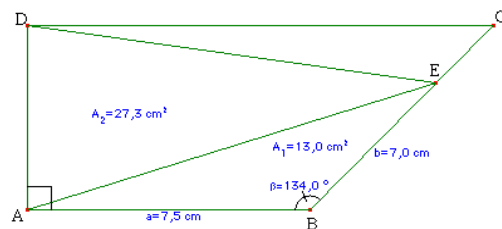


Fig. 1.3

Numerical solution: A formulation for the particular problem is systematically developed by assembling the area of the trapezium from three triangular areas ($A_1 = 12$, A_2 , A_3). These three triangular areas as well as the residual sides of the trapezium are expressed in terms of given quantities and additional auxiliary quantities. We use a figure to develop a formulation.

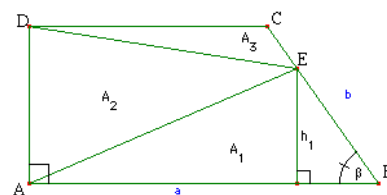


Fig. 1.4

Most simply we expand one of the figures already constructed during the graphical solution process (Fig. 1.4). The problem requires to solve for A_2 and to eliminate the auxiliary variables A_T , A_3 , c , d , and h . The command “NSolveReal” is used to determine approximate solutions (input 1.1). For A_2 we obtain, given to six significant digits, the approximate value 14.6071 (output 1.1).

$$\text{NSolveReal} \left[\left\{ \begin{aligned} &A_T == 12 + A_2 + A_3, \\ &A_T == \frac{(9+c)}{2}d, 12 == \frac{1}{2}9h_1, A_3 == \frac{(d-h_1)}{2}c, \\ &c == 9 - 5 \text{Cos}[55^\circ], d == 5 \text{Sin}[55^\circ], \\ &\{A_2\}, \\ &\{A_T, A_3, c, d, h_1\} \end{aligned} \right. \right]$$

$$\{A_2 == 14.6071\}$$

Input and Output 1.1

The solution is independent of the order in which the equations of the formulation are given. In this case we chose a so-called *top-down* formulation. There are 6! different possibilities for a formulation by sequencing the equations.

Algebraic solution:

To obtain a solution for the class of all problems which are represented by our particular problem, we need to generalise the formulation such that particular numerical values are replaced by variable names. Then the exact command “SolveReal” can be applied to this formulation (input 1.2). In a matter of seconds the general solution is returned (output 1.2), an equation for A_2 in terms of a , b , β and A_1 .

$$\text{SolveReal} \left[\left\{ \begin{aligned} &A_T == A_1 + A_2 + A_3, \\ &A_T == \frac{(a+c)}{2}d, A_1 == \frac{1}{2}ah_1, A_3 == \frac{(d-h_1)}{2}c, \\ &c == a - b \text{Cos}[\beta], d == b \text{Sin}[\beta], \\ &\{A_2\}, \\ &\{A_T, A_3, c, d, h_1\} \end{aligned} \right. \right]$$

$$\{A_2 == \frac{b(a^2 \text{Sin}[\beta] - 2 \text{Cos}[\beta]A_1)}{2a}\}$$

Input and Output 1.2

One can gain further information by considering some of the auxiliary variables, which are to be eliminated, as elements of the set of wanted quantities, or by declaring the sought-after quantity as an auxiliary variable. Thereby we change the original problem by manipulating its formulation and gain additional aspects of how to treat the problem. An example is shown in input and output 1.3.

$$\text{SolveReal} \left[\left\{ \begin{aligned} &A_T == A_1 + A_2 + A_3, \\ &A_T == \frac{(a+c)}{2}d, A_1 == \frac{1}{2}ah_1, A_3 == \frac{(d-h_1)}{2}c, \\ &c == a - b \text{Cos}[\beta], d == b \text{Sin}[\beta], \\ &\{A_2, A_T\}, \\ &\{A_3, c, d, h_1\} \end{aligned} \right. \right]$$

$$\{A_2 == \frac{b(a^2 \text{Sin}[\beta] - 2 \text{Cos}[\beta]A_1)}{2a},$$

$$A_T == -\frac{1}{2}b(-2a + b \text{Cos}[\beta]) \text{Sin}[\beta]\}$$

Input and Output 1.3

Example 2:

A ladder of a length of 5 meter is to be leaned against a wall. This is hindered by a plinth of equal height and depth (1m). At which distance from the wall has the ladder to be erected to reach as high as possible at the wall?

Graphical solution:

We model the situation in plane geometry. On a horizontal ray we measure 5 cm (scale 1:100) from a movable point A, the base of the ladder, such that the top of the ladder drags along the vertical wall. Furthermore the drawing includes the square cross section of the plinth (Fig. 2.1). The distance from the base of the ladder to the wall is denoted a and b is the height of the top of the ladder above the ground. We drag A until the ladder touches the free corner of the square and obtain one graphical solution (Fig. 2.2). The second solution, which is reflection symmetric with respect to the bisector of the right angle between “ground” and “wall”, is eliminated according to the requirements of the problem (Fig. 2.3).

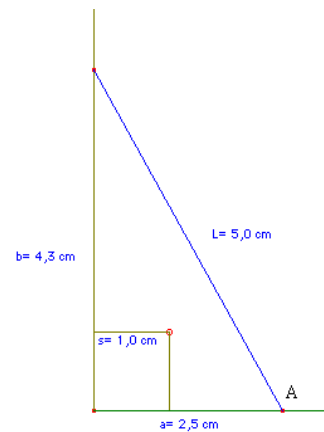


Fig. 2.1

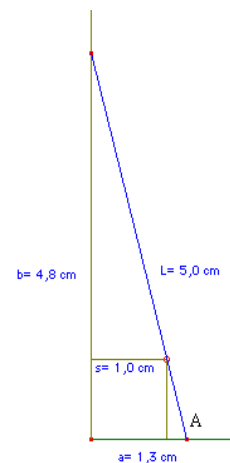


Fig. 2.2

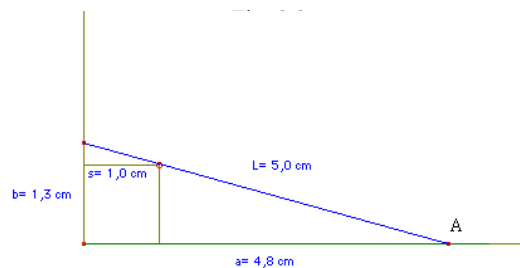


Fig. 2.3

Supplement: Using an animation, various ladder positions can be illustrated dynamically (without hindrance by the plinth) by “snapping” the point A (Fig. 2.4). The result of the animation shows an envelope enclosing the various ladder positions; it is just one of the four branches of an asteroïd (explanation?).

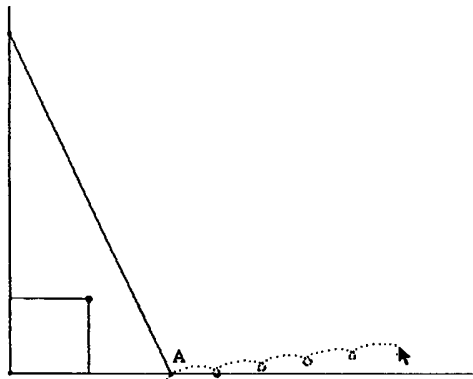


Fig. 2.4

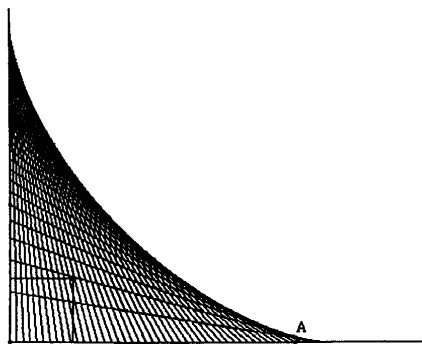


Fig. 2.5

The modelling could be done in a different manner by gliding the ladder through the free corner of the plinth. Thereby the top of the ladder can come to lie behind the wall, which is unreal (Fig. 2.6). Solutions are obtained from the intersections of the linearly interpolated envelope and the wall (Fig. 2.7). We can also animate the various ladder positions dynamically.

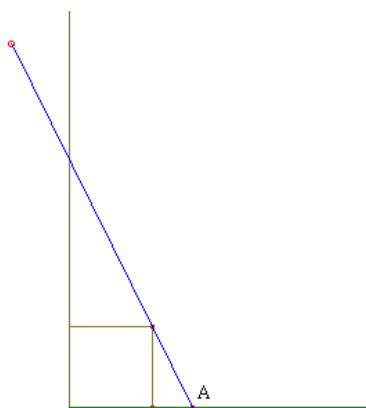


Fig. 2.6

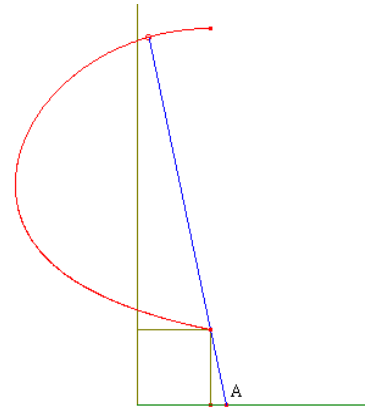


Fig. 2.7

With this we have exhausted the heuristic principle to find solutions among the set of figures obtained by relaxing one of the conditions given.

Numerical solution:

Using the similarity of triangles and Pythagoras’ theorem we extract from Fig. 2.2 a formulation for our particular problem consisting of two equations. These can be solved approximately for a and b using “NSolveReal”. There is no need for auxiliary variables (input 2.1). Output 2.1 shows four solutions, two of which are eliminated because of their negative values and only one of which represents the sought-after optimal solution with approximate values improved upon the graphical solution.

$$\text{NSolveReal} \left[\left\{ \frac{1}{b} == \frac{a-1}{a}, \quad a^2 + b^2 == 5^2 \right\}, \right. \\ \left. \{a, b\} \right]$$

$$\{ \{a == -4.9304, b == 0.831377\},$$

$$\{a == 0.831377, b == -4.9304\},$$

$$\{a == 1.26052, b == 4.8385\},$$

$$\{a == 4.8385, b == 1.26052\} \}$$

Input and Output 2.1

A reasoning for *four* solutions: The formulation leads to an algebraic equation in a of order 4, which is not evident from the computer numeric and algebra. That could be a starting point for a traditional treatment of such kind of problems.

In a different form of the formulation, in which we associate variable names s and L with the given data, we can express the equations in general terms straightaway without using particular values; the result is identical (input and output 2.2).

$$\text{NSolveReal} \left[\left\{ s == 1, L == 5, \frac{s}{b} == \frac{a-s}{a}, a^2 + b^2 == L^2 \right\}, \right. \\ \left. \{a, b\} \right],$$

$$\{ \{a == -4.9304, b == 0.831377\},$$

$$\{a == 0.831377, b == -4.9304\},$$

$$\{a == 1.26052, b == 4.8385\},$$

$$\{a == 4.8385, b == 1.26052\} \}$$

Input and Output 2.2

From this we obtain a formulation for the general solution by removing the association between variables and values. The exact solution returns, among other things, the functional dependence of the wanted quantities in terms of the given quantities (input and output 2.3).

$$\text{NSolveReal} \left[\left\{ \frac{s}{b} == \frac{a-s}{a}, \quad a^2 + b^2 == L^2 \right\}, \right. \\ \left. \{a, b\} \right], \\ \left\{ \left\{ a == \frac{2s \left(L^2 - (s + \sqrt{L^2 + s^2}) \left(-2s + \sqrt{L^2 - 2s(s + \sqrt{L^2 + s^2})} \right) \right)}{\left(s + \sqrt{L^2 + s^2} - \sqrt{L^2 - 2s(s + \sqrt{L^2 + s^2})} \right)^2}, \right. \right. \\ b == \frac{1}{2} \left(s + \sqrt{L^2 + s^2} - \sqrt{L^2 - 2s(s + \sqrt{L^2 + s^2})} \right) \left. \right\}, \\ \left\{ a == \frac{2s \left(L^2 + (s + \sqrt{L^2 + s^2}) \left(2s + \sqrt{L^2 - 2s(s + \sqrt{L^2 + s^2})} \right) \right)}{\left(s + \sqrt{L^2 + s^2} + \sqrt{L^2 - 2s(s + \sqrt{L^2 + s^2})} \right)^2}, \right. \\ b == \frac{1}{2} \left(s + \sqrt{L^2 + s^2} + \sqrt{L^2 - 2s(s + \sqrt{L^2 + s^2})} \right) \left. \right\}, \\ \left\{ a == \frac{2s \left(L^2 + (-s + \sqrt{L^2 + s^2}) \left(-2s + \sqrt{L^2 + 2s(-s + \sqrt{L^2 + s^2})} \right) \right)}{\left(-s + \sqrt{L^2 + s^2} + \sqrt{L^2 + 2s(-s + \sqrt{L^2 + s^2})} \right)^2}, \right. \\ b == \frac{1}{2} \left(s - \sqrt{L^2 + s^2} - \sqrt{L^2 + 2s(-s + \sqrt{L^2 + s^2})} \right) \left. \right\}, \\ \left\{ a == \frac{2s \left(L^2 - (s + \sqrt{L^2 + s^2}) \left(2s + \sqrt{L^2 + 2s(-s + \sqrt{L^2 + s^2})} \right) \right)}{\left(s - \sqrt{L^2 + s^2} + \sqrt{L^2 + 2s(-s + \sqrt{L^2 + s^2})} \right)^2}, \right. \\ b == \frac{1}{2} \left(s - \sqrt{L^2 + s^2} + \sqrt{L^2 + 2s(-s + \sqrt{L^2 + s^2})} \right) \left. \right\} \left. \right\},$$

Input and Output 2.3

Among the four returned solutions the first one is correct (to that we establish the approximate values of the exact particular solution, for example, according to input 2.2). To obtain a simple condition for real solutions one has to solve

$$L^2 - 2s(s + (L^2 + s^2)^{1/2}) \geq 0$$

for, e.g., L ; calculus with inequalities can thereby be avoided if we solve $L^2 - 2s(s + (L^2 + s^2)^{1/2}) = 0$ for L using “SolveReal” and test the inequality by inserting solutions for L and s . The result is $L \geq 2\sqrt{2} \cdot s$. This can also be understood as follows: the ladder cannot reach the wall if its length is less than twice the diagonal of the square plinth cross section. The height b and the denominator of a , which equals $(2b)^2$, vanish only for $s = 0$, that is, in case the square cross section of the plinth degenerates to a point.

The particular solution can now be generalised, for example, by allowing the plinth to have a rectangular cross section. Fig. 2.8 shows a corresponding graphical solution etc. The formulation needs to be generalised to

$$\frac{s_2}{b} = \frac{a - s_1}{a}, \quad L^2 = a^2 + b^2$$

and solved for a and b .

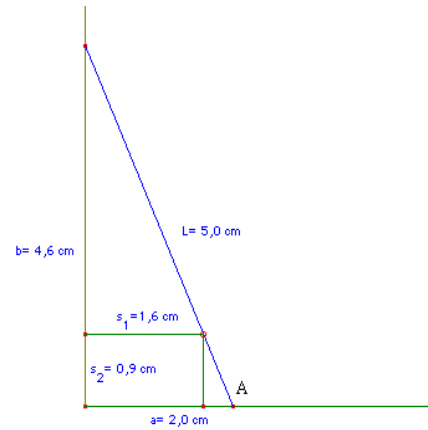


Fig. 2.8

Example 3:

A square pyramid has a volume of 50 cubic centimetres and a surface of 100 square centimetres. Determine the base length and height of the pyramid (reversal of a stereometric standard problem).

Graphical solution:

We construct a cavalier projection of a square pyramid with variable base length (movable corner A) and variable height (movable apex X) – Fig. 3.1. Using the calculator from our construction tool set and measurements for a , h and h_T (the height of a side triangle which is hidden in the figure) we generate terms for volume V and surface S , the values of which vary with a and h . (Using a 3D-graphics package suitable for geometry at school we could compute volume and surface of a spatial object “square pyramid” directly.) We drag A and X more or less systematically until approximate values for $V = 50.0 \text{ cm}^3$ and $S = 100 \text{ cm}^2$ are achieved and read for the base length and the height the values: $a = 6.2 \text{ cm}$, $h = 3.9 \text{ cm}$ (Fig. 3.2).

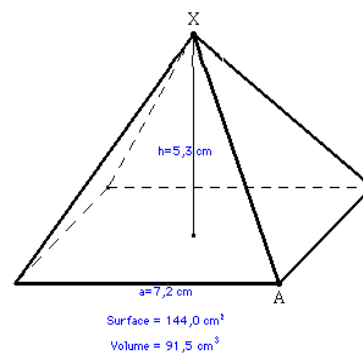


Fig. 3.1

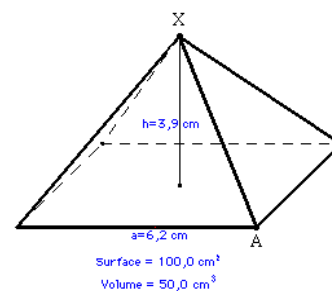


Fig. 3.2

Numerical solution:

Either we use a sketch or one of the figures already made for the graphical solution (Fig. 3.3). – Starting from the given data one develops a formulation for the solution in a manner as systematic as possible; in this case in the form of a so-called *top-down* formulation (input 3.1).

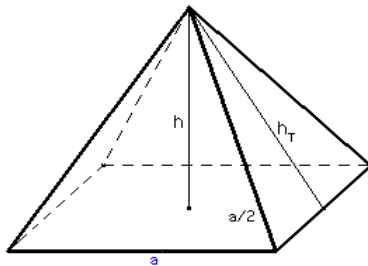


Fig. 3.3

```

NSolvePositive[{V == 50, S == 100,
  V ==  $\frac{1}{3}$  B h,
  S == B + LS,
  B == a2,
  LS == 4T,
  T ==  $\frac{a}{2}$  hT,
  hT2 == ( $\frac{a}{2}$ )2 + h2},
  {a, h},
  {B, LS, T, hT}}
```

```

{{h == 3.92375, a == 6.18294},
 {h == 12.7429, a == 3.43092}},
```

Input and Output 3.1

```

SolveReal[{V ==  $\frac{1}{3}$  B h,
  S == B + LS,
  B == a2,
  LS == 4T,
  T ==  $\frac{a}{2}$  hT,
  hT2 == ( $\frac{a}{2}$ )2 + h2},
  {a, h},
  {B, LS, T, hT}}
```

```

{{h ==  $\frac{S^2 - \sqrt{S^4 - 288SV^2}}{24V}$ , a ==  $-\frac{\sqrt{S^2 + \sqrt{S^4 - 288SV^2}}}{2\sqrt{S}}$ },
 {h ==  $\frac{S^2 - \sqrt{S^4 - 288SV^2}}{24V}$ , a ==  $\frac{\sqrt{S^2 + \sqrt{S^4 - 288SV^2}}}{2\sqrt{S}}$ },
 {h ==  $\frac{S^2 + \sqrt{S^4 - 288SV^2}}{24V}$ , a ==  $-\frac{\sqrt{S^2 - \sqrt{S^4 - 288SV^2}}}{2\sqrt{S}}$ },
 {h ==  $\frac{S^2 + \sqrt{S^4 - 288SV^2}}{24V}$ , a ==  $\frac{\sqrt{S^2 - \sqrt{S^4 - 288SV^2}}}{2\sqrt{S}}$ }},
```

Input and Output 3.2

The sought-after variables are a , h and the auxiliary variables are B , LS , T and h_T . Because the determination and elimination, respectively, of six variables requires six (essential) equations we have satisfied a necessary condition for the solvability of the system of equations. We obtain four solutions of which only two correspond to positive solutions of a and h (output 3.2). – This raises the question of why there are four solutions. Resolving the formulation for a leads to a biquadratic equation. Doing the algebra by hand one would better solve for h . One of the solutions is already known. The other solution has been overlooked while constructing the graphical solution. (Warning! – Be not always satisfied with graphical solutions gained by dragging figures.) Now we can graphically illustrate this solution by correspondingly dragging A and X (Fig. 3.4). There is a flat but broad pyramid and a higher but narrower one, both of which match the given data for surface and volume.

Algebraic solution:

To get a grasp of the class of all problems represented by the particular solution we need to determine the exact general solution for given S and V . From this we can obtain the conditions for solvability, that is, relations between S and V for which real solutions, and, according to the requirements of the problem, real positive solutions exist.

We only need to remove the “N” from “NSolveReal” and relax the assignments $V = 50$, $S = 100$ (input 3.2). The exact resolution of the general formulation leads to four solutions, of which only the second and the fourth are acceptable (output 3.2). (If, for other problems, we could not easily recognise the correct solutions among the general ones, we only had to apply the approximating command “N” to the exact solutions while substituting given data.)

Conditions for real solutions:

$$S^4 - 288 \cdot S \cdot V^2 \geq S, \text{ or } S^3 \geq 2 \cdot (12 \cdot V)^2.$$

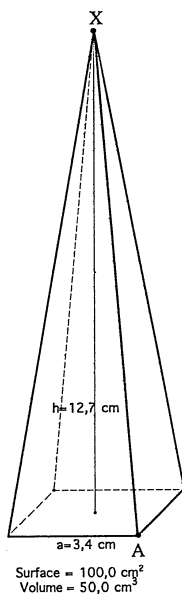


Fig. 3.4

Conditions for positive real solutions:

$$S^2 - (S^4 - 288 \cdot S \cdot V^2)^{1/2} > S, \text{ or } 288 \cdot S \cdot V^2 > 0,$$

which is satisfied for $S > 0$ and $V > 0$.

(Because current computer algebra systems cannot yet readily solve non-linear inequalities one would have to strengthen calculus involving inequalities in class.)

3. Concluding remarks

Remark 1:

Ideally a computer application should be able to execute all discussed modes of solution without the need to change the user interface, because such a change can, despite multi-tasking, cause difficulties for naïve users. There is still demand for software development in this regard.

Remark 2:

Computer-aided treatment of (geometric) calculation problems depends essentially on the chosen application. The question of which computer application best aids the solution of a particular kind of problem needs competent teachers and competent designers of syllabi for an answer. This as long as there are no didactic expert systems available. A difficulty is posed by the current monism as regards computer applications, that is, the attempt to solve (almost) all mathematical problems of relevance to school with a single computer application.

Remark 3:

The idea, illustrated in this work, of a comprehensive treatment of geometric calculation problems is an example of a change induced by a medium not only of methods but also of aims and subjects. However, here the conservative opinion is advocated that it is sufficient, initially, to realise new methods and aims with traditional, suitably complex problems.

Remark 4:

Computer-aided teaching of mathematics, particularly in secondary education plays a marginal role due to lack of corresponding curricula. This prevents a fruitful competition of old and new standards in the treatment of mathematical problems. Consequently ideas for the integration of new media in the teaching of mathematics fail to materialise and the crisis to justify mathematics instruction is intensified.

4. References

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