Aporism: Uncertainty about Mathematics

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Abstract: Neither absolutism nor aposteriorism have questioned the progressive elements associated with the applications and the social functions of mathematical knowledge. Aporism raises this question by discussing the thesis of the formatting power of mathematics. This thesis unites linguistic relativism applied to mathematics and the idea that technology is a structuring principle in society. We are no longer surrounded by "nature", instead we live in a techno-nature. Mathematical abstractions can be projected outside the sphere of mathematics, and in this way they modulate and eventually constitute fundamental categories of techno-nature.

The Vico paradox expresses the difficulties of specifying the nature and function of technological actions: We are not even able to grasp and to understand what we have ourselves constructed. A critique cannot be guaranteed by scientific (or mathematical) thinking itself. Critique becomes a much more complex activity including reflections on technological actions. A critique includes ethical considerations, and therefore a critique of mathematics is also ethical.

Kurzreferat: *Aporismus: Unsicherheit bzgl. der Mathematik.* Weder Absolutismus noch Aposteriorismus haben die progressiven Elemente, die mit den Anwendungen und sozialen Funktionen des mathematischen Wissens assoziiert werden, in Frage gestellt. Der Aporismus stellt diese Frage, indem er die These der formenden Macht der Mathematik diskutiert. Diese These vereinigt den auf die Mathematik angewandten linguistischen Relativismus und die Idee, daß Technologie ein Strukturprinzip der Gesellschaft ist. Wir sind nicht länger von "Natur" umgeben, stattdessen leben wir in einer "Techno-Natur". Mathematische Abstraktionen können nach außerhalb der mathematischen Sphäre projiziert werden, und auf diesem Wege regulieren und konstituieren sie eventuell sogar grundlegende Kategorien der Techno-Natur.

Das Vico-Paradox bringt die Schwierigkeiten, Natur und Funktion technologischer Aktionen zu spezifizieren, zum Ausdruck: wir sind nicht einmal fähig, das zu begreifen und zu verstehen, was wir selbst geschaffen haben. Kritik kann nicht durch wissenschaftliches (oder mathematisches) Denken selbst garantiert werden. Kritik wird zu einer viel komplexeren Tätigkeit, die Reflexionen über technologische Aktionen miteinschließt. Kritik enthält auch ethische Betrachtungen, und deshalb ist eine Kritik der Mathematik auch eine ethische.

ZDM-Classification: A40, E20, M10

A paradox

"In the past 100 years, we have seen enormous advances in our knowledge of nature and in the development of new technologies. ... And yet, this same century has also shown us despicable human behaviour. Unprecedented means of mass destruction, of insecurity, new terrible diseases, unjustified famine, drug abuse, and moral decay are matched only by an irreversible destruction of the environment. Much of this paradox has to do with the absence of reflections and considerations of values in academics, particularly in the scientific disciplines, both in research and in education. Most of the means to achieve these wonders and also these horrors of science and technology have to do with advances in mathematics." This is how Ubiratan D'Ambrosio, in "Cultural Framing of Mathematics Teaching and Learning", introduces a section about mathematics and society (D'Ambrosio 1994, 443).

According to the Enlightenment, scientific development and human progress are closely related. Therefore it seems a paradox that science can be related to human destruction. This paradox questions the optimistic assumption that science also sustains progress in an economic and political sense. Has science come to play a dual role? Is science related not only to human progress but also to human disaster? Does mathematics play a double-role, representing both reason and unreason in social development?

In the philosophy of science, including the philosophy of mathematics, mathematical knowledge and ethics have been kept apart. Mathematics has represented knowledge in its paradigmatic form, while ethics has been seen as an expression of emotions; and up till now it has not been considered important to interrelate these two domains. After mentioning a few classical positions in the philosophy of mathematics, I shall, however, suggest an interpretation of mathematics which invites ethical considerations.

Apriorism and absolutism

Apriorism claims that the source for mathematical knowledge is found in human rationality prior to any empirical experience. It is not necessary to make any sensory observations in order to obtain mathematical knowledge. This apriorism has been expressed by many philosophers: René Descartes, Immanuel Kant, Gottlob Frege, Bertrand Russell and Rudolf Carnap, just to mention a few. The justification for their apriorism has, however, been elaborated in different ways.[1]

Apriorism goes with *absolutism* which claims that it is possible to identify certain truths beyond any possible doubt. Thus, it has been a main concern for Russell to safeguard human knowledge from what he found to be barbarous attacks of skepticism. In particular Russell wanted to make mathematics a bastion for absolutism by making it an extension of logic.

Absolutism interprets "pure mathematics" as "innocent mathematics", as mathematics gets a share of divinity. We cannot imagine a proved mathematical proposition to be false; it expresses a necessary truth. God's imaginations and creations will, therefore, be in accordance with mathematics. (The Pythagorean Theorem also applies in Paradise.) As necessary truths cannot be otherwise, they cannot be questioned from an ethical point of view. Necessary truths also represent eternity.

The works of René Descartes represent a harmonious unification of rationality, critique and mathematical methodology. He suggests human rationality to be omnipotent: If reason is exercised in a proper way, no fact can be so remote that it cannot be grasped by human rationality. By means of reason, human beings are able to reach all possible truths.

How then can this rationality be exercised? Descartes suggests a method which reveals many similarities to Euclid's presentation of geometry. We have to identify some simple truths as absolute, because any deduction must find a start in axioms. These axioms will guarantee the truth of the deduced theorems. In order to identify such basic truths, a universal doubt has to be applied: Whatever is possible to doubt should be eliminated from the stock of held beliefs, only what is impossible to doubt should be retained. As it seems possible to doubt anything, this spring cleaning may empty the mind completely. Descartes, however, finds a fixed point for his rationalism in the statement, "Cogito ergo sum". To apply a universal doubt includes a mental activity, and carrying out an activity means that the person himself or herself must exist. For Descartes it is not possible to doubt the truth of "cogito ergo sum". This axiom he establishes as a foundation of knowledge. The steps forward are then taken by reason which deduces new truths from already established truths. Deductive reasoning becomes the constructive element in building up knowledge (Descartes 1993).

As critique is interpreted by rationalism as logical integrity, it is not possible to imagine any science more critical than mathematics. A critique addresses wrongly assumed opinions and beliefs, but necessary truths are beyond the scope of criticism. Mathematics becomes a subject beyond criticism, as critical thinking itself follows the pattern of mathematical thinking. In this field human reason has become transparent to itself. Mathematics represents pure and innocent knowledge which squares with God's knowledge. As a consequence, mathematics does not raise any ethical problems.

Logical positivism: Ethics as nonsense

Logical positivism excludes any possibility for an ethical discussion of mathematics, as ethical considerations become privatised. They represent irrationalism. Not all logical positivists adopted this interpretation of ethics, but following for instance Alfred J. Ayer's interpretation, we see ethics relegated as non-science, and ultimately as nonsense (see Ayer 1946, Chapter VI: Critique of Ethics and Theology).

The key idea in the positivist interpretation of science is expressed by the principle of verification according to which a proposition has a meaning if, and only if, it can be verified. This is a strong criterion of meaning. The purpose is to eliminate all formulations which might appear as meaningful and scientific but which in fact express nonsense. The principle of verification should eliminate all elements of dogmatism and metaphysics from theories which claim to be scientific.[2]

The principle of verification is brought into harmony with assumptions about language, logic and mathematics. In *Principia Mathematica,* Alfred N. Whitehead and Bertrand Russell try to show how all mathematical concepts and truths can be reduced to logical concepts and truths. This idea is to a large extent accepted by logical positivism. As formal logic (and therefore mathematics) provides a grammar for science, logical positivism assumes an affinity between logic and the principle of verification in the sense that the statements which are acceptable, according to the principle of verification, are precisely those statements which can be formulated in the language of science (Carnap 1959 and 1937).

The principle of verification suggests that ethical statements have no epistemic significance; they only represent personal expressions. As ethical statements cannot be verified, they have no meaning. They can only be seen as expressions of taste, and their semantic content becomes similar to the content of expressions like "Ummm" and "Ahh". Ethical neutrality in science is therefore a consequence of logical integrity.

Critical thinking becomes similar to scientific thinking. According to logical positivism, the notion of critique can be analysed in terms of a proper scientific methodology. A critical practice becomes a scientific practice, which must be kept separate from any ethical considerations. As logic and mathematics can be interpreted as the "grammar of science", they also come to represent the standards of critique. As a consequence, it does not make sense to raise any ethical critique of mathematics. This would simply corrupt logic.

Aposteriorism and fallibilism

Aposteriorism challenges apriorism. According to aposteriorism mathematical knowledge is grounded in experience.[3] This claim is made, for instance, by John Stuart Mill who finds that mathematical truths are inductive truths (Mill 1970, Book II, Chapter 6). The necessity of mathematical statements is only an expression of the regularity of certain sensory impressions. We have so many times observed instances of, say, $3 + 4 = 7$ that we assume this statement to be a necessary truth. Such ideas were presented by David Bloor in *Knowledge and Social Imagery* and by Philip Kitcher in *The Nature of Mathematical Knowledge.* Both elaborate on an a posteriori interpretation of mathematics with reference to Mill; and Kitcher presents a Mill Arithmetic which is supposed to capture the idea that mathematics can be seen as an idealised theory of actual operations.

Aposteriorism is accompanied by *fallibilism.* A classical argument in epistemology is that sense experiences may cause mistakes, and when knowledge is grounded on such a source, it cannot produce necessary truths. Our senses only reveal a world of contingencies, not a world of necessities. Imre Lakatos' fallibilistic interpretation of mathematics is inspired by Karl Popper's critical rationalism.

A main focus in Popper's philosophy of science is expressed by the title of his first important work, *The Logic of Scientific Discovery,* and the picture he outlines of scientific development is captured by the title of another of his works, *Conjectures and Refutations.* A scientific theory has the logical status of a conjecture, and it has no hope of obtaining a more prominent status. The only other possibility for a conjecture is to be refuted. The methodology of critical rationalism is: if we have a conjecture not yet falsified, then we must try to refute it. This procedure will make sure that conjectures which survive are hard-tested conjectures. This is the only way to ensure scientific progress. Science consists of conjectures not yet falsified, and therefore these conjectures might be true, although we will never come to know. This zig-zag route between conjectures and refutations is identified as a critical activity.

Critique, then, comes to characterise scientific process.

How Lakatos develops Popper's philosophy of science into a philosophy of mathematics is captured by the title of Lakatos' main work, *Proofs and Refutations* with the subtitle, *The Logic of Mathematical Discovery.* Mathematics follows a line of development similar to other sciences, although in this case the dialectics is constituted by proofs and refutations. Progress is ensured by means of quasi-empirical thought experiments. Concepts are modified or stretched with reference to non-formal considerations. Lakatos installs fallibilism in the heart of the last bastion of absolutism. He also emphasises his fallibilistic philosophy by not defining the notion of "mathematical truth" anywhere in *Proofs and Refutations.*[4]

Aposteriorism turns mathematical knowledge into "genuine" human knowledge by making it fallible. However, in Lakatos' philosophy no ethical questions have been raised. In fact, the fallibilism suggested by Lakatos includes a strong protection of mathematics. Although the logic of proofs and refutations is not transparent, it is not seen as producing obscure consequences. Instead it represents a sound methodology which can serve as a scientific programme (Lakatos 1970).

According to critical rationalism, critique is defined by the logic of the internal development of science. If this development follows the pattern of conjectures and refutations or, in case of mathematics, the pattern of proofs and refutations, then scientific development need not be subjected to any other form of critique.

Critical rationalism does not identify a critical activity outside scientific development. Proper scientific progress represents human progress. Therefore, Lakatos' fallibilism does not raise questions to mathematics beyond questions concerning the internal development of mathematics itself.

Critical rationalism represents a philosophy of mathematics which does not help to recognise the paradox mentioned by D'Ambrosio. Lakatos' fallibilism refers to micro-fallacies, not to social macro-fallacies. It does not invite an ethical examination of the *context* of mathematical development.

Aporism

It is possible to step beyond aposteriorism towards a new perspective of mathematics which I call *aporism.*[5] The Greek word *aporeo* means "being in a loss" or "being without resources". I have chosen this word in order to emphasise that the interpretation of mathematics is not based on a certain philosophical clarification of the nature of mathematics. Aporism, instead, represents an uncertainty about how to understand and criticise the "social agency" of mathematics. Aporism is an expression of a concern for decoding also the horrors which might be associated with applications of mathematics.[6]

Aporism acknowledges the possibility that pure reason may turn into perverted forms in which the ideal harmony between reason, scientific development and human and social progress is broken. As part of the rationalistic perspective, reason ensures the progressive qualities of knowledge, but aporism accepts the possibility that pure reason develops pathological cases, and that some of these are connected to the development of mathematics. In other words: the very notion of "pure reason" becomes problematised.[7]

Aporism elaborates on the paradox mentioned by D'Ambrosio: On the one hand, mathematics is a precondition for the wonders of technology, on the other hand, mathematics appears to be part of a destructive force also associated with technology. The paradox indicates that reason, expressed in applications of mathematics, may play a double role. Reason, in the shape of "instrumental reason", becomes problematic.

Aporism, however, represents a real uncertainty, as it does not simply suggest a bleak perspective of mathematics: The role of mathematics might in fact be progressive and strong.[8]

The formatting power of mathematics

Aporism relates to different ideas about mathematics. The first I refer to is the *thesis of the formatting power of mathematics*. This thesis has been expressed by the subtitle of the book *Descartes' Dream* written by Philip Davis and Ruben Hersh. The subtitle is: *The World According to Mathematics*. Mathematics is not only a language for talking about the world and a means for describing and explaining phenomena. Mathematics is also a tool for rearranging the world. It becomes a source for technological action and design. Social reality is a social construction which can only be understood if we are able to grasp the role of mathematics in social affairs.[9]

In epistemology, language has previously been seen as unproblematic. Phenomena described in any language are not affected by the language that describes them. Different languages only provide the same phenomenon with different labels. Language is a passive tool of which the user has control. This perspective of language (in accordance with the Greek tradition) has dominated philosophy until the beginning of this century – Bertrand Russell admitted that only after 1918 he began to look at language as a source of epistemological difficulties (Russell 1993, 108).

According to linguistic relativism, as originally formulated by Edward Sapir and Benjamin Lee Whorf, differences between languages also imply differences between perspectives (Sapir 1929, Whorf 1956). Language is not a neutral reflector of reality but also an active agent in the interpretation of phenomena, and this being so, language becomes a structuring code. The grammar of language is reflected in the world.

This thesis about "symbolic power" can be connected to the perspective of technology suggested by Jacque Ellul (Ellul 1964). Ellul sees technology not only as a means for struggling against nature but also as a means to organise our lives. Technology does not simply reflect human needs and human interests. (The assumption of rationalism was precisely that fabrications of human rationality represent human interests.) We are not in control of technology, as we are supposed to be in control of a tool. Technology cannot be compared to a tool. It becomes a principle for organisation, but also technology itself becomes an organisation. Therefore, technology concerns not only our relationship with nature, but it relates to all sorts of human activities. We become enclosed by technology.

Linguistic relativism and the assumption that technology encapsulates human activities can be related. Both language and technology penetrate the whole of civilisation. We are deluged in language, our means of understanding, and we are enveloped by technology, our means of survival. The hermeneutic condition, i.e. that we are captured by our own preconceptions, unable to escape the hermeneutic circle, becomes true in a double sense: Language and technology exercise formatting powers.[10]

The speech act theory emphasises that we are acting by means of language (Searle 1969). Mathematics is a language and can be seen as a source of actions of a quite different kind than those "innocent" actions normally discussed in the speech act theory. The mathematical language expresses its symbolic power through technology and performs a social structuring.[11]

The thesis of the formatting power of mathematics unites linguistic relativism applied to mathematics and the idea that technology is a structuring principle in society. We are no longer surrounded by "nature", instead we live in a *techno-nature.*[12]

Abstraction

Mathematics has to do with abstractions, but from what source do they emerge? The answer of apriorism is that mathematical categories are established prior to any sensory experience; therefore, mathematical abstractions grow from pure rationality. Aposteriorism suggests that abstractions are grounded on sensory experiences, and Bloor indicated that abstractions express social conventions (Bloor 1976, 86–87). Here, however, I shall focus on a different question: What is done by means of mathematical abstractions?

Abstractions can be projected outside the sphere of mathematics. We can talk about *realised abstractions* to be understood as abstractions made real.[13] It is not so that mathematics, whatever source abstractions might have, has to be understood first of all in terms of the origin of abstractions. Previously, the main concern of the philosophy of mathematics has been to identify the foundation of mathematics and the nature of mathematical progress, but aporism also tries to clarify the social and technological "activity" of mathematics. Therefore, aporism also considers how mathematical abstractions materialise as technological actions.

Such abstractions can enter social life in different forms. Davis and Hersh provide a long list of examples of prescriptive use of mathematics which leads to some sort of human or technological action: "We are born into a world with so many instances of prescriptive mathematics in place that we are hardly aware of them, and, once they are pointed out, we can hardly imagine the world working without them. Our measurements of space and mass, our clocks and calendars, our plans for buildings and machines, our monetary system, are prescriptive mathematisations of great antiquity. To focus on more recent instances ... think of the income tax. This is an enormous mathematical structure superposed on an enormous pre-existing mathematical financial structure. ... In American society, there are plentiful examples of recent and recently reinstated prescriptive mathematisation: exam grades, IQ's, life insurance, taking a number in a baker shop, lotteries, traffic lights ... telephone switching systems, credit cards, zip codes, proportional representation voting ... We have introduced these systems, often for reasons known only to a few; they regulate and alter our lives and characterise our civilisation. They create a description before the pattern itself exists" (Davis and Hersh l988, 120–121).

These examples of prescriptive use of mathematics also illustrate realised abstractions. All aspects of human life seem to be affected by mathematical thinking, and the formatting power of mathematics can be analysed in terms of realised abstractions.

Modulation and constitution

Following Kant in *Critique of Pure Reason,* we might think of space and time as fundamental forms of intuition. As mathematics represents features of these forms, it will by necessity apply to our experiences. I shall not interpret mathematics with reference to fundamental forms of intuition but, instead, I shall see mathematics in relation to fundamental domains of technological action.[14] These domains can also be seen as categories of our techno-nature, and in this sense they apply to our "life-world".

The thesis of the formatting power of mathematics can now be specified in the following way: *The fundamental categories of the techno-nature are continually modulated and eventually constituted by mathematical abstractions.* The modulation and constitution are expressions of realising abstractions.

When we talk about applications of mathematics, we often think of a preexisting non-mathematised situation which is modelled by means of mathematics. In this case it seems possible to separate the model from the "reality" to which mathematics is applied. Many applications of mathematics, however, do not have this nature. Mathematics grows into the situation and becomes part of the categories in which the activities take place. This is the reason why, in many cases, I prefer to talk about modulation and constitution instead of application.

In his lecture at The 7th International Congress on Mathematical Education in Québec, Thomas Tymoczko mentioned the relationship between mathematics and *war* (Tymoczko 1994). His point was that war and mathematics are interrelated in an intimate way. We may talk about modern warfare as constituted by mathematics – not in the sense that mathematics is the cause of war; but we cannot imagine modern warfare to take place without mathematics as an integral part (see also Højrup and Booss-Bavnbek 1994).

Davis and Hersh mention clocks and calendars as examples of the prescriptive use of mathematics. Our whole management of *time* is formed by mathematics. This seems a most direct example of the phenomenon that a domain of technological action is constituted by mathematics: "The mechanical clock extends the domain of quantification and measurability. ... Applying measure and number to time means measuring and quantifying all other areas, in particular those where time and space are related to one another. The measurability of time pushes forward the development of natural science as an (empirical) science of measurement (and hence objective science) and mathematics as the theory of measurement. The problems of constructing precise and accurate measuring instruments become a concern of mathematicians. ... The clock, used from the beginning as an instrument of social order and social coordination, changes the organisation of social life by following rigid 'objective' determination, organisation and control of various social interactions" (Keitel, Kotzmann and Skovsmose 1993, 256).

Concerning mathematics and *economy,* Tymoczko makes the following point: "Business does not just apply various already existing mathematical theories to facilitate an activity that is, in principle, independent from such mathematical application (although it can do that). Business could not exist in anything like its historical form without some mathematics. Certainly we cannot imagine a modern economy struggling along without mathematics then suddenly becoming more efficient because of the introduction of mathematics!" (Tymoczko 1994, 330). Many other studies make the same point. Thus, Davis and Hersh have described the income tax as a mathematical structure superposed on an already existing mathematical financial structure.[15]

As mathematics modulates and eventually constitutes financial structures, mathematics influences other categories of the techno-nature. This way mathematics becomes part of society's deep-structures.[16] I see mathematical modulation and constitution as deeper and much stronger than mathematical application. Maybe applications can be grasped, discussed and evaluated, but modulations and constitutions form the very categories of the techno-nature, and therefore they also penetrate our categories for understanding.

The Vico-paradox

According to Giambatista Vico, the rationalist idea that it is possible to come to understand nature and the whole universe expresses a blasphemy: How can humankind imagine that, by its limited resources, it could come to understand the creations made by an almighty and omniscient God? Each individual human being has only limited knowledge and limited power. God, as the creator of the universe, can understand how it works, but only the creator will be able to understand his work. What human beings can hope to understand is what they themselves have been able to create.

The Greek *techne* refers to human creation. Following Vico's line of ideas, we should expect it possible for the human mind to grasp technology which is the paradigm of human creations. But when we consider the functions of technology we are lost. Humankind is not in control of technology, not even from a conceptual point of view. We are unable to express effects of technology, whether intended or unintended. This I want to call the *Vico paradox:* Not even what we ourselves have constructed are we able to grasp and to understand.[17]

We no longer live in "nature". Our environment is structured and organised into a "techno-nature". Science has provided us with means for describing and predicting natural phenomena, which can be used for technological inventions. But when we face techno-nature, which includes our own constructions, then natural sciences fail. Scientific knowledge of nature is not sufficient for interpreting the totality of nature and human construction. Neither sciences nor "critique of culture" provide us with the means for clarifying the effects of science. Religion and magic have been interpreted (for instance with inspiration from logical positivism) as expressions of an insufficient understanding of nature. The question, however, is: What phenomena will express our insufficient understanding of techno-nature?

The thesis of the formatting power of mathematics is first of all a sociological thesis about the role of mathematics, while the Vico-paradox expresses an epistemological thesis about the (lack of) possibilities for grasping the nature of this role. At present we have no access to an adequate conceptual framework for analysing what we are doing by means of mathematics. Realising this dilemma is an essential point of aporism. Devastating theoretical limitations of sociology and social philosophy are expressed by the fact that neither Giddens (1984) nor Habermas (1984, 1987) make any reference to mathematics as playing a role in the technological society.[18]

Why then is it so difficult to understand what mathematics is doing? Applications of mathematics (in the traditional sense of explicit mathematical modelling) express only a minor part of the social role of mathematics. Modulations and constitutions are fundamental phenomena in our technological society. Domains of technological action are deeply structured by mathematics. This makes it difficult to analyse and to reflect upon the functions of mathematics. This epistemic difficulty is expressed by the Vico-paradox which, however, is also a challenge. It could be a pseudo-paradox.[19]

Aporism, critique and ethics

According to the thesis of the formatting power of mathematics, human reason, as expressed by mathematics, modulates and constitutes fundamental categories of the techno-nature. Reason is not only a thinking-tool but a source for technological action as well. According to the Vico-paradox we are unable to fully reflect on the implications of what we are doing. Our reflections are partial and insufficient, but they are essential.

This situation, acknowledged by aporism, implies that the distinction between critique and ethics is obscure. According to rationalism, critical thinking can be identified with logical (deductive) reasoning, which means that mathematics represents critique. Logical positivism tried to conquer critique by equalling critical thinking with scientific methodology. A similar attempt was made by critical rationalism which connected critique directly with scientific progress. These strategies exemplify attempts to relate a critical activity directly to a well specified scientific activity.

From the perspective of aporism these strategies are problematic. The formatting power of mathematics and the

Vico-paradox show that a critique cannot be guaranteed by scientific (or mathematical) thinking. Critique becomes a much more complex activity including reflections on technological actions. A critique (of action) includes ethical considerations, and therefore a critique of mathematics is also an ethical task.

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Annotations

1 For an introduction to apriorism (from the perspective of logical positivism) see Ayer (1946, 71–87) and Hahn (1959).

2 For a discussion of the principle of verification, see Hempel (1959). The principle of verification also became a main target for the critique of logical positivism: How is this principle itself verified?

3 For an introduction to aposteriorism see the chapter "A Renaissance of Empiricism in the Recent Philosophy of Mathematics", in Lakatos (1978, 24–42).

4 This fallibilism has played an important role as a suggestion for the missing philosophy of progressive mathematics education, including constructivism; see for instance Ernest (1991).

5 This notion has been suggested to me by Irineu Bicudo from Universidade Estadual Paulista, Rio Claro, Brazil.

6 In what follows I shall interpret aporism within an ethical perspective, but it is also possible to elaborate on a logical interpretation. This has been suggested to me by Tasos Patronis, The University of Patras, Greece. A starting point, then, could be the incompleteness theorem of Kurt Gödel which expresses the fact that a formal representation of a mathematical theory cannot provide a full picture of this theory. This point can be related to the "problem of consciousness": A consciousness cannot fully reflect its own states.

7 Several authors have presented ideas related to aporism; thus Mogens Niss states "... it is a striking fact that although the social significance of mathematics seems to be ever increasing in scope and density, the place, rôle and function of mathematics are largely invisible to – and unrecognised by – the general public, decision makers and politicians" (Niss 1994, 371). See also Borba and Skovsmose (1997).

8 Aporism can be interpreted as a suggestion for a working philosophy of a critical mathematics education, see Skovsmose (1994) and Skovsmose and Nielsen (1996).

9 The social role of mathematics in technology has been discussed by many authors. See for instance Booss-Bavnbek (1995); Booss-Bavnbek and Pate (1989); Davis (1989); Højrup and Booss-Bavnbek (1994); Keitel (1989, 1993); Keitel, Kotzmann and Skovsmose (1993); Keitel et al. (Eds.) (1989) and Restivo (Ed.) (1993). For a discussion of the thesis of the formatting power of mathematics, see Skovsmose (1994).

10 Also the growing risk structures of society have a source in technology. Technology not only influences the actual structures of society but moreover the likelihood that certain potential events may in fact occur. From where do those risk structures emerge? Mathematical formalisation can be seen as a source of such structures. Formalisation produces a way of looking at reality and therefore a new way of acting, and formalisation-based actions can be risk-producing. For a discussion of risks and mathematical modelling, see Booss-Bavnbek (1991).

11 The notion of "symbolic power" has been discussed by Pierre Bourdieu (1991), while Anthony Giddens has made social structurations a key term in his interpretations of sociology, see Giddens (1984).

12 The notion of "second nature" instead of "techno-nature" has been used in Skovsmose (1994) and in Keitel, Kotzmann and Skovsmose (1993).

13 For a discussion of the notions "thinking abstraction", "realised abstraction" and "real abstraction", see Keitel, Kotzmann and Skovsmose (1993).

14 The list of such domains can be elaborated in many different ways. Let me just mention: time, space, economy, war, science, and communication.

15 How mathematics constitutes and modulates economic affairs is also discussed in Swetz (1987) and in Damerow et al. (1974). See also Fischer (1993).

16 This also provides an interpretation to the notions of "implicit mathematics" and "frozen mathematics" understood as mathematics which is integrated in fundamental human affairs; see for instance Keitel (1993).

17 The Vico-paradox is discussed in Skovsmose (1994). For a previous introduction, see Jensen and Skovsmose (1986).

18 Naturally, I fully acknowledge the many attempts to study the relationship between mathematics, technology and society. I do not claim such studies to be superficial. My point is only that concepts for understanding the techno-nature are "weak", compared to concepts and theories for studying nature.

19 This epistemic difficulty is a challenge for critical mathematics education. See Skovsmose (1994) and Skovsmose and Nielsen (1996).

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