On the Notion of Space in Geometry

Vagn Lundsgaard Hansen, Lyngby

Abstract: The development of the notion of space in geometry is traced from the early axiomatization in Euclid's Elements over the discovery of non-Euclidean geometries to geometry of manifolds in relativity theory and in gauge and string theories in contemporary physics. The notion of space is considered in a historic-philosophical perspective including a short discussion of the contributions of artists to visualization of spatial objects.

Kurzreferat: *Über den Raumbegriff in der Geometrie.* Die Entwicklung des Raumbegriffs in der Geometrie wird skizziert von der frühen Axiomatisierung in den Elementen von Euklid über die Entdeckung nichteuklidischer Geometrien zur Geometrie der Mannigfaltigkeiten in der Relativitätstheorie sowie in der Eichtheorie und in der Stringtheorie der modernen Physik. Der Raumbegriff wird in einer historisch-philosophischen Perspektive betrachtet, wobei auch Beiträge von Künstlern zur Visualisierung räumlicher Objekte angesprochen werden.

ZDM-Classification: A30, E20, G10

What is space? There is no immediate answer to this question. For the layman space is where we live, or slightly more sophisticated, where we exist and have our sensations. For a philosopher, the nature of space is a challenging problem. In the fundamental philosophical work *Kritik der reinen Vernunft* in 1781, Immanuel Kant considers space and time to be a priori given forms of appearance ("Anschauungsformen") necessary for having sensations. What is space, or, more generally, a "space", for a mathematician, and more precisely for a geometer? This is the question to be considered here, in a historic-philosophical perspective.

1. Euclid's Elements: Axiomatisation of the notion of space

Geometry derives from the Greek word *geometria*, which means measurement of land. The word was used by the Greek historian Herodotus in the fifth century B.C. in his great epic on the Persian wars in which he writes that "geometria" was used in ancient Egypt to find the right distribution of land after the floods of the Nile.

As a framework for the description and measurement of figures, geometry was developed empirically in the early cultures of Egypt and Mesopotamia (often identified with Babylonia) several thousand years ago. Geometry as pure mathematics, which encompasses a collection of abstract statements about ideal figures and proofs of these statements, was founded around 600 B.C. in the Greek culture by Thales, who according to legend proved several theorems in geometry.

Classical Greek geometry has persisted mainly through the famous 13 books written by Euclid around 300 B.C. which are known as Euclid's *Elements*. In these books the mathematical and in particular the geometrical knowledge possessed by the Greeks at the time of Euclid is summarized and systematized in such a clear way that the exposition has put a stamp on mathematical writings ever since. The geometrical content is now known as *Euclidean geometry*.

Other peaks in classical Greek geometry was reached in works by Archimedes – to be singled out is his determination of the area of the surface of a sphere – and the work on conic sections by Apollonius from around 200 B.C.

Euclid's Elements contain definitions, postulates and theorems. The basic definitions are not particularly precise. For example a *point* is defined as an object without parts, and a *line* as an object without width. A line corresponds to what we nowadays call a segment of a curve. A *straight* line is defined as a line which runs directly along its points, and hence corresponds to what we nowadays call a line segment. Since one cannot define a notion by listing all the properties it does not possess, these definitions leave much to be desired and should be taken as a point of departure only. It was, however, also a position on the existence of *atoms*, indivisible elements, in the discussion among Greek philosophers about this notion; a point in Euclid's Elements is an atom.

In other definitions, Euclid defines notions such as angle and parallelism and geometrical figures such as the circle and the different types of triangles and quadrilaterals. Here we shall only state Euclid's definition of a *right angle*: "When a straight line is erected on another line, so that the angles next to each others have the same measures, then any of the angles having the same measures is *right*; and the erected line is said to be *vertical* to the other line."

In a certain sense postulates and theorems have the same status in Euclid, but with the difference that postulates are facts directly accepted as true without proofs, whereas theorems have to be proved by rational arguments from definitions and postulates.

Constructions of geometrical figures play an important role in Greek geometry, in providing the necessary proofs of existence. In particular, the constructions that can be performed with ruler and compass were considered important. Which rules must be satisfied in this connection? Which constructions are admissible? In the first three of his postulates, Euclid therefore writes down some basic admissible constructions with ruler and compass.

In the English translations of Euclid's Elements, the postulates (for plane geometry) are formulated in the following manner.

Let there be postulated:

1. *That one can draw a straight line from any point to any other point.*
2. *That one can extend a finite straight line continuously in a straight line.*
3. *That one can construct a circle with any centre and any radius.*
4. *That all right angles are equal to one another.*
5. *That, if a straight line intersects two straight lines and make the interior angles on the same side less than two right angles, then the two lines, if they are extended indefinitely, meet on that side on which are the angles less than two right angles.*

Subtle difficulties are hidden behind the above system of postulates, and the system is not sufficient to characterize Euclidean geometry. Not until two thousand years after Euclid did David Hilbert (1862–1943) succeed in formulating a complete set of axioms for Euclidean geometry in his book *Grundlagen der Geometrie* from 1899. Hilbert divided his axioms into five classes dealing respectively with *incidence* (about connections and intersections between points, lines and planes), *ordering* (about points between others on line segments), *congruence* (about similarity of line segments, angles and triangles), *parallelity* and *continuity* (about density of points on lines).

The methods which were developed to handle geometrical questions by Euclid, Apollonius and their many successors based on purely geometrical notions is now known under the name *synthetic geometry*.

A decisive new step in the history of geometry was taken with the development of *analytic geometry* (coordinate geometry) in the book *La Géométrie* by René Descartes (1596–1650) in 1637. With the emphasis in coordinate geometry on algebraic methods, the synthetic methods in geometry for some centuries receded in the background.

In his work, Descartes criticized the line of approach to geometry in Euclid's Elements and in Apollonius' work on conic sections; on the one hand he considered this line of approach to be too abstract and on the other hand as too dependent on the consideration of concrete figures. Independently of Descartes, also Pierre de Fermat (1601–1665) developed a coordinate geometry first published in 1679, although the basic work had been done in 1629. Fermat, in contrast to Descartes, felt in line with Greek thinking and he viewed his work as merely another way of formulating the work of Apollonios.

The appearance of *differential and integral calculus* by the end of the seventeenth century had enormous influence on the development of geometry, and a new branch of geometry, *differential geometry*, was born.

Fundamentally, however, the new developments in geometry mentioned above were still based on the Euclidean notion of space.

2. The influence of artists on the geometrical notion of space

From the dawn of mankind artists have contributed in a very literal sense to shaping our notion of space. In the oldest cultures, visual art not only factually but also visually was plane, i.e., without depth. This is especially evident in the Sumerian culture from around 3000 B.C., in ancient Egypt, and in the oldest Greek culture. On Greek vase paintings from the later period there are often spatial elements; for example in the well known motive "Ajax and Achilleus at the board game" from around 530 B.C. (see e.g. Sparkes 1996), in which a bilateral symmetry is realized by a 180° rotation in space around an axis in the motive. Examples of such realizations of a bilateral symmetry do, however, already exist in the Sumerian culture (see e.g. Weyl 1952, p.9). In the making of sculptures, artists of course always had to relate to physical space.

The spatial dimension in visual art, i.e., depth, was introduced much later, in close connection with the development of a theory of *perspective*. The laws of perspective were first formulated in the Renaissance by Alberti in 1435 and further developed by Leonardo da Vinci and Albrecht Dürer. From a mathematical point of view the theory of perspective exploits the so-called *central projection*, in which one considers a fixed *image plane* in space and a fixed *point of sight* outside the image plane. Any given point in space different from the point of sight is projected into the image plane by drawing the line from the point of sight through the given point to intersection with the image plane.

From a mathematical point of view the theory of perspective later was incorporated as one of the main methods in *descriptive geometry,* which is the theory of how a spatial figure can be represented mathematically correctly in a drawing plane by different projections, and how one can construct and determine intersections of spatial figures from such plane projections. Descriptive geometry was founded as a mathematical science by Gaspard Monge (1746–1818) in the work *Géométrie descriptive* published after a series of lectures on the subject in 1795. The subject is a forerunner for *projective geometry*, having its point of departure in a number of properties of geometrical figures left unchanged by central projection discovered in the seventeenth century by G. Desargues and B. Pascal among others.

3. On the notion of space in mathematics and philosophy

Geometry as pure mathematics, which encompasses a collection of abstract statements about ideal figures, was, as mentioned earlier, founded around 600 B.C. in the Greek culture by Thales. The ideal geometrical figures will here be referred to as geometrical forms. A *figure*, in other words, is the concrete realization of an abstract *form*. In a certain sense one can say that the artists in their works realize the abstract forms in concrete figures; thereby the artist gets into an intellectual relationship with the geometer, who looks for the abstract form behind the concrete figure.

Plato (427–348 B.C.) supposed that such shapes had an independent life in the world of ideas, which he assumed to be real. Figures in the world of sensations are imperfect images of the ideas. Aristotle (384–322 B.C.), who was the most important of Plato's pupils and immediate successors, did not accept the concept of a world of ideas. Concrete figures according to Aristotle consist of form and matter, and geometry has to do with the form of the figures. Thus mathematics has to do with the abstract aspects of concrete objects. It is, among other things, due to this discrimination between the concrete and the abstract that mathematicians at all later times have felt in great debt to the Greeks.

Many philosophers have taken a position on the notion of space, and we cannot mention all points of view here. Of particular importance for posterity has been Kant's discussion of the notion of space.

Immanuel Kant (1724–1804) views space and time as *a priori* given *forms of appearance*, since he considers space and time as necessary conditions of perception, which in itself cannot be perceived. In *Kritik der reinen Vernunft* in 1781, in the first section of *Die transzendentale Ästhetik,* Kant writes on space:

"Space is not an empirical concept which has been derived from outer experiences. For in order that certain sensations be referred to something outside me (that is, to something in another region of space from that in which I find myself), and similarly in order that I may be able to represent them as outside and alongside one another, and accordingly as not only different but as in different places, the representation of space must be presupposed. The representation of space cannot, therefore, be empirically obtained from the relations of outer appearance. On the contrary, this outer experience is itself possible at all only through that representation." (Kant 1781, p.38)

With his view on the notion of space, Kant excludes the possibility of more than one such notion. And for Kant, this a priori given notion of space is the one exhibited in Euclid's Elements.

Since space is a necessary condition for having sensations and since space is described in mathematical terms, Kant comes close to saying that everything that can be experienced through our senses can be described mathematically. Thereby he is in line with the Greek philosophers and with great physicists such as Galileo Galilei (1564–1642) and Isaac Newton (1642–1727).

4. Dimension and the geometrical notion of space

As a mathematical notion, *dimension* is derived from the properties related to length, area and volume. A geometrical figure, with length only (as a curve), is said to have dimension 1; if the figure can be ascribed an area (as a surface) it has dimension 2; a body in space with a volume has dimension 3. In general, a geometrical figure is said to have dimension n if the position of a point in the figure on small pieces can be specified by n coordinates. As an example, the position of a point on a curve can be determined by the distance along the curve measured from a fixed point on the curve. One can think of 4 dimensions as 3 spatial parameters and 1 time parameter. Higher dimensions can be imagined by adding further parameters, e.g. temperature, pressure, etc.

A geometrical figure, where the position of a point in the figure on small pieces can be described by n coordinates, is called an n-dimensional *manifold*. In geometry one studies the properties of manifolds and for a mathematician every manifold represents a *space*. The Dutch mathematician Freudenthal has even defined geometry as the branch of mathematics occupied with "grasping space" (Freudenthal 1973). And in its widest sense, the mathematical notion of space is inseparately attached to the notion of manifold here.

In addition to dimension also notions such as *orientation*, *curvature* and *symmetry* are fundamental notions in the mathematical description of "space" (Hansen 1993a). An orientation provides the object under consideration with preferred directions and rotations. Curvature has a direct relation to *metric* quantities such as length and volume. Symmetry measures the degree of *homogeneity* in the object.

5. Non-Euclidean geometries

Right from the beginning, mathematicians have speculated whether Euclid's 5th postulate is really necessary, or whether it is a consequence of the four other postulates. Down through the centuries, vain attempts were made to deduce Euclid's 5th postulate from the other postulates. During these efforts, many equivalent formulations of the postulate were found. The most famous equivalent formulation is due to Playfair 1795 and is known as Playfair's axiom, or

The parallel axiom: For any given line in the plane and any given point outside the line, there passes exactly one line through the given point that does not intersect the given line.

As a consequence of his definitions, and the five postulates, Euclid shows in the Elements that the following theorem holds.

The sum of the angles in a triangle: The sum of the angles in a triangle is equal to the sum of two right angles (180°).

The most famous attack on the problems in connection with Euclid's 5th postulate is due to the Italian mathematician Girolamo Saccheri (1667–1733). In the book *Euclides ab omni naevo vindicatus* (Euclid freed of any spot) in 1733, Saccheri attempted to prove the theorem about the sum of the angles in a triangle, making only use of the first four of Euclid's postulates. He tried to obtain a contradiction by making assumptions corresponding to the sum of the angles in a triangle being greater than, respectively less than, the sum of two right angles. It can be proved that the theorem about the sum of the angles in a triangle is in fact equivalent to Euclid's 5th postulate. Saccheri did not succeed in giving a correct proof of the theorem as desired.

Around 1830 came the breakthrough, when the Russian mathematician Nicolai Ivanovich Lobachevsky (1793–1856) in 1829 and the Hungarian mathematician János Bolyai (1802–1860) in 1832 independently of each other announced that they could construct geometries satisfying all the properties known from Euclidean geometry except the parallel axiom. As it turned out, Euclid had made a clever decision when he formulated the 5th postulate in the description of his geometry. Karl Friedrich Gauss (1777–1855) had in fact obtained similar results already back in 1816 but had kept his findings to himself since they deviated so strongly from accepted philosophical thinking of the time.

In a paper in 1887 Henri Poincaré (1854–1912) described a particularly well known model of a non-Euclidean geometry: *the hyperbolic plane* (see e.g. Hansen 1997). The points in Poincaré's model of the hyperbolic plane are the points within the boundary of a Euclidean disc Φ, and the *hyperbolic lines* are the Euclidean circular arcs in Φ that intersect the boundary circle of Φ at right angles. As hyperbolic lines we also include all Euclidean diametres in Φ; one can think of these hyperbolic lines as Euclidean circles with "infinitely large" radius. Angles are measured as the corresponding Euclidean angles. The sum of angles in a hyperbolic triangle is less than 180°.

In this century Poincaré's model has been used by the Dutch artist M. C. Escher in four circular woodcuts with the common title: *Circle limit* (see e.g. Ernst 1976). In Escher's picture *Circle limit III* the fish swim along circular arcs similar to hyperbolic lines (see e. g. Coxeter 1996) and it is easy to find examples of circular arcs breaking the parallel postulate. In the picture *Circle limit IV* Escher has constructed a fascinating mosaic of devils and angels. All the line segments are strictly hyperbolic, and all devils and all angels have the same non-Euclidean size. When they appear to be of different sizes it is due to the fact that the model distorts distances – in the same way as Greenland in a standard atlas of the world appears larger than Australia (which is not the case). There is infinitely far out to the boundary of the disc measured by hyperbolic length in Poincaré's model.

Due to the fine agreement between theory and observations in nature people had in the course of centuries become accustomed to thinking of Euclid's postulates as self evident truths. The construction of *non-Euclidean geometries* raised the question which kind of geometry describes the physical world in the best possible way.

6. The qualitative aspects of the notion of space

In a paper in 1679 Gottfried Wilhelm Leibniz (1646–1716) suggested a new type of investigations of geometrical figures, which he called *analysis situs*, or *geometria situs*. It remains obscure what he had in mind, but clearly he was unsatisfied with coordinate geometry as a means to handle geometrical figures. In a letter written in Catania in 1836, Johann B. Listing, a student of Gauss, suggested that the name analysis situs be substituted by *topology*, the name used nowadays. The name topology is generally attached to the study of those qualitative properties of geometrical figures that remain invariant by bending, streching, contraction or any other continuous deformation that does not create new points or let existing points melt together. To geometry similarly one counts those quantitative properties of geometrical figures that remain invariant by transformations preserving lengths and angles. In the 20th century, topology has grown into a very important mathematical subject.

We shall discuss the topology of closed surfaces as an illustration of many of the mathematical concepts and ideas presented above.

Intuitively, a surface is a geometrical object, which looks locally like the plane. In order to understand what this means, one can think about an atlas of the globe of the Earth. The plane charts in the atlas each provides the *local* information about a small part of the globe of the Earth, while the complete atlas of charts provides the *global* information about the globe of the Earth. In topology one transfers this picture and the terminology directly and say that a geometrical object locally looks like the plane if it can be described by an atlas of plane charts such that the local information in overlapping charts is compatible. Such objects are also called *2-dimensional manifolds* since as earlier mentioned there are natural generalizations to higher dimensions.

It is always possible to choose a sense of direction around any point in a surface. If these senses of directions can be chosen consistently along the surface it is said to be an *orientable* surface; in the opposite case it is said to be a *non-orientable* surface. A model for all non-orientable surfaces was discovered by A.F. Möbius (1790–1868). A so-called *Möbius band* is obtained by taking a rectangular strip and gluing a pair of opposite ends, after first giving the strip a half twist. In general, a surface is non-orientable if it contains a closed strip equivalent to a Möbius band.

A surface, which consists of a single piece (it is connected), has bounded extension and no edges (it is without boundary), is called a *closed surface*. Every orientable, closed surface is topologically equivalent to a sphere or to the surface of a ball with handles. The number of handles on the ball is called the *genus* of the surface. Correspondingly, every non-orientable, closed surface is topologically equivalent to a sphere from which a number (the genus of the surface) of "curved" circular discs have been removed and substituted by Möbius bands sewed onto the surface along the boundaries of the circular holes. A particularly well-known non-orientable, closed surface was discovered by Felix Klein (1849–1925) and is now known as the *Klein bottle*. It has genus 2, since it can be split into two Möbius bands. Any model of a non-orientable, closed surface realized in 3-dimensional space must by necessity exhibit self-intersections.

The first complete exposition of the topological classification of the closed surfaces was given in a survey paper on topology in 1908 by the German mathematician Max Dehn (1878–1952) and the Danish mathematician Poul Heegaard (1871–1948).

The main idea in the proof of the classification theorem goes back to Bernhard Riemann (1826–1866) and proceeds by a process which modern topology has dubbed *surgery* (see e.g. Hansen 1993a, Chapter 2).

It turns out that an orientable closed surface of genus at least 2 can be smoothly "wrapped up" in the hyperbolic plane (see e.g. Hansen 1993b), whereby it can be given constant negative curvature. The curves of (locally) shortest length, the *geodesics*, in the resulting geometry are the images of the hyperbolic lines in the hyperbolic plane. A closed surface of genus at least 2 thereby gets the structure of a so-called *hyperbolic* space form. Correspondingly, an orientable closed surface of genus 1 – a *torus* – can be smoothly “wrapped up” in the Euclidean plane whereby it can be given constant curvature zero; a so-called *parabolic* space form. In the resulting geometry, the geodesics are the images of the straight lines in the Euclidean plane. A closed surface of genus 0 is topologically equivalent to a sphere and can be given constant positive curvature; a so-called *elliptic* space form. In this case, the geodesics are the great circles on the sphere.

The above 2-dimensional space forms have natural higher dimensional analogues which play a significant role in the study of the geometry of manifolds in higher dimensions.

Manifolds occur naturally in mathematical models in e.g. physics, chemistry, biology and mathematical economy. In the applications, the geodesics in geometrical structures on the manifolds often represent important phenomena.

7. Geometry of manifolds in modern physics

With the discovery of non-Euclidean geometries began one of the golden periods in the interaction between mathematics and physics, which in the beginning of this century led Albert Einstein (1879–1955) to develop his *theory of relativity* (see e.g. Hansen 1993a, Chapter 5).

The concept of an *event* is fundamental in relativity. It is an idealized occurrence in the physical world without extension in neither space nor time. The collection of all possible events in the universe, those that have occurred in the past, those occurring now as well as those to occur in the future, together form a basic set M called a *space-time* in which particles move along so-called *world lines.* The general idea is then to equip the space-time M with as adequate a mathematical structure as possible to describe physical phenomena.

Hermann Minkowski (1864–1909) suggested a space-time M with a linear structure of dimension 4 which was used by Einstein in his *special theory of relativity* in 1905. In the *Minkowski space-time*, the 3 spatial axes have no coupling to the time axis. In order to incorporate gravitation, Einstein had, however, in his *general theory of relativity* in 1915, to substitute the Minkowski space-time with a curved space-time manifold M, which locally looks like the Minkowski space-time but in which the 3 spatial axes cannot globally be separated from the time axis. From a philosophical point of view it is noteworthy that this is the first time in history that the *geometry* of space-time is proposed as the cause of *physical* phenomena.

There are four known fundamental forces in nature, namely gravity, electromagnetism and the weak and the strong interactions of elementary particles. In order to make a theory that unifies these four forces in nature, physicists have to couple internal symmetries in elementary particles, like *spin*, with external symmetries of the particles in the space-time manifold. This coupling takes place in a so-called *fibre bundle* E over the space-time manifold M. The bundle space E is foliated in fibres F, one copy F_x of F for each point x in M, such that the possible internal symmetries for a particle at x is described by the elements in F_x. The resulting theories are known as *gauge theories*.

There are difficulties in combining gravity with the three other forces in a unified theory. Therefore physicists have developed an extension of gauge theories in which particles are represented by open or closed strings in the space-time manifold. There is some hope that out of these so-called *string theories* will emerge the Grand Unified Theory (GUT) which is the ultimate goal. In order to avoid certain undesirable features from a physical point of view in the mathematical models occurring in string theories, mathematics dictates the space-time manifold to be of dimension 26.

With gauge theories, and more generally, string theories, we have reached the frontiers of research in mathematical physics and a natural end of this essay. The question about the nature of “space” will certainly occupy philosophers also in the future, for the question is perhaps even deeper than the question about the “origin of life”. And mathematicians most certainly will invent new spaces and new structures which might have an impact on the so-called real world, whatever that is.

8. References

Coxeter, H. S. M.: The Trigonometry of Escher’s Woodcut Circle Limit III. – In: The Mathematical Intelligencer 18(1996)No. 4, p. 42–46

Ernst, B.: The Magic Mirror of M. C. Escher. – New York: Random House, 1976

Freudenthal, H.: Mathematics as an Educational Task. – Dordrecht: Reidel, 1973

Hansen, V. L.: Geometry in Nature. – Wellesley, MA: A. K. Peters, 1993a

Hansen, V. L.: Jakob Nielsen (1890–1959). – In: The Mathematical Intelligencer 15(1993b)No. 4, p. 44–53

Hansen, V. L.: The dawn of non-Euclidean geometry. – In: Int. J. Math. Educ. Sci. Technol. 28(1997)No. 1, p. 3–23

Kant, I.: Kritik der reinen Vernunft. – 1781, Second edition (B-version), p. 38. English translation by Norman Kemp Smith in 1929

Sparkes, B. A.: The Red and the Black. Studies in Greek Pottery. – London: Routledge, 1996

Weyl, H.: Symmetry. – Princeton University Press, 1952

Author

Hansen, Vagn Lundsgaard, Prof. Dr., Department of Mathematics, Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark. E-mail: hansen@mat.dtu.dk