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On two-dimensional recurrence

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1. Introduction.

We consider the question of the existence of smooth flows, on two--manifids, whose fixed points are hyperbolic and which have nontrivial recurrent trajectories. The problem is treated by smoothing continuous flows. This paper continues the work begun in [1] and extended in [3]. As an application of the main result, we improve the example of nonorientable nontrivial recurrence constructed in [4].

1.1. Definition. Let M be a C^∞ two-manifold and let $\varphi, \psi\colon R \times M \to M$ be continuous flows on M. We say that φ and ψ are *topologically equivalent* if there is a homeomorphism of M that takes trajectories of φ onto trajectories of ψ, preserving the natural orientation of the orbits.

1.2. Definition. A *simple Cherry flow* is a continuous flow on a compact connected two-manifold M without boundary such that:

(a) φ has only finitely many fixed points which are either (topological) saddle points or sources.

(b) Let $p_1, p_2, \ldots, p_m$ be the source-fixed-points of φ and let $u_1, u_2, \ldots, u_m$ be their basins of repulsion (= their *unstable manifolds*). Then, each u_j contains a unique trajectory θ_j connecting p_j to a (unique) saddle point $q_j \in \partial u_j$ (see Fig. 1).

(c) $\bigcup_{i=1}^{m} u_i$ is dense in M

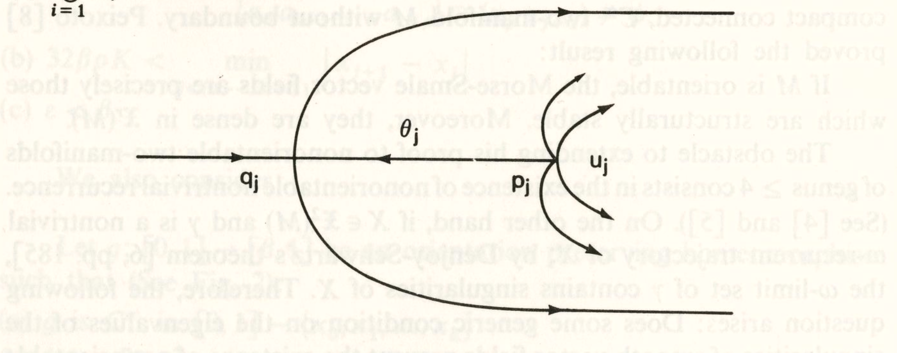

Fig. 1

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1.3. Definition. Let $\varphi: \mathbb{R} \times M \to M$ be a continuous flow on a two--manifold M. A trajectory γ of φ is said to be *nonorientable nontrivial recurrent* if given $p \in \gamma$, $\gamma - \{p\}$ has two connected components, γ_1 and γ_2, and moreover, for any segment S passing through p and transversal to φ, there exists connected components $ab \subset \gamma_1 - S$ and $cd \subset \gamma_2 - S$ such that both $ab \cup S$ and $cd \cup S$ contain a onesided simple closed curve.

Now we state our theorems

Theorem A. *Let φ be a simple Cherry flow on a compact connected two--manifold M without boundary. Then, there exists a smooth flow ψ, topologically equivalent to φ, such that:*

(a) *the fixed points of ψ are hyperbolic*

(b) *the eigenvalues of the saddle points of ψ can be chosen arbitrarily with the following restriction: if X_ψ denotes the smooth vector field induced by ψ and p is a saddle point, then the divergence of X_ψ at p is ≤ 0.*

Theorem B. *On any nonorientable two-manifold of genus ≥ 4, there exist simple Cherry flows having nonorientable nontrivial recurrent trajectories.*

It will be proved that in general, simple Cherry flows have nontrivial recurrent trajectories. Flows with this sort of trajectory are not always smoothable; for example, Denjoy [2] constructed a C^1 flow on the torus which is not topologically equivalent to any C^2 flow. His example does not satisfy the condition (c) of the definition of simple Cherry flows.

Now we explain the motivation for Theorem B. We denote by $\mathfrak{X}^r(M)$, $r = 1, 2, \ldots, \infty$, the space of C^r-vector fields (with the C^r-topology) on a compact connected, C^∞ two-manifold M without boundary. Peixoto [8] proved the following result:

If M is orientable, the Morse-Smale vector fields are precisely those which are structurally stable. Moreover, they are dense in $\mathfrak{X}^r(M)$.

The obstacle to extending his proof to nonorientable two-manifolds of genus ≥ 4 consists in the existence of nonorientable nontrivial recurrence. (See [4] and [5]). On the other hand, if $X \in \mathfrak{X}^2(M)$ and γ is a nontrivial ω-recurrent trajectory of X, by Denjoy-Schwartz's theorem [6, pp. 185], the ω-limit set of γ contains singularities of X. Therefore, the following question arises: Does some generic condition on the eigenvalues of the singularities of smooth vector fields prevent the existence of nonorientable nontrivial recurrence? In this work, as a direct consequence of Theorems A and B, we give a negative answer to this question.

To simplify matters, we prove Theorem A only when the Cherry flow (M, φ) satisfies the following:

1.4. M is a compact connected two-manifold without boundary; φ is a simple Cherry flows on M with precisely one source fixed point and there is no trajectory connecting two saddle points.

At the end of the proof of Theorem A, we indicate how to deal with the general case. The main tool in the proof of Theorem A is Lemma 2.4. It will be seen that the proof of Theorem B follows easily from [4].

2. The Packing Lemma.

In this section, we give some preliminars and then we prove Lemmas (2.3) and (2.4). Lemma (2.3) is used to prove Lemma (2.4) (the packing Lemma).

Let μ be the usual Borel measure of $\mathbb{R}$ (that is, if (a, b) is an interval of $\mathbb{R}$, then $\mu((a, b)) = b\text{-}a$). Let E and F be intervals of $\mathbb{R}$; *we say that* $E < F$ if $\forall\, e \in E$, $\forall f \in F$, $e < f$. $\mathbb{N}$ will denote the set of positive integers.

We need the following:

2.1. Let $x_0, x_1, \ldots, x_K$ be fixed elements of $[0, 1]$ satisfying $0 = x_0 < x_1 < \ldots < x_{K-1} < x_K = 1$. Let $\varepsilon, \beta \in (0, 1)$, ρ be a positive integer and $\{a_i\}_{i \in \mathbb{N}}$ be a sequence of different elements of $\bigcup_{j=0}^{K-1} (x_j, x_{j+1})$ *such that*:

(a) $\forall i \in \mathbb{N}$, $\forall j \in \{0, 1, \ldots, K-1\}$, we have that
$$\{a_i, a_{i+1}, \ldots, a_{i+\rho}\} \cap (x_j, x_{j+1}) \neq \phi.$$

(b) $32\beta\rho K < \min_{j \in \{0, \cdots, K-1\}} |x_{j+1} - x_j|$

(c) $\varepsilon < \beta$.

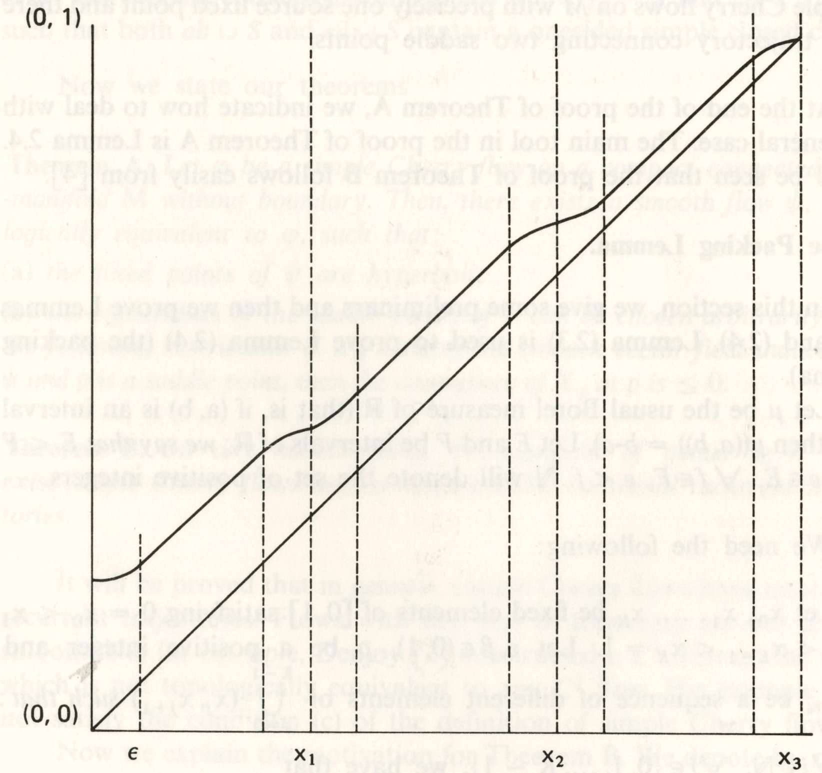
We also consider

Let g: $[0, 1] \to [\beta, 1]$ be an orientation preserving homeomorphism such that (See Fig. 2):

(a) g is C^∞ in $[0, 1] - \{x_0, x_1, \ldots, x_K\}$

(b) $\forall x \in [0, 1] - \{x_0, x_1, \ldots, x_K\}$, $g'(x) \leq 1$ and moreover $g'(x) = 1$ if and only if $x \in (0, 1) - \bigcup_{i=0}^{K} (x_i - \varepsilon,\ x_i + \varepsilon)$.

(c) $\forall j \in \{0, 1, \ldots, K\}$, g' is decreasing in $(x_j - \varepsilon, x_j) \cup [0, 1]$ and increasing in $(x_j, x_j + \varepsilon) \cap [0, 1]$.



graph of g when K = 3

Fig. 2

Lemma 2.3. *Assume (2.1) and (2.2). Given $p \in \mathbb{N}$, we can construct a finite sequence of open non-empty pairwise disjoint intervals $B_1 = A_{1p}$, $B_2 = A_{2p}$, $\ldots$, $B_p = A_{pp}$ contained in $[0, 1]$ and such that:*

(i) $\mu(B_1) = \beta$

(ii) $\{B_1, B_2, \ldots, B_p\}$ *is ordered by* $\{a_1, a_2, \ldots, a_p\}$ *and* $\{x_0, x_1, \ldots, x_K\}$*. That is*

(ii.1) *If* $a_i \in (x_j, x_{j+1})$*, for some* $i \in \{1, 2, \ldots, p\}$ *and* $j \in \{0, 1, \ldots, K-1\}$*, then* $B_i \subset (x_j, x_{j+1})$*; and*

(ii.2) *If* $a_i < a_t$*, when* i, $t \in \{1, 2, \ldots, p\}$*, then* $B_i < B_t$.

(iii) $\{B_1, B_2, \ldots, B_p\}$ *is modelled by g. That is,* $\forall i \in \{2, 3, \ldots, p\}$, $\mu(\beta_i) = \mu(g(B_{i-1}))$

(iv) $\{B_1, B_2, \ldots, B_p\}$ *is piled up* $\{x_0, x_1, \ldots, x_K\}$*. That is:*

(iv.1) $\bigcup_{i=1}^{p} B_i$ *is the finite union of pairwise disjoint sets* $I_{x_0}, I_{x_1}, \ldots, I_{x_K}$ *such that* $\forall i \in \{0, 1, \ldots, K\}$, I_{x_i} *is either the empty set or a closed interval containing* x_i.

(iv.2) *Given* $j \in \{1, 2, \ldots, K\}$*, let* $\{B_{j1}, B_{j2}, \ldots, B_{jl_j}\} = \{B_i / i \in \{1, 2, \ldots, p\}$ *and* $B_i \subset (x_{j-1}, x_j)\}$.

If $B_{j1} \cup B_{j2} \cup \ldots \cup B_{jl_j} \not\supset [x_{j-1}, x_{j-1} + \varepsilon] \cup [x_j - \varepsilon, x_j]$*, then any element of* $\{B_{j1}, B_{j2}, \ldots, B_{jl_j}\}$*, with the possible exception of one, is contained in* $[x_{j-1}, x_{j-1} + \varepsilon] \cup [x_j - \varepsilon, x_j]$.

Proof. Let us denote by $\tilde{x}_j = 2pj$, $j = 0, 1, \ldots, K$, and define an auxiliary map $G: [0, 2pK] \to [\beta, 2pK]$ satisfying

(1) $\forall j \in \{0, 1, \ldots, K\}$, G restricted to $(\tilde{x}_j - \varepsilon, \tilde{x}_j + \varepsilon) \cap [0, 2pK]$ is, modulo a translation (rigid movement) in its domain and image, equal to g restricted to $(x_j - \varepsilon, x_j + \varepsilon) \cap [0, 1]$. Moreover,

$$\forall x \in [0, 2pK] - \bigcup_{j=0}^{K} (\tilde{x}_j - \varepsilon, \tilde{x}_j + \varepsilon), \; G'(x) = 1.$$

We recall that,

$$\forall x \in [0, 1] - \bigcup_{j=0}^{K} (x_j - \varepsilon, x_j + \varepsilon), \; g'(x) = 1.$$

Certainly, the map $G: [0, 2pK] \to [\beta, 2pK]$, defined as follows, satisfies (1).

$$G(x) = \begin{cases} g(x_j + (x - \tilde{x}_j)) + \tilde{x}_j - x_j, & \text{if } x \in [\tilde{x}_j - \varepsilon, \tilde{x}_j + \varepsilon] \cup [0, 2pK] \\ & \text{for some } j \in \{0, 1, \ldots, K\}. \\ g(x_j + \varepsilon) + x - (x_j + \varepsilon), & \text{if } x \in [\tilde{x}_j + \varepsilon, \tilde{x}_{j+1} - \varepsilon] \text{ for some} \\ & j \in \{0, 1, \ldots, K-1\}. \end{cases}$$

Now, let $H: [0, 1] \to [0, 2pK]$ be a homeomorphism such that, $\forall j \in \{0, 1, \ldots, K\}$, $H(x_j) = \tilde{x}_j$. Let us write, $\tilde{a}_i = H(a_i)$, $\forall i \in \mathbb{N}$. We claim that:

(2) There exists a finite sequence of open non-empty, pairwise, disjoint intervals $\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_p$ contained in $[0, 2pK]$ and such that (See the statement of this Lemma).

(i′) $\mu(\tilde{B}_1) = \beta$

(ii′) $\{\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_p\}$ is ordered by $\{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_p\}$ and $\{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_K\}$.

(iii′) $\{\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_p\}$ is modelled by G.

(iv′) $\{\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_p\}$ is piled up on $\{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_K\}$.

In fact, since, $\forall j \in \{0, 1, \ldots, K-1\}$, $\mu((\tilde{x}_j, \tilde{x}_{j+1})) = 2p$, we may construct a set $\{\tilde{B}'_1, \tilde{B}'_2, \ldots, \tilde{B}'_p\}$ of open, non-empty pairwise disjoint intervals of $[0, 2pK] - \bigcup_{j=0}^{K} [\tilde{x}_j - \varepsilon, \tilde{x}_j + \varepsilon]$, ordered by $\{\tilde{a}_1, \ldots, \tilde{a}_p\}$ and $\{\tilde{x}_0, \ldots, \tilde{x}_K\}$, and such that, $\forall i \in \{1, 2, \ldots, p\}$, $\mu(\tilde{B}'_i) = \beta$. Notice that the set $\{\tilde{B}'_1, \tilde{B}'_p, \ldots, \tilde{B}'_p\}$ is modelled by G because $\forall x \in [0, 2pK] - \bigcup_{j=0}^{K} [\tilde{x}_j - \varepsilon, \tilde{x}_j + \varepsilon]$, $G'(x) = 1$. For each $j \in \{1, 2, \ldots, K\}$, let $\{\tilde{B}'_{j1}, \tilde{B}'_{j2}, \ldots, \tilde{B}'_{jl_j}\} = \{\tilde{B}'_i \mid \tilde{B}'_i \subset (\tilde{x}_{j-1}, \tilde{x}_j)$ and $i \in \{1, 2, \ldots, p\}\}$ and assume that $\tilde{x}_{j-1} < \tilde{B}'_{j1} < \tilde{B}'_{j2} < \ldots < \tilde{B}'_{jl_j} < \tilde{x}_j$. Now, we move $\tilde{B}'_{11}$ towards $\tilde{x}_0$ (in a continuous way) so that the resulting new sequence, which we still denote by $\{\tilde{B}'_1, \tilde{B}'_2, \ldots, \tilde{B}'_p\}$, satisfies (i) – (iii). Certainly $\tilde{B}'_{11} = \tilde{B}'_\sigma$, for some $\sigma \in \{1, 2, \ldots, p\}$, and so when $\tilde{B}'_\sigma$ is being moved towards $\tilde{x}_0$, the measures of the intevals $\tilde{B}'_{\sigma+1}, \tilde{B}'_{\sigma+2}, \ldots, \tilde{B}'_p$ are being readjusted (in a continuous way). We stop moving $\tilde{B}'_{11}$ when $\tilde{B}'_{11} = (\tilde{x}_0, \tilde{x}_0 + \beta)$. Next, we move $\tilde{B}'_{12}$ towards $\tilde{B}'_{11}$ in such a way that, at any stage of the process, $\{\tilde{B}'_1, \tilde{B}'_2, \ldots, \tilde{B}'_p\}$ satisfies (i) – (iii). We stop moving $\tilde{B}'_{12}$ when $\tilde{B}'_{11} = (\tilde{x}_0, z)$ and $\tilde{B}'_{12} = (z, v)$, for some $z, v \in (\tilde{x}_0, \tilde{x}_1)$. Notice that $z \leq \tilde{x}_0 + \beta$ because, $\forall x \in [0, 2pK] - \{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_K\}$, $G'(x) \leq 1$. Let $\tilde{B}'_{11} = \tilde{B}'_{\sigma_1}$ and $\tilde{B}'_{\sigma_2} = \tilde{B}'_{12}$, with $\sigma_1, \sigma_2 \in \{1, 2, \ldots, p\}$; it may happen that $\sigma_2 < \sigma_1$ and so we may have $z < x_0 + \beta$. Observe that (2.1), item (c), tell us that $\mu(\tilde{B}'_{11})$ (resp. $\mu(\tilde{B}'_{12})$) varies in a monotonic nonincreasing way when we move either $\tilde{B}'_{11}$ or $\tilde{B}'_{12}$. Since the sequence $\{\tilde{B}'_1, \tilde{B}'_2, \ldots, \tilde{B}'_p\}$ is finite, by continuing in this manner, we will find the intervals $\{\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_p\}$ satisfying (2).

Now, we will prove that

(3) $$\forall j \in \{0, 1, \ldots, K-1\},\ \mu\Big(\bigcup_{\tilde{B}_i \subset (x_j, x_{j+1})} \tilde{B}_i\Big) < \mu([x_j, x_{j+1}]).$$

Certainly (1) – (3) will imply the existence of the B_i's, $i = 1, 2, \ldots, p$, satisfying the required conditions in this Lemma.

In order to prove (3), we first claim that

(4) $$\overline{\bigcup_{i=1}^{p} \tilde{B}_i} \not\supset \bigcup_{j=0}^{K} ([\tilde{x}_j - \varepsilon, \tilde{x}_j + \varepsilon] \cap [0, 2pK]).$$

Otherwise, since G is an isometry in $[0, 2pK] - \bigcup_{j=0}^{K} ([\tilde{x}_j - \varepsilon, \tilde{x}_j + \varepsilon] \cap [0, 2pK])$ and $G([0, 2pK]) = [\beta, 2pK]) = [\beta, 2pK]$, we have that

(5) $$\mu\Big(\bigcup_{i=1}^{p} \tilde{B}_i\Big) = \mu\Big(G\Big(\bigcup_{i=1}^{p} \tilde{B}_i\Big)\Big) + \beta$$

therefore

$$\sum_{i=1}^{p} \mu(\tilde{B}_i) = \mu\Big(\bigcup_{i=1}^{p} \tilde{B}_i\Big) = \mu\Big(G\Big(\bigcup_{i=1}^{p} \tilde{B}_i\Big)\Big) + \mu(\tilde{B}_1) = (\text{by } (5))$$

$$= \sum_{i=1}^{p} \mu(\tilde{B}_i) + \mu(G(\tilde{B}_p)) \ (\text{by } ((2), (\text{iii}'))).$$

This implies that $\mu(G(\tilde{B}_p)) = 0$ and, since $\{\tilde{B}_1, \ldots, \tilde{B}_p\}$ is modelled by G, that $\mu(\tilde{B}_1) = 0$ which is a contradiction. This proves (4).

It follows from (4) that there exists $m \in \{0, 1, \ldots, K-1\}$ such that

(6) $$\overline{\bigcup_{\tilde{B}_i \subset (x_m, x_{m+1})} \tilde{B}_i} \not\supset [\tilde{x}_m, \tilde{x}_m + \varepsilon] \cup [\tilde{x}_{m+1} - \varepsilon, \tilde{x}_{m+1}].$$

Now, we prove (3). Let $j \in \{0, 1, \ldots, K-1\}$ and $\Sigma = \{i \in \{1, 2, \ldots, p\} / \tilde{B}_i \subset (\tilde{x}_j, \tilde{x}_{j+1})\}$. Let $\theta_1 = \min\{t \in \Sigma\}$. Let us suppose that the positive integers $\theta_1, \theta_2, \ldots, \theta_s$ have been defined. If $\Sigma - \bigcup_{j=1}^{s} \bigcup_{i=0}^{\rho} (\theta_j + i) \neq \phi$, we proceed inductively to define

$$\theta_{s+1} = \min\Big\{t \mid t \in \Sigma - \bigcup_{j=1}^{s} \bigcup_{i=0}^{\rho} (\theta_j + i)\Big\}.$$

Certainly, there exist positive integers $\sigma, \theta_1, \theta_2, \ldots, \theta_\sigma$ such that

$$\Sigma - \bigcup_{j=1}^{\sigma} \bigcup_{i=0}^{\rho} (\theta_j + i) = \phi.$$

By (6), because $\{\tilde{B}_1, \ldots, \tilde{B}_\rho\}$ is piled up on $\{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_K\}$ ((2), item (iv′)), by ((2.1), (a)) and ((2.2), (b)), we have that given θ_i, $i \in \{1, 2, \ldots, \sigma\}$, there exists $t_i \in \{0, 1, \ldots, \rho\}$ such that

(7) $$\mu(\tilde{B}_{\theta_i}) = \mu(\tilde{B}_{\theta_i+1}) = \ldots = \mu(\tilde{B}_{\theta_i+t_i}), \quad \text{but} \quad \mu(\tilde{B}_{\theta_i+t_i}) > \mu(\tilde{B}_{\theta_i+t_i+1})$$

this implies that

(8) $$\tilde{B}_{\theta_i+t_i} \subset \bigcup_{i=0}^{K} [\tilde{x}_i - 2\varepsilon, \tilde{x}_i + 2\varepsilon] \cap [0, 2pK].$$

Therefore

$$\mu\Big(\bigcup_{B_i\subset(x_j,x_{j+1})}\tilde{B}_i\Big)\leq\sum_{\substack{i=1\\s=0}}^{\substack{s=\rho\\i=\sigma}}\mu(\tilde{B}_{\theta_i+s})\leq$$

$$\leq 2\rho\sum_{i=1}^{\sigma}\mu(\tilde{B}_{\theta_i+t_i})\leq \text{(See (7), ((2), (item (iii)') and}$$

notice that, $\forall x\in[0,1]-\{\tilde{x}_0,\tilde{x}_1,\ldots,\tilde{x}_K\}$; $G'(x)\leq 1$)

$$\leq 2\rho\sum_{i=0}^{K}\mu([\tilde{x}_i-2\varepsilon,\tilde{x}_i+2\varepsilon]\cap[0,2\rho K])\leq \text{(See (8))}$$

$$\leq 16\rho K\varepsilon<\frac{1}{2}\mu([x_j,x_{j+1}])\qquad\text{(See (2.1))}$$

this proves (3) and therefore the Lemma.

Lemma 2.4. (*Packing Lemma*). *Assume* (2.1) *and* (2.2). *There exists a sequence* $\{A_i\}_{i\in\mathbb{N}}$ *of open, non-empty, pairwise disjoint subintervals of* $[0,1]$

(i) $\mu(A_1)=\beta$

(ii) *If* $a_i\in(x_j,x_{j+1})$, *for some* $i\in\mathbb{N}$ *and* $j\in\{0,1,\ldots,K-1\}$, *then* $A_i\subset(x_j,x_{j+1})$. *Moreover, if* $a_s<a_t$, *for* $s,t\in\mathbb{N}$, *then* $A_s<A_t$.

(iii) $\forall i\in\mathbb{N}$, $\mu(A_{i+1})=\mu(g(A_i))$

(iv) *The connected components of* $\overline{\bigcup_{i=1}^{\infty}A_i}$ *are* $K+1$ *closed intervals* W_{x_0}, $W_{x_1},\ldots,W_{x_{K-1}}$, W_{x_K} *containing* $[x_0,x_0+\varepsilon]$, $[x_1-\varepsilon,x_1+\varepsilon]$, …, $[x_{K-1}-\varepsilon,x_{K-1}+\varepsilon]$ *and* $[x_K-\varepsilon,x_K]$, *respectively*

(v) $\mu\Big(\overline{\bigcup_{i=1}^{\infty}A_i}\Big)=\sum_{i=1}^{\infty}\mu(A_i)$

Proof. For each $p\in\mathbb{N}$, let $A_{ip}=\{u_p^i,v_p^i\}$, $i=1,2,\ldots,p$, be as in Lemma 1. Certainly, there is a sequence $\{a_{i1}\}_{i\in\mathbb{N}}$ of natural numbers such that $\lim u_{a_{i1}}^1$ and $\lim v_{a_{i1}}^1$ exist. Assume that we have chosen sequences $\{a_{ij}\}_{i\in\mathbb{N}}$, $j=1,2,\ldots,k$, and proceed inductively to select $\{a_{i(k+1)}\}_{i\in\mathbb{N}}$ as being a subsequence of $\{a_{ik}\}_{i\in\mathbb{N}}$, and such that $\lim u_{a_i(k+1)}^{k+1}$ and $\lim v_{ai(k+1)}^{k+1}$ exist. Denote, $\forall n\in\mathbb{N}$ and $\forall i\in\mathbb{N}$, $a_n=a_{nn}$, $\lim_n u_{a_n}^i=u_i$ and $\lim_n v_{a_n}^i=v_i$. Under these conditions it follows easily from Lemma (2.3) that the open intervals of the sequence $\{\tilde{A}_i=(u_i,v_i)\}_{i\in\mathbb{N}}$ are pairwise disjoint, satisfy properties (i), (ii) and (iii), and consequently, they are non-empty (We do not know if they satisfy properties (iv) and (v) as well).

Now we claim that

$$\mu\Big(\bigcup_{i=0}^{K}[x_i-\varepsilon,x_i+\varepsilon]\cap[0,1]-\bigcup_{i=1}^{\infty}A_i\Big)=0\tag{1}$$

In fact, let $\Sigma=[0,1]-\bigcup_{i=1}^{\infty}A_i$. Because $g([0,1])=[\beta,1]$ and (iii), we have that

$$1=\mu(\Sigma)+\sum_{i=1}^{\infty}\mu(A_i)$$

and that

$$1=\beta+\mu(g(\Sigma))+\sum_{i=1}^{\infty}\mu(g(A_i))=\mu(g(\Sigma))+\sum_{i=1}^{\infty}\mu(A_i).$$

Therefore $\mu(\Sigma)=\mu(g(\Sigma))$. This together with the fact that,

$$\forall x\in\bigcup_{i=0}^{K}((x_i-\varepsilon,x_i+\varepsilon)\cap\Sigma),\ g'(x)<1,$$

imply that (1) is true.

As a consequence of (1) and because g is an isometry in $[0,1]-\bigcup_{i=0}^{K}(x_i-\varepsilon,x_i+\varepsilon)$, we only need to pile up the intervals of $\{A_i\}_{i\in\mathbb{N}}$ which are contained in $[0,1]-\bigcup_{i=1}^{K}(x_i-\varepsilon,x_i+\varepsilon)$ in order to get that $\{A_i\}_{i\in\mathbb{N}}$ satisfy not only (i), (ii) and (iii), but also (iv) and (v).

3. Smoothability of simple Cherry flows.

We shall need same terminology and notation. Let $\varphi:\mathbb{R}\times M\to M$ be a continuous flow on a two-manifold M. The *positive* (*negative*) *semi-trajectory* of $x\in M$ is the set

$$\gamma_x^+=\{\varphi(t,x)/t\in[0,\infty)\}\ (\gamma_x^-=\{\varphi(t,x)/t\in(-\infty,0]\}).$$

The *trajectory* of $x\in M$ is the set $\gamma_x=\gamma_x^+\cup\gamma_x^-$. A point $y\in M$ is an ω-limit point (α-*limit point*) of $x\in M$, if there is a sequence of real numbers $t_k\to\infty$ ($t_k\to-\infty$) such that $\varphi(t_k,x)\to y$. The set of ω-limit points (α-limit points) of x is denoted $\omega(x)$ ($\alpha(x)$).

We say that either $x\in M$ or γ_x is *non-trivial ω-recurrent* (α-*recurrent*) if $x\in\omega(x)$ ($x\in\alpha(x)$) but γ_x is neither a fixed point nor a closed orbit.

Lemma 3.1 (Peixoto). *If ψ is a flow on a compact two-manifold M, having only fintely many fixed points which are saddle points and Γ is a segment (circle) transverse to ψ, then the domain of definition of the Poincaré map $T: \Gamma \to \Gamma$ induced by ψ is the finite union of intervals (x, y). The endpoint x (resp. y) lies on a trajectory tending to a saddle point, as $t \to \infty$, and/or is an endpoint of Γ.*

Proof. (See [8, Lemma 3, pp. 106] or [7, pp. 157]).

Lemma 3.2. *Let (M, φ) be a simple Cherry flow which satisfies (1.4). Then φ posseses a non-trivial ω-recurrent trajectory.*

Proof. Let p be the source-fixed-point of φ. We first prove that φ has no closed trajectory. Assume that θ is a closed trajectory of φ; since $W^u(p)$ is dense in M, θ is a sink. Let $W^s(\theta)$ denote the basin of attraction of θ (= its stable manifold) and Γ_1 (resp. Γ_2) be a circle transverse to φ contained in $W^u(p)$ (resp $W^s(\theta)$). By the same proof as that of Lemma (3.1), since $\overline{W^u(p)} = M$ and because there is a unique saddle point connected to p by a trajectory (Def. (1.2)), we have that both the Poincaré map $T_{12}: \Gamma_1 \to \Gamma_2$ induced by φ and its inverse T_{12}^{-1} are defined everywhere except at one point (which belongs to a saddle separatrix). However, this is impossible because φ has no trajectory connecting two saddle points (See (1.4)).

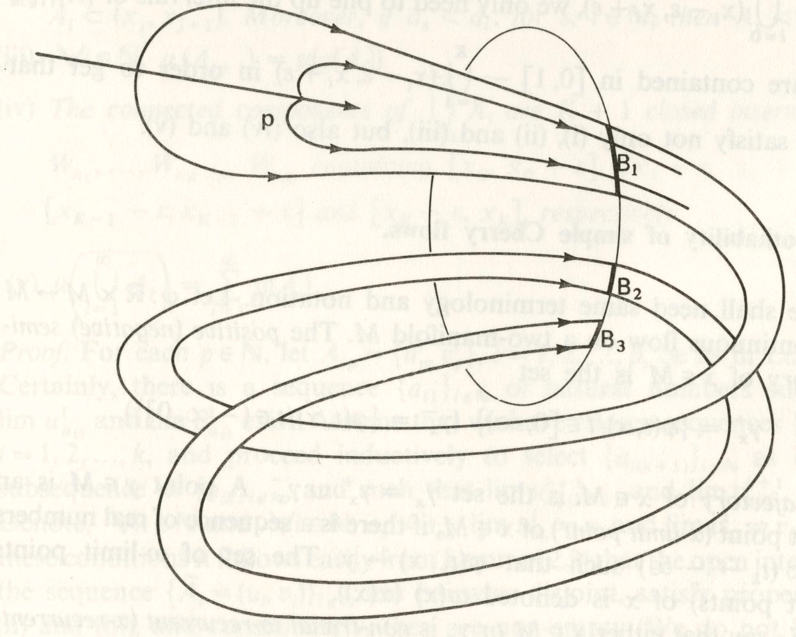


Fig. 3

Now, assume that φ does not have any nontrivial ω-recurrent trajectory. Let $x \in W^u(p) - \{p\}$ such that $\omega(x)$ is not a fixed point (See Definition (1.2)). By [8, proof of Lemma 3] or [7, proposition 2.3, pp. 151], $\omega(x)$ is either a closed orbit or a graph made up of fixed points and trajecries connecting them. This is a contradiction because φ has neither closed orbits nor trajectories conecting saddle points.

In the remainder of this section, (M, φ) will denote a simple Cherry flow which verifies (1.4). Let p be the sourcefixed-point and q be a non--trivial ω-recurrent point of φ. There exists a circle C transverse to φ and passing through q (See [5, Lemma 2] or [9, Proposition 7.1, pp. 66]). Certainly, the sets $B_i = \{x \in C/$there exists a (positive) trajectory γ leaving the source p such that γ crosses C exactly $i - 1$ times before passing through $x\}$, $i = 1, 2, \ldots, n, \ldots$ are open, non-empty, pairwise disjoint subintervals of C (See fig. 3).

Lemma 3.3. *Let $T: C \to C$ be the Poincaré map induced by φ. T is defined everywhere except possibly at finitely many points which belong to stable separatrices of saddle points of φ. We denote $x_0 = x_K$ and assume that (x_i, x_{i+1}), $i = 0, 1, \ldots, K-1$, are the connected components of the domain of T.*

Proof. This follows at once from Lemma (3.1) and from the fact that $W^u(p)$ is dense in C.

Lemma 3.4. *Let B_i, $i = 1, 2, \ldots, n, \ldots$, and $x_0, x_1, \ldots, x_{K-1}$ be as above. Fix $a_i \in B_i$, $i = 1, 2, \ldots, n, \ldots$. There exists a positive integer ρ such that $\forall i \in \mathbb{N}$, $\forall j \in \{0, 1, \ldots, K-1\}$, we have that*

$$\{a_i, a_{i+1}, \ldots, a_{i+\rho}\} \cap (x_j, x_{j+1}) \neq \phi.$$

Proof. Suppose that $C = \mathbb{R}/\mathbb{Z}$ and identify $\mathbb{R}/\mathbb{Z}$ with $[0, 1)$ in the canonical way. Let "<" denote the usual linear order of $[0, 1) = \mathbb{R}/\mathbb{Z}$ and assume that

$$0 = x_0 < x_1 < \ldots < x_{K-1} < 1 = 1 = x_K \text{ (See Lemma (3.3)).}$$

Let q_1 be the saddle point connected to p by a trajectory. By (1.4) (φ has no trajectory connecting saddle points) and by Lemma (3.1), we have that, $\forall\ i \in \mathbb{N}$, if s is an endpoint of B_i, then s belongs to an unstable separatrix λ_1 of q_1 and moreover (See definition of the B_i's) λ_1 intersects positively C precisely $i - 1$ times before reaching s. Hence

(1) The closed intervals of $\{\overline{B}_i\}_{i \in \mathbb{N}}$ are pairwise disjoint.

By Lemma (3.3), if $s \in C$ is not in a stable separatrix of φ, then, $\forall n \in \mathbb{N}$, T^n is defined in s. Now, we prove that

(2) $\forall\ s \in C - \bigcup_{i=1}^{\infty} B_i$, either s belongs to a stable separatrix of φ or $\{T^n(s)/n \in \mathbb{N}\}$ is dense in $C - \bigcup_{i=1}^{\infty} B_i$.

Otherwise, there exists $s \in C - \bigcup_{i=1}^{\infty} B_i$ and an open interval $(x, y) \subset C$ which contains a point of $C - \bigcup_{i=1}^{\infty} B_i$ and is disjoint from the closure of $\{T^n(s)/n \in \mathbb{N}\}$. Under these conditions, since $\bigcup_{i=1}^{\infty} B_i$ is dense C and by (1) it follows easily that

(3) There are infinitely many B_i's contained in (x, y).

From now on, assume that x (resp y) is an endpoint of some B_i. Thus, Lemma (3.1), (3) and the fact that $\overline{\bigcup_{i=1} B_i} \supset (x, y)$ imply that

(4) There exist, $\sigma_1, \sigma_2, \ldots, \sigma_{n-1} \in (x, y)$ $x = \sigma_0 < \sigma_1 < \ldots < \sigma_{n-1} < \sigma_n = y$ such that for $p \in (x, y)$, $p \in \{\sigma_1, \ldots, \sigma_{n-1}\}$ if and only if γ_p^+ goes to the set {saddle points of φ} $\cup \{x\} \cup \{y\}$ before reintersecting (x, y) (recall that $p \in \gamma_p^+$). Moreover, the Poincaré map $S: (x, y) \to (x, y)$ induced by φ is defined in (σ_i, σ_{i+1}), $\forall\ i \in \{0, 1, \ldots, n-1\}$, (Notice that in general $S \neq T$).

Therefore

(5) $\forall i \in \{0, 1, \ldots, n-1\}$, there exists $\ell_i \in \mathbb{N}$ such that, $\forall s \in \{1, 2, \ldots, \ell_i\}$, $T^s((\sigma_i, \sigma_{i+1})) \cap (x, y) = \phi$, but $T^{\ell i+1}((\sigma_i, \sigma_{i+1})) \subset (x, y)$. And

(6) There exists $\ell_n \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{\ell_n} B_i\right) \cap (x, y) = \phi,$$

but $B_{\ell_{n+1}} \subset (x, y)$.

Denote

$$\mathfrak{a}_1 = \bigcup_{i=1}^{\ell_n} B_i \quad \text{and} \quad \mathfrak{a}_2 = \bigcup_{i=0}^{n-1} \bigcup_{s=0}^{\ell_i} (T^s((\sigma_i, \sigma_{i+1})).$$

By using (4) (6) we obtain that, $\forall t \in \mathbb{N}$, $B_{\ell_n+1+t} \subset \mathfrak{a}_2$. Hence (notice that $\overline{\bigcup_{i=1}^{\infty} B_i} = C$), we find that.

(7) $\overline{\mathfrak{a}_1 \cup \mathfrak{a}_2} = C$.

Now we observe that if $T^m(s)$ belong to $\mathfrak{a}_2$, for some $m \in \mathbb{N}$, it will exist $\tilde{m} \in \mathbb{N}$, $\tilde{m} \geq m$, such that $T^{\tilde{m}}(s) \in (x, y)$ which is contradiction. So,

(8) $\{T^n(s)/n \in \mathbb{N}\} \subset C - (\mathfrak{a}_1 \cup \mathfrak{a}_2)$.

Because $\mathfrak{a}_1 \cup \mathfrak{a}_2$ is formed by finitely many intervals whose endpoints are contained in saddle separatrices and by (7), it is seen that s belongs to a saddle point separatrix and $\{T^n(s)/n \in \mathbb{N}\}$ is finite (recall that, $\forall n \in \mathbb{N}$, T^n is defined in s). This contradiction proves (2).

Given $s \in C$ and a neighborhood $V(s)$ of s in C we denote:

$$V^*(s) = \begin{cases} V(s) - s, & \text{if } s \text{ belongs to a stable separatrix of } \varphi. \\ V(s) & \text{otherwise} \end{cases}$$

Since $\bigcup_{i=1}^{\infty} B_i$ is dense in C, by (1) and by (2), it can be easily seen that, $\forall s \in C$, there exists a positive integer $\tilde{p} = \rho(s)$ and a neighborhood $V(s)$ of s in C such that, $\forall y \in V^*(s)$ and $\forall j \in \{0, 1, \ldots, K-1\}$, we have that $\{y, T(y), \ldots, T^{\tilde{p}}(y)\} \cup (x_j, x_{j+1}) \neq \phi$. The proof of Lemma (3.4) follows easily from this and from the compactness of C.

3.5. Proof of Theorem A when (M, φ) satisfies (1.4).

Let $C, T: C \to C$ and $B_1, B_2, \ldots, B_i$ be as after (3.2). Let $x_0, x_1, \ldots, x_k$ be as in Lemma (3.3).

Denote by $\mathcal{O}$ the canonical positive orientation of $C = \mathbb{R}/\mathbb{Z}$. Let $a, b \in C$, $a \neq b$, we define the interval $(a, b) = \{z \in C - \{a\}/$if "<" denotes the linear order induced by the orientation $\mathcal{O}$ in $C - \{a\}$, then $z < b\}$. Observe that $(a, b) \cap (b, a) = \phi$ and $(a, b) \cup (b, a) = C - \{a, b\}$. The notation $a < c < b$ will mean that $c \in (a, b)$.

Choose, $\forall i \in \mathbb{N}$, $a_i \in B_i$. By Lemma (3.4) the sequence $\{a_i\}_{i \in \mathbb{N}}$, of elements of $\bigcup_{i=0}^{K-1} (x_i, x_{i+1})$, satisfies (a) of (2.1). Certainly, we can find $\beta > 0$ verifying (b) of (2.1), g: $[0, 1] \to [\beta, 1]$ as in (2.2), and then $\{A_i\}_{i \in \mathbb{N}}$ and $W_{x_i} = [c_i, d_i]$, $i \in \{0, 1, \ldots, K\}$ as in Lemma (2.4).

Let $\sigma = \sum_{i=0}^{K} \mu(W_{x_i})$. We define the following local isometry bijective functions

$$F: \left(\bigcup_{i=0}^{K} [c_i, d_i)\right) \to [0, \sigma) \quad \text{and} \quad L: \left(\bigcup_{i=0}^{K} g([c_i, d_i))\right) \to [\beta, \sigma)$$

as follows: $F(s) = s - c_j + \sum_{i=0}^{j-1} \mu([c_i, d_i))$, if $s \in [c_j, d_j)$ and $L(s) = \beta + s - g(c_j) + \sum_{i=0}^{j-1} \mu(g([c_i, d_i)))$, if $s \in g([c_j, d_j))$. Let $\tilde{g} = L \circ g \circ F^{-1}$: $[0, \sigma) \to [\beta, \sigma)$. It is clear that

(i) $\tilde{g}$ is a homeomorphism which is C^∞ everywhere except possibly at points of $\{F(x_0), F(x_1), \ldots, F(x_K)\}$. Moreover, given W_{x_i}, $i = 0, 1, \ldots, K$, there exist translations S_i and R_i of $\mathbb{R}$ (rigid movements) such that $S_i \circ g = \tilde{g} \circ R_i$ in W_{x_i}.

(ii) $\mu(A_i) = \mu(F(A_i))$, $\mu\left(\overline{\bigcup_{i=1}^{\infty} F(A_i)}\right) = \sum_{i=1}^{\infty} \mu(F(A_i)) = \sigma$, and $\bigcup_{i=1}^{\infty} F(A_i) = [0, \sigma]$.

(iii) $\tilde{g}$ induz a map $\tilde{g}$: $\mathbb{R}/_{\sigma\mathbb{Z}} \to \mathbb{R}/_{\sigma\mathbb{Z}}$.

(1) To simplify matters, let us suppose $\sigma = 1$ and therefore, $\tilde{g} = g$, $F(A_i) = A_i$ ($\forall i \in \mathbb{N}$), $\sum_{i=1}^{\infty} \mu(A_i) = 1$ and $F(x_i) = x_i$ $\forall i \in \{0, 1, \ldots, K\}$).

Given $i \in \mathbb{N}$, if $T|_{B_i}$ (T restricted to B_i) is orientation reversing (resp. preserving) we define $\tilde{T}_i = H \circ (-I) \circ g$ (resp. $\tilde{T}_i = H \circ g$), where $-I: \mathbb{R} \to \mathbb{R}$ denotes the map $x \to -x$ and H denotes the unique translation of $\mathbb{R}$ (rigid movement) satisfying $H \circ (-I) \circ g(A_i) = A_{i+1}$ (resp. $H \circ g(A_i) = A_{i+1}$).

We will prove that $\sum_{i=1}^{\infty} (\tilde{T}_{i/A_i})$ extends to a smooth function defined in $\bigcup_{0}^{K-1} (x_i, x_{i+1})$ in such a way that, $\forall i \in \{0, 1, \ldots, K-1\}$, $\tilde{T}$ restricted to (x_i, x_{i+1}) is either of the form $H \circ g$ or $H \circ (-I) \circ g$, where $H: \mathbb{R} \to \mathbb{R}$ is a translation. We only consider the case in which $i = 1$ and T is orientation preserving in (x_1, x_2). Let $A_k = (u_k, v_k)$ and $\tilde{T}_k = H_k \circ g$, $k = 1, 2, \ldots, n, \ldots$. Certainly, we only have to prove that if $A_k, A_j \subset (x_1, x_2)$, then $H_k = H_j$. Let us suppose that $u_j < v_j < u_k < v_k$. Hence:

$$
\begin{aligned}
| T_j(u_k) - \tilde{T}_j(v_j) | &= | H_j\, g(u_k) - H_j\, g(v_j) | = \\
&= | g(u_k) - g(v_j) | = (H_j \text{ is a translation}) \\
&= \mu(g(v_j, u_k)) = (\mu \text{ is the canonical measure of } \mathbb{R}) \\
&= \mu\Big(g\Big(\bigcup_{A_\ell \subset (v_j, u_k)} A_\ell\Big)\Big) = (\text{Lemma } (2.4), (v)) \\
&= \sum_{A_\ell \subset (v_j, u_k)} \mu(g(A_\ell)) = \sum_{A_\ell \subset (v_j, u_k)} \mu(A_{\ell+1}) \quad (\text{Lemma } (2.4), (iii)).
\end{aligned}
$$

Recall that a_i is a point of B_i. Certainly, given $a_s, a_j, a_t \in (x_1, x_2)$, with $a_s < a_j < a_t$, we have that $a_{s+1} < a_{j+1} < a_{t+1}$. This implies that $A_\ell \subset (v_j, u_k)$ if and only if $A_{\ell+1} \subset (v_{j+1}, u_{k+1})$. Therefore

$$
\sum_{A_\ell \subset (v_j, u_k)} \mu(A_{\ell+1}) = \sum_{A_{\ell+1} \subset (v_{j+1}, u_{k+1})} \mu(A_{\ell+1}) =
$$
$$
= | u_{k+1} - v_{j+1} | = | \tilde{T}_k(u_k) - \tilde{T}_j(v_j) |.
$$

That is, we have proved that

$$
(2) \qquad | \tilde{T}_j(u_k) - \tilde{T}_j(v_j) | = | \tilde{T}_k(u_k) - \tilde{T}_j(v_j) |.
$$

Since T is to be orientation preserving in (x_1, x_2) and by (2), it can be easily that

$$
\tilde{T}_j(u_k) = \tilde{T}_k(u_k), \text{ i.e., } H_j(g(u_k)) = H_k(g(u_k))
$$

which implies that $H_j = K_k$.

Now we claim that there exists a homeomorphism $h: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $h \circ T = \tilde{T} \circ h$ (i.e., T and $\tilde{T}$ are topologically conjugated). In fact, let $h_1: B_1 \to A_1$ be any orientation preserving homeomorphism. Let us suppose that $h_k: B_k \to A_k$ has been defined and proceed inductively to define $h_{k+1}: B_{k+1} \to A_{k+1}$ by $h_{k+1} = \tilde{T} \circ h_k \circ T^{-1}{}_{|B_{k+1}}$. Notice that if $B_i < B_j < B_k$ i.e., $\forall\, b_i \in B_i$, $\forall\, b_j \in B_j$, $\forall\, b_k \in B_k$, $b_i < b_j < b_k$) then $A_i < A_j < A_k$ (See Lemma (2.4), (ii)). Therefore, since $\{B_i\}_{i \in \mathbb{N}}$ and $\{A_i\}_{i \in \mathbb{N}}$ are dense in C, the map $\bigcup_{i=1}^{\infty} h_i$ can be extended to a homeomorphism $h: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which conjugates T and $\tilde{T}$.

Since T is topologically conjugate to $\tilde{T}$ and by using (M, φ) as a model, we can construct a manifold $\tilde{M}$ containing $\mathbb{R}/\mathbb{Z}$ and a C^∞ flow $(\tilde{M}, \psi)$ such that the Poincaré map $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ induced by ψ is precisely $\tilde{T}$. Actually, this construction can be made in such a way that ψ is topologically equivalent to φ (See [3, Proposition 1]).

The map g has very few restrictions, so ψ can be chosen to satisfy all the properties in this theorem.

3.6. Sketch of the proof of Theorem A in the general case.

It is not difficult to extend the proof in (3.5) to the case in which the simple cherry flow has no trajectories connecting two saddle points (See [3, section 3]). Under these conditions, the arguments of [3, section 5] work to prove theorem A for any simple cherry flow.

3.7. Proof of Theorem B.

Let M be a nonorientable two-manifold of genus ≥ 4. By [4], there exists a smooth flow φ defined on M having finitely many fixed points which are saddles and possessing a nonorientable nontrivial recurrent trajectory γ such that $\omega(\gamma) = M$.

Now we blow up γ; that is, we split open M along γ and insert the union of the unstable manifold of a source p, a saddle point q and its separatrices $\lambda_1, \lambda_2, \lambda_3, \lambda_4$; the way is indicated in Fig. 4 before blowing up γ, and in Fig. 5 after blowing up γ (See [3, section 3]). Certainly, the resulting flow is a continuous Cherry flow.

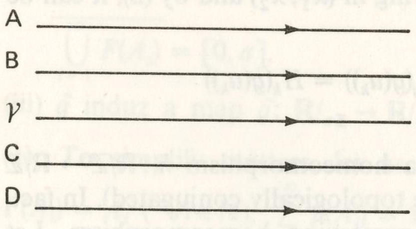


Fig. 4

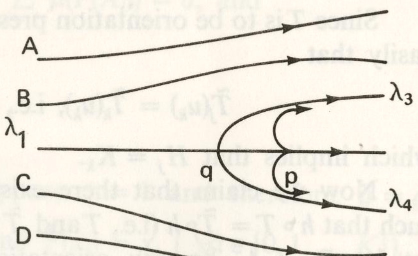


Fig. 5

References

[1] T.M. Cherry, *Analytic quasi-periodic curves of discontinuous type on a torus*, Proc. London Math. Soc. v. 44 (1938), 175-215.

[2] A. Denjoy, *Sur les courbes définies par les équations differentielles à la surface du tore*, J. Math. Pures Appl (9) 11 (1932) pp. 333-375.

[3] C. Gutierrez, *Smoothability of Cherry flows on two-manifolds*, To appear.

[4] C. Gutierrez, *Smooth nonorientable nontrivial recurrence on two-manifolds*, To appear in J. Differential Equations.

[5] C. Gutierrez, *Structural stability for flows on the torus with a cross-cap*, Trans. Amer. Math. Soc. 241 (1978), 311-320.

[6] P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons. Inc. (1973).

[7] Palis e De Melo, *Introdução aos Sistemas Dinâmicos*, Instituto de Matemática Pura e Aplicada, CNPq. Projeto Euclides. Editora Edgard Blücher, Ltda. S.P. Brasil (1978).

[8] M. Peixoto, *Structural stability on two-dimensional manifolds*, Topology, vol. 1 (1962) pp. 101-120.

[9] M. Peixoto, *Teoria Geométrica das Equações Diferenciais*, Sétimo Colóquio Brasileiro de Matemática, Instituto de Matemática Pura e Aplicada do CNPq. Brasil (1969).

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