BOL. SOC. BRAS. MAT., VOL. 10 Nº 1 (1979), 17-26

Characterization of a differentiable structure by its group of diffeomorphisms

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1. Introduction.

A *differentiable structure* on a *topological n-manifold M*, i.e. on a metrizable topological space M which is locally homeomorphic to $\mathbb{R}^n$, is a linear subspace $F \subset C(M)$, $C(M)$ is the vectorspace of continuous real functions on M, the elements of F are called smooth functions, such that:

(i) for each set of n elements $f_1, \ldots, f_n \in F$ and each C^∞ function $\hat{f} : \mathbb{R}^n \to \mathbb{R}$, $\hat{f}(f_1, \ldots, f_n) \in F$;

(ii) if $g : M \to \mathbb{R}$ is continuous and if for each $x \in M$ there is an $f \in F$ such that, on some neighbourhood of x, f and g coincide, then $g \in F$;

(iii) for each $x \in M$, there is a neighbourhood U of x in M and a set of n elements $f_1, \ldots, f_n \in F$ such that, for each $f \in F$, there is a C^∞ function $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ with $f \mid U = \hat{f}(f_1, \ldots, f_n) \mid U$.

See, e.g. Helgason [2]. Such a differentiable structure determines a *group of diffeomorphisms* $\mathrm{Diff}(M) = \{\varphi : M \to M \mid \varphi$ is a homeomorphism and for each $f : M \to \mathbb{R}, f \in F$ iff $f\varphi \in F\}$. It has been argued, notably in [5], that geometric structures, including "diffenrentiable structures", should be studied by analyzing the transformation group of "structure preserving" bijections, in the case of differentiable structures: the group of diffeomorphisms. From this point of view it is important to know whether the group of diffeomorphisms actually determines the differentiable structure (or that one transformation group may be the group of diffeomorphisms of different differentiable structures). The purpose of this paper is to show that indeed the group of diffeomorphisms determines the differentiable structure:

Theorem 1. *Let $\Phi : M_1 \to M_2$ be a bijection between two smooth n-manifolds such that $\lambda : M_2 \to M_2$ is a diffeomorphism iff $\Phi^{-1} \cdot \lambda \cdot \Phi$ is a diffeomorphism. Then Φ is a diffeomorphism.*

(*) Recebido em 24/11/78

An equivalent formulation of this result is: if M be a smooth manifold, $\mathrm{Diff}(M)$ is equal to its normalizer in the group of all bijections of M to itself, i.e., if $h: M \to M$ is a bijection such that $h^{-1} \cdot \mathrm{Diff}(M) \cdot h = \mathrm{Diff}(M)$, then $h \in \mathrm{Diff}(M)$.

It should be mentioned that for some other geometric structures the corresponding automorphism groups are not equal to their normalizer. For example the normalizer of the Euclidean group, in dimensions > 1, consists of all similarity transformations (this normalizer has *one* more dimension than the Euclidean group; this is related with the fact that, if one wants to specify an Euclidean structure without giving the group, one has to specify *one* unit (of length)). Also the normalizer of the group of all volume preserving, or special, affine automorphisms of $\mathbb{R}^n$, $n > 1$, is the full affine group (again the difference in dimension is *one* and *one* unit, of volume, is involved). I don't know any reference for these statements; the proofs are not very hard, see the appendix of this paper. Also it seems to be so that if a physical theory is invariant under a transformation group G, the minimal number of "independent physical units" (like the unit of time, length.....) is equal to the difference of the dimension of the normalizer of G and the dimension of G, if there are no "dimension constants"; for more confusion on this point see [1] Ch. IV, V.

The problem, treated in this paper, occured to me when reading Sourian's definition in [7] of a differentiable structure as the pseudo-group of its local diffeomorphisms; for details see definition (19.6) in this reference.

In this paper "smooth" or differentiable" will always refer to "C^∞"; for M an orientable differentiable manifold, $\mathrm{Diff}^+(M)$ denotes the group of orientation preserving diffeomorphisms of M to itself.

2. Reduction to $\mathbb{R}^n$

In this section we show that our main theorem follows from

Theorem 2. *Let $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ be a bijection such that $\lambda: \mathbb{R}^n \to \mathbb{R}^n$ belongs to $\mathit{Diff}^+(\mathbb{R}^n)$ iff $\Phi^{-1}\lambda\Phi \in \mathit{Diff}^+(\mathbb{R}^n)$. Then Φ is a diffeomorphism.*

The proof of theorem 2 occupies the sections 3 and 4. Before proving theorem 1, assuming theorem 2, we have to introduce some general notions.

Definition 2.1. Let X be a set and G be a group of bijections of X to itself. *G induces a topology T_G* on X: T_G is generated by the sets $X \supset U_g = \{x \in X \mid g(x) \neq x\}$, $g \in G$. Clearly, the elements of G are homeomorphisms of (X, T_G). From G and T_G one obtains a *pseudo-group* G, the elements of which are homeomorphisms $\varphi: U \to V$, U and V open in (X, T_G), such that for each $x \in U$, there is a $g_x \in G$ such that φ and g_x coincide in some neighbourhood of x. If W is an open subset of (X, T_G) then the *restriction* of G to W consists of those $\varphi: U \to V$ in G with $U, V \subset W$; this restiction is denoted by $G \mid W$. The *restriction* of G to W consists of the elements of $G \mid W$ whose domain and range are both equal to W; it is denoted by $G \mid W$.

Remark 2.2. If we apply the previous definitions to the case where X is a smooth n-manifold and G is its group of diffeomorphisms, we see that T_G is the usual topology. If we take $W \subset X$ to be diffeomorphic with $\mathbb{R}^n$, then $G \mid W$ consists either of all diffeomorphisms of W or of all orientation preserving diffeomorphisms of W. The first case occures if S is non-orientable or if X admits an orientation reversing diffeomorphism; the second case occures if X is orientable but admits no orientation reversing diffeomorphism, see [3, exercise 15, pag. 140].

Proof of theorem 1. Let Φ be as in the assumption of theorem 1. By the above remark, Φ is a homeomorphism. We want to prove that Φ is a diffemorphism; it is enough to do this locally. Let $W_i \subset M_i$, $i = 1,2$ be open subsets such that $\Phi(W_1) = W_2$ and such that W_1 is diffeomorphic with $\mathbb{R}^n$. Denote $\Phi \mid W_1 : W_1 \to W_2$ by Φ_W. Then it follows from the above remark and definitions that $\lambda: W_2 \to W_2$ belongs to $\mathrm{Diff}(M_2) \mid W_2$ iff $\Phi_W^{-1} \lambda \Phi_W$ belongs to $\mathrm{Diff}(M_1) \mid W_1$. Since Φ is a homeomorphism, it transforms orientation preserving diffeomorphisms λ to orientation preserving ones. So if we would know that W_2, with the differentiable structure of M_2, is diffeomorphic with $\mathbb{R}^n$ the differentiability of Φ_W would follow from theorem 2. W_2 is of course homeomorphic with W_1 and hence with $\mathbb{R}^n$, but, e.g. for $n = 4$, I don't know of a proof that this implies that W_2 is diffeomorphic with $\mathbb{R}^n$. Using the special properties of Φ_W we can prove W_2 to be diffeomorphic with $\mathbb{R}^n$: take a smooth closed n-disc D_2 in W_2, $D_1 = \Phi_W^{-1}(D_2)$, and take $\varphi \in \mathrm{Diff}(M_1) \mid W_1$ such that $D_1 \subset \mathrm{int}(\varphi(D_1))$ and $W_1 = \bigcup_{i \geq 0} \varphi^i(D_1)$. Then $\Phi_W \varphi \Phi_W^{-1} = \tilde{\varphi}$ is an element of $\mathrm{Diff}(M_2) \mid W_2$ such that $D_2 \subset \mathrm{int}(\tilde{\varphi}(D_2))$ and $W_2 = \bigcup_{i \geq 0} \tilde{\varphi}^i(D_2)$. Since each $\tilde{\varphi}^i(D_2)$ is diffeomorphic with D_2, a smooth closed n-disc, $\tilde{\varphi}^i(D_2)$ int $(\tilde{\varphi}^{i-1}(D_1))$ is diffeomorphic with $S^{n-1} x [0,1]$. Hence W_2 is diffeomorphic with $\mathbb{R}^n$ and the proof is complete.

3. The one-dimensional case.

In this section we prove theorem 2 for the case $n = 1$.

Lemma 3.1. *Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a bijection such that $\lambda\colon \mathbb{R} \to \mathbb{R}$ belongs to $\mathrm{Diff}^+(\mathbb{R})$ iff $\Phi^{-1}\lambda\,\Phi \in \mathrm{Diff}^+(\mathbb{R})$. Then Φ and Φ^{-1} have a non-zero first derivative in each point of $\mathbb{R}$.*

Proof. From the previous section we know that Φ is a homeomorphism. Hence, as a real function, Φ is monotone so by the theorem of Lebesque [6], Φ has almost everywhere a first derivative; the same holds for Φ^{-1}.

From the assumtions it follows that, for each orientation preserving diffeomorphism $\lambda\colon \mathbb{R} \to \mathbb{R}$ and each $y \in \mathbb{R}$, the following limit exists and is positive:

$$\lim_{h\to 0} \frac{\Phi^{-1}\lambda\,\Phi(y+h) - \Phi^{-1}\lambda\,\Phi(y)}{h} = \lim_{h\to 0} \frac{\Phi^{-1}(\lambda\Phi(y+h)) - \Phi^{-1}(\lambda\Phi(y))}{\lambda\Phi(y+h) - \lambda\Phi(y)} \cdot$$

$$\frac{\lambda\,\Phi(y+h) - \lambda\,\Phi(y)}{\Phi(y+h) - \Phi(y)} \cdot \frac{\Phi(y+h) - \Phi(y)}{h}.$$

In order to analyse the above product and its possible limits, we introduce:

$$f_1(h) = \frac{\Phi^{-1}(\lambda\Phi(y+h)) - \Phi^{-1}(\lambda\Phi(y))}{\lambda\Phi(y+h) - \lambda\Phi(y)}$$

$$f_2(h) = \frac{\lambda\Phi(y+h) - \lambda\Phi(y)}{\Phi(y+h) - \Phi(y)}$$

$$f_3(h) = \frac{\Phi(y+h) - \Phi(y)}{h}.$$

We know that $\lim_{h\to 0} f_2(h)$ exists and is positive. Choose y so that Φ has a first derivative, and hence so that $\lim_{h\to 0} f_3(h) = \Phi'(y)$ exists. If $\Phi'(y)$ is zero then $\lim_{h\to 0} f_1(h) = \pm\infty$ and hence Φ^{-1} has no first derivative in $\lambda\Phi(y)$ (for *all* $\lambda \in \mathrm{Diff}^+(\mathbb{R})$); this means that Φ^{-1} is nowhere differentiable which contradicts the theorem of Lebesque. So $\Phi'(y)$ is different from zero. From this it follows that Φ^{-1} has a non-zero first derivative in $\lambda\Phi(y)$ (for all λ) and hence has a non-zero first derivative everywhere. By reversing the argument we find that Φ everywhere has a non-zero first derivative.

Lemma 3.2. *Let $\lambda_1, \lambda_2\colon \mathbb{R} \to \mathbb{R}$ be two orientation preserving diffeomorphisms and $\Phi\colon \mathbb{R} \to \mathbb{R}$ a homeomorphism, satisfying:*

(i) $\lambda_1(0) = 0$;
(ii) $\lambda_1'(0) < 1$;
(iii) *for each* $t \in \mathbb{R}$, $\lambda_1^k(t) \to 0$ *for* $k \to +\infty$;
(iv) Φ *and* Φ^{-1} *have a first derivative in each point of* $\mathbb{R}$;
(v) $\Phi^{-1}\lambda_1\,\Phi = \lambda_2$.

Then Φ is a C^∞ diffeomorphism.

Proof. Without loss of generality we may assume that $\Phi(0) = 0$. Then also $\lambda_2(0) = 0$, $\lambda_2'(0) < 1$ and for each $t \in \mathbb{R}$, $\lambda_2^k(t) \to 0$ as $k \to +\infty$. By [8] there are diffeomorphisms $\varphi_1, \varphi_2\colon (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\varphi_i^{-1}\lambda_i\varphi_i$ is linear.

$$\begin{array}{ccc} \overset{\lambda_2}{\circlearrowleft}\mathbb{R} & \xrightarrow{\ \Phi\ } & \mathbb{R}\overset{\lambda_1}{\circlearrowright} \\ \downarrow \varphi_2 & & \downarrow \varphi_1 \\ \mathbb{R} & \xrightarrow{\ \psi\ } & \mathbb{R} \end{array}$$

We define ψ by $\varphi_1\Phi\varphi_2^{-1} = \psi \cdot \psi$ and ψ^{-1} are differentiable in zero. Let $\Lambda_i = \lambda_i'(0)$, let $\psi(1) = a$ and $\psi(b) = c$ for some $b \in (\Lambda_2, 1)$. Then $\psi(\Lambda_2^k) = \Lambda_1^k \cdot a$ and $\psi(\Lambda_2^k b) = \Lambda_1^k c$.

$$\psi'(0) = \lim_{k\to\infty} \frac{\psi(\Lambda_2^k) - \psi(0)}{\Lambda_2^k} = \lim a.\ \frac{\Lambda_1^k}{\Lambda_2^k} \text{ hence } \Lambda_1 \leq \Lambda_2$$

$$(\psi^{-1})'(0) = \lim_{k\to\infty} \frac{\psi^{-1}(\Lambda_1^k \cdot a) - \psi^{-1}(0)}{\Lambda_1^k \cdot a} = \lim \frac{1}{a} \cdot \frac{\Lambda_2^k}{\Lambda_1^k} \text{ hence } \Lambda_2 \leq \Lambda_1.$$

Also $\psi'(0) = \lim_{k\to\infty} \dfrac{\psi(b \cdot \Lambda_2^k) - \psi(0)}{b \cdot \Lambda_2^k} = \lim \dfrac{c}{b} \cdot \dfrac{\Lambda_1^k}{\Lambda_2^k}$, so $\Lambda_1 = \Lambda_2$ and $a = \dfrac{c}{b}$ or $\psi(b) = b \cdot \psi(1)$.

This means that ψ is linear and hence Φ is a diffeomorphism.

Proof of theorem 2 *with $n = 1$.* Φ, as in the assumptions of theorem 2 satisfies the assumptions of lemma (3.1); hence both Φ and Φ^{-1} have everywhere a first derivative. Take now $\lambda_1\colon \mathbb{R} \to \mathbb{R}$ with $\lambda_1(t) = 1/2\,t$ and $\lambda_2 = \Phi^{-1}\lambda_1\Phi$; then the assumptions in lemma (3.2) are satisfied. Hence Φ is a diffeomorphism.

4. Global coordinates and the proof of theorem 2.

Definition 4.1. A map $f: \mathbb{R}^n \to \mathbb{R}$ is called a *global coordinate* if there is a bijection $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ such that $\lambda: \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism if and only if $\Phi^{-1}\lambda\Phi$ is a diffeormorphism and such that $f = \xi_1 \cdot \Phi$, where $\xi_1: \mathbb{R}^n \to \mathbb{R}^1$ is the function assigning to each $x \in \mathbb{R}^n$ its first coordinate.

The main result of this section is

Proposition 4.2. *Each global coordinate $f: \mathbb{R}^n \to \mathbb{R}$ is smooth.*

Theorem 2 clearly follows from proposition (4.2). More precisely: proposition (4.2) is equivalent with theorem 2. We know that theorem 2, and hence proposition (4.2) hold in dimension 1. Hence we may, and do, assume that $n > 1$ and that theorem 2 holds in all dimensions smaller than n. In the rest of this section, $f: \mathbb{R}^n \to \mathbb{R}$ is a fixed global coordinate, $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ is a fixed bijection such that $f = \xi_1 \cdot \Phi$ and such that $\lambda: \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism if and only if $\Phi^{-1}\lambda\Phi$ is a diffeomorphism.

Lemma 4.3. *f is continuous.*

Proof. For each open $U \subset \mathbb{R}$, there is a diffeomorphism $\kappa: \mathbb{R}^n \to \mathbb{R}^n$ such that $\kappa(x) \neq x$ if and only if $x \in \xi_1^{-1}(U)$. Hence $\Phi\kappa\Phi^{-1}$ is a diffeomorphism of $\mathbb{R}^n$ which moves points only if they are in $(\xi_1\Phi)^{-1}(U) = f^{-1}(U)$. Hence $f^{-1}(U)$ is open; this proves continuity.

Lemma 4.4. *$f^{-1}(0)$ is a smooth co-dimension 1 manifold.*

Proof. Let $\kappa: \mathbb{R}^n \to \mathbb{R}^n$ be the diffeomorphism such that $\Phi^{-1}\kappa\Phi(x_1,..,x_n) = (\frac{1}{2}x_1, x_2, \ldots, x_n)$. Then the fixed point set $\mathrm{Fix}(\kappa)$ of κ equals $f^{-1}(0)$; we want to show that it is a normally hyperbolic invariant submanifold for κ. Then by [4] the lemma follows.

Choose $p \in f^{-1}(0)$; we want to analyse $(d\kappa)_p$. Let $A \subset T_p(\mathbb{R}^n)$ be the linear subspace of those $v \in T_p(\mathbb{R}^n)$ such that $((d\kappa)_p - id)\,v = 0$. The dimension m of A is either n or $n-1$, otherwise $f^{-1}(0) = \mathrm{Fix}(\kappa)$ would, near p, be contained in a submanifold of co-dimension greater then one, in which case $f^{-1}(0)$ would not be able to separate $f^{-1}(-\infty, 0)$ and $f^{-1}(0, +\infty)$. In order to prove the lemma, we only have to show that $(d\kappa)_p$ has one eigenvalue with norm different from zero. This will be done by contradiction; so we assume all eigenvalues of $(d\kappa)_p$ to be one and derive a contradiction.

We have $\|f\kappa^k(p+h) - f\kappa^k(p)\| = \frac{1}{2^k} \cdot \|f(p+h) - f(p)\|$. Choose a compact neighbourhood K of p in $\mathbb{R}^n$; the sequence $\{\varepsilon_i\}$ is determined by

$$\varepsilon_i = \max\{\delta \mid \text{for each } h, \|h\| \leq \delta,\ \kappa^{-1}(p+h) \in K \text{ for all } 0 \leq j \leq i\}.$$

For each $\alpha > 1$, $\alpha^i \varepsilon_i \to \infty$ as $i \to \infty$ (this is a consequence of the assumption on the eigenvalues of $(d\kappa)_p$. If now $\|h\| \leq \varepsilon_i$, then

$$\left\| \frac{f(p+h) - f(p)}{\|h\|} \right\| \leq \frac{1}{2^i} \cdot \frac{\operatorname{diam}(f(K))}{\|h\|}.$$

The righthand-side goes to zero for $i \to \infty$, $\|h\| \to 0$, so $(df)_p$ exists and is zero. We prove that from this we may conclude that $(df_q) = 0$ for all q: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{T}: \mathbb{R} \to \mathbb{R}$ be translations such that $\xi_1 \cdot T = \tilde{T} \cdot \xi_1$. Then $f \cdot (\Phi^{-1} T \Phi) = \xi_1 \Phi(\Phi^{-1} T \Phi) = \xi_1 T \Phi = \tilde{T}\xi_1\Phi = \tilde{T} \cdot f$. Hence also (df) is zero in $(\Phi^{-1} T \Phi)(p)$; this implies df to be zero everywhere, but then f would be constant: the contradiction!

Lemma 4.5. *f is smooth.*

Proof. Since $f^{-1}(0)$ is smooth, $\Phi \mid f^{-1}(0): f^{-1}(0) \to \xi^{-1}(0)$ is a diffeomorphism (apply theorem 1 in the dimension $n-1$). Now we take a map $\tilde{\kappa}: \mathbb{R}^n \to \mathbb{R}^n$ defined by $\tilde{\kappa}(x_1, \ldots, x_n) = (x_1, \lambda_2 x_2, \ldots, \lambda_n x_n)$ with $0 < \lambda_2, \ldots, \lambda_n < 1$ independent over the rationals. Define $\kappa = \Phi^{-1} \cdot \tilde{\kappa} \cdot \Phi$; from the linearization theorem [4, 8, 9] which apply to κ we conclude that

- $\mathrm{Fix}(\kappa)$ is a smooth 1-manifold N which is mapped by f bijectively on $\mathbb{R}$;
- the projection $\pi: \mathbb{R}^n \to N$, defined by $\pi(x) = \lim_{i \to \infty} \kappa^i(x)$ is smooth and satisfied $(f \mid N \cdot \pi) = f$;
- for each diffeomorphism $\varphi: N \to N$ there is a diffeomorphism $\tilde{\varphi}: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi = \tilde{\varphi} \mid N;$$
$$\varphi \cdot \pi = \pi \cdot \tilde{\varphi}.$$

From this in turn it follows that $f \mid N$ satisfies the assumptions of theorem 1 (for $n = 1$) and hence $f \mid N$ is smooth. Now $f = (f \mid N) \cdot \pi$ is also smooth.

Appendix.

In this appendix we want to determine the normalizer of the Euclidean group (i.e. the group of orientation and distance preserving affine transformations of $\mathbb{R}^n$), the special affine group (i.e. the group of volume preserving affine transformations of $\mathbb{R}^n$) and the affine group.

Lemma A.1. *Let $f: \mathbb{R} \to \mathbb{R}$ be a bijection with $f(0) = 0$. Then f is linear over Q (the rationals) iff for each translation $\varphi_a: \mathbb{R} \to \mathbb{R}$ ($\varphi_a(x) = x + a$) $f^{-1} \varphi_a f$ is again a translation, (and hence is $\varphi_{f^{-1}(a)}$).*

Proof. If f is linear over Q, then also f^{-1} is linear over Q and $f^{-1} \varphi_a f(x) = f^{-1}(f(x) + a) = x + f^{-1}(a)$; hence $f^{-1} \varphi_a f = \varphi_{f^{-1}(a)}$.

On the other hand, if for each $a \in \mathbb{R}$, $f^{-1} \varphi_a f = \varphi_{f^{-1}(a)}$, we have with $x = f^{-1}(a)$, $f^{-1} \varphi_{f(x)} f(y) = \varphi_x(y)$ or $f^{-1}(f(x) + f(y)) = x + y$ or $f(x) + f(y) = f(x + y)$;

hence f is additive, or linear over Q.

Lemma A.2. *Let $f: \mathbb{R} \to \mathbb{R}$ be a bijection with $f(1) = 1$. Then f is multiplicative, i.e., $f(x \cdot y) = f(x) \cdot f(y)$, iff for each scalar multiplication $\varphi_\mu: \mathbb{R} \to \mathbb{R}$, $\varphi_\mu(x) = \mu \cdot x$, $f^{-1} \varphi_\mu f$ is also a scalar multiplication (and hence is $\varphi_{f^{-1}(\mu)}$).*

Proof. The same way as lemma (A.1).

Theorem A.3. *The normalizer of the Euclidean or special affine group of $\mathbb{R}^1$ consists of all bijections of $\mathbb{R}^1 \to \mathbb{R}^1$ which are affine over Q.*

The normalizer of the affine group of $\mathbb{R}^1$ is that affine group itself.

Remark A.4. For $n \geq 2$, the straight line through $p, q \in \mathbb{R}^n$, $p \neq q$ can be characterized terms of the Euclidean, the special affine, or the affine group as follows:

(a) (Euclidean) the line through p, q is the set of all $r \in \mathbb{R}^n$ such that, whenever φ_p, φ_q are Euclidean transformations such that:

$$\varphi_p(p) = p,\ \varphi_q(q) = q,$$
$$\varphi_p(r) = \varphi_q(r),$$
$$\text{then } \varphi_p(r) = \varphi_q(r) = r.$$

(b) ((Special) affine) the line through p, q is the set of all $r \in \mathbb{R}^n$ such that for any (special) affine transformation $\varphi_{p,q}: \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi_{p,q}(p) = p$ and $\varphi_{p,q}(q) = q$, $\varphi_{p,q}(r) = r$.

Corolary A.5. *Let $n \geq 2$. The normalizer of the Euclidean, the special affine or the affine group of $\mathbb{R}^n$ consists of maps $f: \mathbb{R}^n \to \mathbb{R}^n$, mapping straight lines to straight lines.*

Proposition A.6. *Any bijection $f: \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, which maps straight lines to straight lines, is an affine map.*

Proof. Clearly, f maps planes to planes; so it is sufficient to show that f, restricted to a plane, is affine. This means that it is enough to show the theorem for $n = 2$.

In this case we may assume that $f(0,0) = (0,0)$, $f(0,1) = (0,1)$ and $f(1,0) = (1,0)$. Then there are bijections $g_1, g_2: \mathbb{R} \to \mathbb{R}$ such that $f(t,0) = (g_1(t), 0)$ and $f(0,t) = (0, g_2(t))$. Since f has to map parallel lines to parallel lines, $g_1(t) = g_2(t) = g(t)$ for all t, and $f(t_1, t_2) = (g(t_1), g(t_2))$.

Let $l_{\lambda,\mu}$ be the line $\{(t_1, \lambda t_1 + \mu)\}$. The image of this line under f is $\{(g(t_1), g(\lambda t_1 + \mu)\} = \{(s_1, g(\lambda \cdot g^{-1}(s_1) + \mu)\}$ which has to be again a straight line (for all $\lambda \cdot \mu$). Hence, for each affine map $\varphi: \mathbb{R} \to \mathbb{R}$, $g \varphi g^{-1}$ is affine; by (A.3) this implies that g is affine. This proves the proposition.

From the above results we immediately obtain

Theorem A.7. *For $n \geq 2$ the normalizer of the Euclidean, resp. special affine, affine group of $\mathbb{R}^n$ is the group of similarity transformations, resp. the affine group, resp. the affine group.*

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