Free resolutions of certain triply generated ideals

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Introduction.

Since the work of Hilbert on algebraic forms, the theory of finite resolutions has seen many developments. More recently, the theory was enriched by the work of Eagon, Northcott, Hochster, Buchsbaum, Eisenbud, Peskine, Szpiro, to mention a few.

I have learned from D. Buchsbaum that the theory bears intimate relation to some rather delicate arithmetical questions. This paper is a little inquest into the arithmetic of ideals generated by three elements in UFD's. Using the *gcd* of two elements we build a complex that, in a basic situation, turns out to be exact and furnishes a resolution of the original ideal (Theorem 1.3). There are some corollaries for ideals generated by monomials in a regular sequence – these are identical to results of D. Taylor [6] and the thesis of Gemeda [3].

In a second section we show that the hypotheses of Theorem 1.3 have to do with an "almost" self-duality in a sense that is made precise. In such a situation we have properties that are similar to those enjoyed by R-sequences and sequences that generate Gorenstein ideals in low codimension (see [2]). But not quite so. We put forward in the third section that the hypotheses of Theorem 1.3 are general enough as to include the ideals generated by the gradient of some singular (projective) plane curves.

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1. The basic complex.

Let R be a UFD (unique factorization domain). We will denote by $gcd(a, b)$ the greatst common divisor of two elements $a, b \in R$, defined only up to unit factors. This gives rise to a map $R \times R \to R$ defined (up to unit factors) by $(a, b) \mapsto a/gcd(a, b)$ This map will be denoted by p.

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Definition 1.1. Let $f = \{f_1, f_2, f_3\}$ be an ordered sequence of elements in a UFD R. We define new (ordered) sequences

$$\begin{aligned} p_{12}(f) &= \{p(f_1,f_2), p(f_2,f_1), f_3\} \\ p_{13}(f) &= \{p(f_1,f_3), f_2, p(f_3,f_1)\} \\ p_{23}(f) &= \{f_1, p(f_2,f_3), p(f_3,f_2)\}. \end{aligned}$$

We call $p_1(f) = p_{12}(f)$ the *first p-transform* of f. Iterating, we define the *second p-transform* $p_2(f) = p_{13}(p_1(f))$ and the *third p-transform* $p_3(f) = p_{23}(p_2(f))$. It is mainly $p_3(f)$ that will occupy us.

Note the obvious inclusions of ideals $(f) \subset (p_1(f)) \subset (p_2(f)) \subset (p_3(f))$.

Also note that $p_3(f)$ is obtained by simply "canceling out" all possible common factors of the elements of the sequence f taken two at a time. As a consequence, we have:

Lemma 1.2. *If $p_3(f)$ as above is an R-sequence then any ordering of its elements form an R-sequence.*

Proof. This follows from [5, Theorem 118].

Remark. A question can be raised here as to how the p-transforms depend, say, on the order of the elements of the given sequence. It is easy to see by examples that if the three elements have a non-unit common factor then p_3 does not commute with the obvious action of the symmetric group S_3 on R × R × R. On the other hand, one can see that if t is the greatest common factor of the three elements of f then

$$p_3(f/t) = \left\{\frac{tf_1}{gcd(f_1,f_2)\,gcd(f_1,f_3)}, \frac{tf_2}{gcd(f_2,f_3)\,gcd(f_2,f_1)}, \frac{tf_3}{gcd(f_3,f_1)\,gcd(f_3,f_2)}\right\}$$

where f/t stands for the sequence $\{f_1/t, f_2/t, f_3/t\}$. From this it immediately follows that $p_3(f/t)$ does commute with the action of S_3.

Given a sequence $f = \{f_1, f_2, f_3\}$ of elements of R, we now build a complex of free R-modules over $R/(f)$. Namely, we have

$$(*) \qquad 0 \longrightarrow R \xrightarrow[s_3]{} R^3 \xrightarrow[s_2]{} R^3 \xrightarrow[s_1]{} R,$$

where, in matrix notation,

$$s_3 = \begin{pmatrix} t \cdot p(f_3,f_1)/gcd(f_2,f_3) \\ -t \cdot p(f_2,f_3)/gcd(f_2,f_1) \\ t \cdot p(f_1,f_2)/gcd(f_3,f_1) \end{pmatrix}$$

$$s_2 = \begin{pmatrix} -p(f_2,f_1) & -p(f_3,f_1) & 0 \\ p(f_1,f_2) & 0 & -p(f_3,f_2) \\ 0 & p(f_1,f_3) & p(f_2,f_3) \end{pmatrix}$$

$$s_1 = (f_1\ f_2\ f_3)$$

Here, t stands for the greatest common divisor of f_1, f_2, f_3.

It is immediate to check that (*) is indeed a complex. We now have

Theorem 1.3. *Let R be a noetherian UFD and let $f = \{f_1, f_2, f_3\}$ be a given sequence in R. If $p_3(f/t)$ is an R-sequence then the complex (*) is exact (and hence furnishes a free resolution of the R-module $R/(f)$. Moreover, one has*

$$ann_R H_1(f) = (f_1, f_2, f_3, \frac{1}{t^2}\, gcd(f_1,f_2)\, gcd(f_1,f_3)\, gcd(f_2,f_3)),$$

where $H_1(f)$ stands for the first homology group of the Koszul complex on the generators f_1, f_2, f_3.

Proof. A direct attack requires a somewhat lengthy calculation with gcd's. We apply instead the criterion of Buchsbaum-Eisenbud [1, Cor. 1].

Thus, we are to check first whether the s_i have the right rank:

$rk(s_3) = 1$; indeed, the entries of s_3 are the elements of $p_3(f/t)$ up to order and sign. As we are assuming that $p_3(f/t)$ is an R-sequence, then some entry of s_3 is $\neq 0$.

$rk(s_1) = 1$; by the same token, some $f_i \neq 0$.

$rk(s_2) = 2$; as above, some $f_i \neq 0$. Say, $f_1 \neq 0$. Then the 2 × 2 minor

$$p(f_1,f_2)\,p(f_1,f_3) = \frac{f_1}{gcd(f_1,f_2)} \cdot \frac{f_1}{gcd(f_1,f_3)} \neq 0$$

Next we check the depth condition. Namely, let $I(s_i)$ be the ideal generated by the $k_i x k_i$ minors of $s_i (k_i = rk(s_i))$. Then:

$depth I(s_1) \geq 1$; this is obvious since some $f_i \neq 0$.

$depth I(s_3) \geq 3$; this follows from the assumption that $p_3(f/t)$ is an R-sequence.

$depth I(s_2) \geq 2$; to prove this, it suffices to find three generators of $I(s_2)$ with no non-unit common factor (cf. Remark below. We claim that the minors

$$p(f_1,f_2)\,p(f_1,f_3) = \frac{tf_1}{gcd(f_1,f_2)\,gcd(f_1,f_3)} \cdot \frac{f_1}{t},$$

$$p(f_2,f_1)\,p(f_2,f_3) = \frac{tf_2}{gcd(f_2,f_1)\,gcd(f_2,f_3)} \cdot \frac{f_2}{t}$$

$$p(f_3,f_1)\,p(f_3,f_2) = \frac{tf_3}{gcd(f_3,f_1)\,gcd(f_3,f_2)} \cdot \frac{f_3}{t}$$

have no non-unit common factor. Indeed, let q be such an irreducible common factor. A little inspection, using the assumption that $p_3(f/t)$ is an R-sequence and that $f_1/t, f_2/t, f_3/t$ have non-unit common factor, immediately shows that q must be a common divisor of, say, the elements

$$\frac{f_1}{t}, \frac{f_2}{t}, \frac{\frac{f_3}{t}}{\frac{gcd(f_1,f_3)}{t}\cdot\frac{gcd(f_3,f_2)}{t}}.$$

But this is impossible too.

Having shown the exactness of the complex, we now go over to computing $ann_R H_1(f)$.

First we claim that $annH_1(f) = t \cdot annH_1(f/t)$. In fact, recall that $H_1 = Z_1/B_1$, where Z_1 is the module of relations and B_1 is the module generated by the trivial relations. Now, sinze $Z_1(f) = Z_1(f/t)$ it is clear that $t \cdot annH_1(f/t) \subset annH_1(f)$. Conversely, let $G' \in R$ be such that $G'Z_1(f) \subset B_1(f)$. Then, using the fact that t is relatively prime to at least one among $p(f_i,f_j)$, we get that t divides G'. Thus $G' = Gt$ and, a fortiori, $G \in annH_1(f/t)$.

Thus, assume $t = 1$. In this case, a straightforward calculation yields $annH_1(f) \subset (gcd(f_1,f_2),f_3) \cap (gcd(f_1,f_3),f_2) \cap (gcd(f_2,f_3),f_1)$. We finally claim that this intersection equals

$$(f_1,f_2,f_3, gcd(f_1,f_2)\, gcd(f_1,f_3)\, gcd(f_2,f_3)).$$

This will prove our contention as the latter is clearly contained in $ann_1 H(f)$.

Checking the claim is a burdensome nevertheless straightforward job with gcd's. First we show the equality

$$(gcd(f_1,f_2),f_3) \cap (gcd(f_1,f_3),f_2) = (f_2,f_3, gcd(f_1,f_2)gcd(f_1,f_3)).$$

Thus, let $G = T_1gcd(f_1,f_2) + T_2f_3 = U_1gcd(f_1,f_3) + U_2f_2$. Then we get $(T_1 - U_2p(f_2,f_1))\,gcd(f_1,f_2) = (U_1 - T_2p(f_3,f_1))\,gcd(f_1,f_3)$. Therefore, we must have $T_1 = U_2p(f_2,f_1) + Ugcd(f_1,f_3)$ for some $U \in R$. Substituting for T_1 above yields

$$G = U_2p(f_2,f_1)\,gcd(f_1,f_2) + Ugcd(f_1,f_2)\,gcd(f_1,f_3) + T_2f_3 = \\ = U_2f_2 + T_2f_3 + Ugcd(f_1,f_2)\,gcd(f_1,f_3).$$

The reverse inclusion is apparent. Next we verify the equality

$$(f_2,f_3, gcd(f_1,f_2)gcd(f_1,f_3)) \cap (gcd(f_2,f_3),f_1) = \\ = (f_1,f_2,f_3, gcd(f_1,f_2)\,gcd(f_1,f_3)\,gcd(f_2,f_3)).$$

Thus, let

$$G = K_1f_2 + K_2f_3 + K_3gcd(f_1,f_2)\,gcd(f_1,f_3) = \\ = L_1gcd(f_2,f_3) + Lf_1. \quad (*)$$

In particular, $gcd(f_1,f_3)$ must divide the expression $K_1f_2 - L_1\,gcd(f_2 f_3) = (K_1p(f_2,f_3) - L_1)\,gcd(f_2,f_3)$, so that, in fact, $L_1 = K_1p(f_2,f_3) + Tgcd(f_1,f_3)$ for some $T \in R$. Hence

$$L_1gcd(f_2,f_3) = K_1f_2 + Tgcd(f_1,f_3)\,gcd(f_2,f_3).$$

Substituting this expression for $L_1gcd(f_2,f_3)$ in (*) yields

$$K_2f_3 + K_3gcd(f_1,f_2)\,gcd(f_1,f_3) = Tgcd(f_1,f_3)\,gcd(f_2,f_3) + L_2f_1.$$

In particular, $gcd(f_1,f_3)\,gcd(f_2,f_3)$ divides $K_3gcd(f_1,f_2)\,gcd(f_1,f_3) - L_2f_1 = (K_3gcd(f_1,f_3) - L_2p(f_1,f_2))\,gcd(f_1,f_2)$ and further has to divide the expression $K_3gcd(f_1,f_3) - L_2p(f_1,f_2)$. Say, then, $K_3gcd(f_1,f_3) = L_2p(f_1,f_2) + Kgcd(f_1,f_3)\,gcd(f_2,f_3)$. Substituting this value in (*) yields our contention.

This finishes the proof of the theorem.

Remark. We made use of the following result: if I is an ideal of depth 1 in a noetherian *UFD* then I has height 1. This is standard and follows, for instance, from [5, Theorem 95].

The theorem admits the following consequences.

Corollary 1.4. *Let R be a noetherian local ring containing a field k. Let $x_1, \dots, x_n$ be an R-sequence contained in the maximal ideal m of R. If $f_1, f_2, f_3 \in m$ are monomials in $x_1, \dots, x_n$ then the complex of Ttheorem 1.3 is a free resolution of $R/(f_1,f_2,f_3)$. Moreover, the following are equivalent:*

(i) *$p_3(f/t)$ generates a proper ideal, where t is the greatest commo:: factor of f_1,f_2,f_3;*

(ii) *The complex of Theorem 1.3 is minimal.*

Proof. Under the hypotheses, $k[x_1, \dots, x_n]$ is k-isomorphic to a polynomial ring $k[X_1, \dots, X_n]$ and R is $k[x_1, \dots, x_n]$-flat [4, Proposition 1]. We may thus assume that $R = k[x_1, \dots, x_n]$. Now, since $p_3(f/t)$ consists as well of monomials in $x_1, \dots, x_n$, it can only generate the unit ideal provided some of its elements is itself 1. In this case and taking in account the definition of p_3, $p_3(f/t)$ is an R-sequence, so Theorem 1.3 applies. Otherwise, $p_3(f/t)$ generates a proper ideal. In this case, we see from [7, Lemma 2.7] that $p_3(f/t)$ is still an R-sequence. So Theorem 1.3 applies once more.

To see that (i) implies (ii) we just recall that the entries of the map s_3 of the complex are just the elements of $p_3(f/t)$, so the complex is minimal there. Also, because $p_3(f/t)$ generates a proper ideal, no f_i can be a multiple of some f_j, $i \neq j$. So, the complex is minimal at s_2 too.

The implication (ii) ⇒ (i) is clear.

Corollary 1.5. *Let $I = (f_1, f_2, f_3)$ be a proper ideal in a noetherian local ring R containing a field. If either p.d.$(I) \geq 3$ or else p.d.$(I) = 2$ and type $(I) \neq 1$, then I cannot be generated by monomials in an R-sequence contained in the maximal ideal of R.*

Here, p.d. stands for the projective dimension of an R-module, while "type" is the rank of the last non-zero module in a free resolution of an R-module (provided it has one).

2. An almost duality.

We keep the same assumptions as in the first section. A sequence $f = \{f_1, f_2, f_3\}$ such that $p_3(f)$ is an R-sequence enjoys a property which should be taken to mean an "almost duality". Namely, the basic free resolution

$$0 \longrightarrow R \xrightarrow[s_3]{} R^3 \xrightarrow[s_2]{} R^3 \xrightarrow[s_1 = (f_1 f_2 f_3)]{} R \longrightarrow R/(f) \longrightarrow 0$$

of Theorem 1.3 satisfies the following conditions.

(1) Any column of s_2 is obtained from a column of the matrix representing $B_1(f/t)$ by dividing out the *gcd* of (the nonzero elements of) that column.

(2) Any row of s_2 is a multiple of a column of $B_1(p_3(f/t))$, while the *gcd* of (the nonzero elements of) the row is the factor that multiplied by a suitable column of s_2 gives a column of $B_1(f/t)$.

We will call almost *self-dual* a sequence $f = \{f_1, f_2, f_3\}$ such that $p_3(f/t)$ is an R-sequence. One way of making such an almost duality transparent is by means of matrices which are nearly alternating. We thus agree to call *almost alternating* a 3×3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix},$$

where the a_{ij} s are nonunits, satisfying the following conditions.

(i) $gcd(a_{11}, a_{21}) = gcd(a_{12}, a_{32}) = gcd(a_{23}, a_{33}) = 1$

(ii) There exist $a, b, c \in R / (0)$ such that the matrix

$$\begin{bmatrix} aa_{11} & ba_{12} & 0 \\ aa_{21} & 0 & ca_{23} \\ 0 & ba_{32} & ca_{33} \end{bmatrix}$$

is skew-symmetric and aa_{11}, aa_{21}, ba_{12} have no non-unit common factor.

We will not distinguish between the sequences $\{f_1, f_2, f_3\}$ and $\{\alpha f_1, \beta f_2, \gamma f_3\}$, where α, β, γ are units. Likewise, no distinction will be made between a matrix A and the matrix obtained from A by multiplying the elements of a column by a unit. Finally, what we are here calling skew-symmetric is a matrix obtained from a usual skew-symmetric matrix by means of column interchange.

The first thing to note is that the elements a, b, c of condition (ii) are uniquely determined up to unit factors. Indeed, (ii) means that

$$\begin{aligned} aa_{11} &= -ca_{33} \\ aa_{21} &= -ba_{32} \\ ba_{12} &= -ca_{23} \end{aligned}$$

and that aa_{11}, aa_{21}, ba_{12} have no (simultaneous) common factor. Condition (i) then easily implies that $a = gcd(a_{32}, a_{33})$, $b = gcd(a_{21}, a_{23})$, $c = gcd(a_{11}, a_{12})$, so that a, b, c are determined by the matrix itself.

The result that we seek reads as follows.

Proposition 2.1. *There is a one-to-one correspondence between almost self-dual sequences with no non-unit common factor and almost alternating matrices.*

Proof. To an almost self-dual sequence $f = \{f_1, f_2, f_3\}$ (no common factor) we associate the matrix $s_2(f)$ of 1-cycles as in Theorem 1.5. Clearly, $s_2(f)$ is almost alternating. Conversely, to an almost alternating matrix we associate the sequence $\{aa_{21}, -aa_{11}, -ba_{12}\}$ as above. The verification that these correspondences are inverse of each other is mere routine.

Note that if $\{f_1, f_2, f_3\}$ is almost self-dual then so is $\{tf_1, tf_2, tf_3\}$, for any $t \in R / (0)$, and both correspond to the same almost alternating matrix. Thus, one cannot recover all almost self-dual sets.

In any case, however, it would be of some worth deciding whether one can recover the ideal generated by an almost self-dual sequence f from the annihilator of $H_1(f)$. This seems to be the case at least for an ideal that can be generated by an almost self-dual set consisting of monomials in an R-sequence.

An observation which can be of some value in the above question is that if $f = \{f_1, f_2, f_3\}$ is an almost self-dual set then ann $H_1(f)$ has no embedded primes if and only if f_1, f_2, f_3 have no (simultaneous) common factor.

We close this brief section by noting that the concept of almost duality can be given a meaning in terms of the dual of the basic free resolution of Theorem 1.3., much in a similar way to the results of [2]. As a matter of fact, an almost self-dual set generates an ideal which behaves very nearly like a Gorenstein ideal, except that Gorenstein ideals cannot be triply generated unless they are already generated by R-sequences. Likewise, the algebra structure with which the free resolution of an almost self-dual set is endowed can be explored. In the case of an ideal generated by monomials in an R-sequence this has been partially indicated in [3].

This latter aspect will be the object of a forthcoming work.

3. Applications.

For simplicity, we will work with the polynomial ring $R = k[X_1, \dots X_n]$ (k a field). However, a good portion of the results of this section can be carried out in the context of noetherian rings.

To start we have the following general.

Lemma 3.1. *Let R be noetherian and let $h_1, \dots, h_s, g, g_1, g_2 \in R$ be elements satisfying the conditions*

(i) *$\{h_1, \dots, h_s, g\}$ is an R-sequence*
(ii) *The radical of $(h_1, \dots, h_s, g)$ is a prime ideal.*
(iii) *$\{h_1, \dots, h_s, g_1, g_2\}$ is an R-sequence in any order.*

Then at least one of the sequences $\{h_1, \dots, h_s, g, g_i\}$, $i = 1, 2$, is an R-sequence.

Proof. Set $I = (h_1, \dots, h_s)$. We whish to show that either g_1 or g_2 is a non-zero-divisor mod (I, g). Assume to the contrary, i.e., both g_1 and g_2 belong to associated primes of (I, g). By (i) and (ii), one must then have $g_1, g_2 \in \in rad(I, g)$. Say, $g_1^m - G_1 g \in I$, $g_2^n - G_2 g \in I$. It follows that $G_1 g_2^n - G_2 g_1^m \in I$. Using (iii) one easily derives $G_1 - G g_1^m \in I$ for some $G \in R$. Substituting in the relation $g_1^m - G_1 g \in I$ yields $(1 - Gg)\, g_1^m \in I$, hence $1 \in (I, g)$. This contradicts (i).

Using the lemma one can prove:

Proposition 3.2. *Let $R = \sum_{i \geq 0} R_i$ be a noetherian graded ring with graded radical $M = M_0 + \sum_{i \geq 1} R_i$ (where M_0 is the Jacobson radical of R_0). Let $h_1, \dots, h_s, g, g_1, g_2 \in M$ be homogeneous elements satisfying conditions (i), (ii), (iii) of the lemma. Then, for some permutation σ of $\{1, 2\}$, $\{h_1, \dots, h_s, g_{\sigma_1}, g, g_{\sigma_2}\}$ is an R-sequence and*

$$(h_1, \dots, h_s, gg_1, gg_2) = (h_1, \dots, h_s, g) \cap (h_1, \dots, h_s, g_{\sigma_1}, g, g_{\sigma_2}).$$

In particular, if height$(M) = s + 2$, the above is a primary decomposition of $(h_1, \dots, h_s, gg_1, gg_2)$, with $M = rad(h_1, \dots, h_s, g_{\sigma_1}, g, g_{\sigma_2})$

Proof. By the lemma, (say) $\{h_1, \dots, h_s, g, g_1\}$ is an R-sequence (in any order because one has a graded situation). We will prove that $\{h_1, \dots, h_s, g_1, gg_2\}$ is then an R-sequence. Since everything lies inside M, it suffices to show that gg_2 is a non-zero-divisor mod(I, g_1), where $I = (h_1, \dots, h_s)$. Suppose $Ggg_2 \in (I, g_1)$. Then $Gg_2 \in (I, g_1)$ as g is a non-zero-divisor mod(I, g_1), and further $G \in (I, g_1)$ because g_2 is a non-zero-divisor mod(I, g_1) by assumption.

Next, for the decomposition, note that one inclusion is obvious. As to reverse inclusion, if $\Phi \in (I, g)$ then $\Phi - Gg \in I$ for some $G \in R$. If further $\Phi \in (I, g_1, gg_2)$ then $(Fg_2 - G)\, g \in (I, g_1)$ for some $F \in R$. As $\{h_1, \dots, h_s, g, g_1\}$ is an R-sequence in M, g is a non-zero-divisor mod (I, g_1). Therefore, $G \in (I, g_1, g_2)$. It follows that $\Phi = \Phi - Gg + Gg \in (I, gg_1, gg_2)$.

Corollary 3.3. *Let $R = k[X, Y, Z]$ (k a field) and let h, g_1, g_2, g be homogeneous elements belonging to (X, Y, Z). Suppose that the following conditions hold:*

(i) *$gcd(h, g) = 1$*
(ii) *$rad(h, g)$ is a prime ideal*
(iii) *$\{h, g_1, g_2\}$ is an R-sequence.*

Then the ideal $J = (h, gg_1, gg_2)$ admits the following primary decomposition

$$J = ann\, H_1 \cap (h, g_{\sigma_1}, gg_{\sigma_2}),$$

for some permutation σ of $\{1, 2\}$, with $(h, g_{\sigma_1}, gg_{\sigma_2})$ the irrelevant component and H_1 the first homology of the Koszul complex generated by h, gg_1, gg_2.

Proof. It immediately follows from Theorem 1.3. and Proposition 3.2.

Applying the results to the gradient ideal $J_f = \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right)$ of a curve in $\mathbb{P}_k^2$ defined by $f \in k[X, Y, Z]$, we see that, provided (1) rad

$\left(\dfrac{\partial f}{\partial X}, g\right)$ is prime or else $g = 1$, where $g = gcd\left(\dfrac{\partial f}{\partial Y}, \dfrac{\partial f}{\partial Z}\right)$, and (2) $\left\{\dfrac{\partial f}{\partial X}, \dfrac{1}{g}\dfrac{\partial f}{\partial Y}, \dfrac{1}{g}\dfrac{\partial f}{\partial Z}\right\}$ is an R-sequence, then J_f coincides with ann $H_1\left(\dfrac{\partial f}{\partial X}, \dfrac{\partial f}{\partial Y}, \dfrac{\partial f}{\partial Z}\right)$ up to an irrelevant component.

Example. (k algebraically closed of char. 0) Consider the family of curves $\{f_{(\alpha)}(Y, Z) + \alpha_{n+1} X^n = 0\}_{(\alpha)}$ in $\mathbb{P}^2_k$, where $f_{(\alpha)}(Y, Z)$ is homogeneous of degree $n \geq 2$. For fixed n, this is a linear subsystem of the complete linear system of curves of degree n. Look at the corresponding family

$$\left\{\left(n\alpha_{n+1}X^{n-1}\frac{\partial f_{(\alpha)}}{\partial Y}\frac{\partial f_{(\alpha)}}{\partial Z}\right)\right\}_{(\alpha)}$$

of gradient ideals. A curve in the family is singular if and only if

$gcd\left(\dfrac{\partial f_{(\alpha)}}{\partial Y}, \dfrac{\partial f_{(\alpha)}}{\partial Z}\right) = g_{(\alpha)} \neq 1$. Now, $\left\{n\alpha_{n+1} X^{n-1}, \dfrac{\partial f_{(\alpha)}}{\partial Y}, \dfrac{\partial f_{(\alpha)}}{\partial Z}\right\}$ is almost self-dual outside a "nowhere dense subset". More precisely, let

$$\begin{array}{ccc} \mathscr{X} & \hookrightarrow & \mathbb{P}^2 \times \mathbb{P}^{n+1} \\ & \searrow & \downarrow \\ & & \mathbb{P}^{n+1} \end{array}$$

be the deformation parametrizing the family $\{f_{(\alpha)}(Y, Z) + \alpha_{n+1}X^n = 0\}$. Then the result reads as follows.

Proposition 3.4. *There is a 2-dimensional (degenerate) come $\mathscr{C}$ in $\mathbb{P}^{n+1}$ such that $s \in \mathbb{P}^{n+1} \setminus \mathscr{C}$ if and only if the gradient of the curve $\mathscr{X}_s$ is almost self-dual.*

Proof. Namely, let $(x) = (x_0 : \ldots : x_n : x_{n+1})$ be homogeneous coordinates of $\mathbb{P}^{n+1}$ and let $(u : v)$ be homogeneous coordinates of the projective line $\mathbb{P}^1$. Consider the rational curve in $\mathbb{P}^n(x_0 : \ldots : x_n)$ which is the image of the map $\mathbb{P}^1 \to \mathbb{P}^n$ defined by

$$(x_0 : \ldots : x_n) = (u^n : \tbinom{n}{1} u^{n-1} \quad v : \tbinom{n}{2} u^{n-2} v^2 : \ldots : v^n).$$

Let $\mathscr{C}$ be the cone over this curve in $\mathbb{P}^{n+1}$. Now, a point $s \in \mathbb{P}^{n+1}$, with coordinates $(\alpha_0 : \ldots : \alpha_n : \alpha_{n+1})$, belongs to $\mathscr{C}$ if and only if the equation of the corresponding fiber $\mathscr{X}_s$ is

$$\begin{aligned} f_{(\alpha)}(Y, Z) + \alpha_{n+1} X^n &= a^n Y^n + \tbinom{n}{1} a^{n-1} b Y^{n-1} Z + \ldots + v^n Z^n + \alpha_{n+1} X^n = \\ &= (aY + bZ)^n + \alpha_{n+1} X^n. \end{aligned}$$

Our claim will, therefore, follow from the following lemma.

Lemma 3.5. *The gradient* $\left\{n\alpha_{n+1} X^{n-1}, \dfrac{\partial f_{(\alpha)}}{\partial Y}, \dfrac{\partial f_{(\alpha)}}{\partial Z}\right\}$ *is not almost selfdual if and only if the equation of the curve has the form* $(aY + bZ)^n + \alpha_{n+1} X^n$.

Proof. The third p-transform of the gradient is the sequence

$$\left\{n\alpha_{n+1} X^{n-1}, \frac{1}{g_{(\alpha)}}\frac{\partial f_{(\alpha)}}{\partial Y}, \frac{1}{g_{(\alpha)}}\frac{\partial f_{(\alpha)}}{\partial Z}\right\}, \quad \text{where} \quad g_{(\alpha)} = gcd\left(\frac{\partial f_{(\alpha)}}{\partial Y}, \frac{\partial f_{(\alpha)}}{\partial Z}\right)$$

as before. Clearly, X^{n-1} is a non-zero-divisor modulo the ideal generated by $\dfrac{1}{g_{(\alpha)}}\dfrac{\partial f_{(\alpha)}}{\partial Y}$ and $\dfrac{1}{g_{(\alpha)}}\dfrac{\partial f_{(\alpha)}}{\partial Z}$, hence the third p-transform is an R-sequence if and only if it generates a proper ideal, i.e., if and only if neither $\dfrac{1}{g_{(\alpha)}}\dfrac{\partial f_{(\alpha)}}{\partial Y}$ nor $\dfrac{1}{g_{(\alpha)}}\dfrac{\partial f_{(\alpha)}}{\partial Z}$ is a constant. As the partial derivatives have the same degree, we see that the gradient fails to be almost self-dual if and only if $\dfrac{\partial f_{(\alpha)}}{\partial Z} = \lambda \dfrac{\partial f_{(\alpha)}}{\partial Y}$, for some $\lambda \in k$.

If this is so then, by Euler's formula, $nf_{(\alpha)} = Y\dfrac{\partial f_{(\alpha)}}{\partial Y} + Z\dfrac{\partial f_{(\alpha)}}{\partial Z} = \dfrac{\partial f_{(\alpha)}}{\partial Y}$ $(Y + \lambda Z)$. Successive differentiation in Y eventually yields

$$n!\, f_{(\alpha)} = \frac{\partial^n f_{(\alpha)}}{\partial Y^n}(Y + \lambda Z)^n, \quad \text{i.e.,} \quad f_{(\alpha)} = \mu(Y + \lambda Z)^n \quad \text{for some} \quad \mu \in k.$$

Conversely, if the equation of the curve is $\mu(Y + \lambda Z)^n + \alpha_{n+1} X^n$ then the third p-transform of the gradient is $\{n\alpha_{n+1} X^{n-1}, 1, \lambda\}$, so the gradient is not almost self-dual. This finishes the proof of the lemma.

Next we set $S = \mathbb{P}^{n+1} \setminus \mathscr{C}$. There is no harm in redenoting the product $\mathfrak{X} \times_{\mathbb{P}^{n+1}} S$ by $\mathfrak{X}$, so we will do that. Thus, we have a deformation $\mathfrak{X} \to S$, where S is a smooth affine variety of dimension $n + 1$ and for every $s \in S$, the gradient of the fiber $\mathscr{X}_s$ is almost selfdual. By Cor. 1.4, for every $s = (\alpha : \alpha_{n+1}) \in S$, the corresponding gradient admits the minimal free resolution

$$(*) \qquad 0 \longrightarrow R \underset{\varphi_3}{\longrightarrow} R^3 \underset{\varphi_2}{\longrightarrow} R^3 \underset{\varphi_1}{\longrightarrow} R$$

where

$$\varphi_3 = \begin{bmatrix} \dfrac{1}{g_{(\alpha)}} \dfrac{\partial f_{(\alpha)}}{\partial Z} \\ \dfrac{1}{g_{(\alpha)}} \dfrac{\partial f_{(\alpha)}}{\partial Y} \\ n\alpha_{n+1} X^{n-1} \end{bmatrix}$$

$$\varphi_2 = \begin{bmatrix} -\dfrac{\partial f_{(\alpha)}}{\partial Y} & -\dfrac{\partial f_{(\alpha)}}{\partial Z} & 0 \\ n\alpha_{n+1} X^{n-1} & 0 & \dfrac{1}{g_{(\alpha)}} \dfrac{\partial f_{(\alpha)}}{\partial Z} \\ 0 & n\alpha_{n+1} X^{n-1} & \dfrac{1}{g_{(\alpha)}} \dfrac{\partial f_{(\alpha)}}{\partial Y} \end{bmatrix}$$

$$\varphi_1 = \left(n\alpha_{n+1} X^{n-1}, \frac{\partial f_{(\alpha)}}{\partial Y}, \frac{\partial f_{(\alpha)}}{\partial Z} \right).$$

Now, the generic curve $\mathscr{X}_\eta$ is smooth. Therefore, the minimal free resolution of its gradient ideal is the Koszul complex on the partial derivatives. In other words, the partial derivatives with respect to Y and Z, of the generic equation $F_{(T)}(Y, Z) + T_{n+1}X^n$, are relatively prime. If we let $S_0 \subset S$ be the such that $s \in S_0$ if and only if the fiber $\mathscr{X}_s$ is smooth, we get a 1-1 correspondence between the fibers of the induced deformation $\mathscr{X} \times_S S_0 \to S_0$ and resolutions of the form (*) with $g_{(\alpha)} = 1$.

Next, we look at the complementary set $\tilde{S}_0 = S \setminus S_0$. Note that, by Sard's theorem, $\tilde{S}_0$ is contained in a proper closed subset of S. The most generic fiber of the map $\mathscr{X} \times_S \tilde{S}_0 \to S_0$ is given by the equation

$$(U_1 Y + V_1 Z)^2 \prod_{i=3}^{n} (V_i Y + V_i Z) + T_{n+1} X^n$$

where T_{n+1}, U_1, V_1, U_3, V_3, etc. are distinct indeterminates. It is clear that this curve admits a minimal free resolution such as (*) with $g_{(\alpha)}$ replaced by $U_1 Y + V_1 Z$ (and α_{n+1} by T_{n+1}). Therefore, if we let $S_1 \subset \tilde{S}_0$ be such that a fiber over a point of S_1 is a curve with exactly one singular point of multiplicity 2, then every fiber of $\mathscr{X} \times_S S_1 \to S_1$ admits a resolution of the form (*) with $g_{(\alpha)} = a_1 Y + b_1 Z$, for some $a_1, b_1 \in k$, and, conversely, every such resolution is the minimal free resolution of the gradient ideal of a fiber of $\mathscr{X} \times_S S_1 \to S_1$.

Next, look at the complementary set $\tilde{S}_1 = \tilde{S}_0 \setminus S_1$. This splits into two components $\tilde{S}_{1,1}$, $\tilde{S}_{1,2}$, namely, the most generic curve of the fibration $\mathscr{X} \times_S \tilde{S}_{1,1} \to \tilde{S}_{1,1}$ is of the form

$$(U_1 Y + V_1 Z)^2 (U_2 Y + V_2 Z)^2 \prod_{i=5}^{n} (U_i Y + V_i Z) + T_{n+1} X^n$$

where T_{n+1}, U_1, V_1, V_2, U_5, V_5, etc. are distinct indeterminates, while the generic curve of $\mathscr{X} \times_S \tilde{S}_{1,2} \to \tilde{S}_{1,2}$ is of the form

$$(U_1 Y + V_1 Z)^3 \prod_{i=4}^{n} (U_i Y + V_1 Z) + T_{n+1} X^n$$

where U_1, V_1, U_4, V_4, etc. are distinct indeterminates. The generic curve on the component $\tilde{S}_{1,1}$ (resp. $\tilde{S}_{1,2}$) has a minimal resolution of the form (*) with $g_{(\alpha)}$ replaced by $(U_1 Y + V_1 Z)(U_2 Y + V_2 Z)$ (resp. $(U_1 Y + V_1 Z)^2$ and α_{n+1} by T_{n+1}. Therefore, if $S_{2,1} \subset \tilde{S}_{1,1}$ (resp. $S_{1,2} \subset \tilde{S}_{1,2}$) denotes the subset such that $s \in S_{2,1}$ (resp. $s \in S_{2,2}$) if and only if $\mathscr{X}$ has exactly two distinct points of multiplicity 2 each (resp. exacly one single point of multiplicity 3), then every fiber of $\mathscr{X} \times_S S_2 \to S_2 = S_{1,1} \cup S_{1,2}$ admits a minimal resolution such as (*), where $g_{(\alpha)}$ is to be replaced by a term of the form $(a_1 Y + b_1 Z)(a_2 Y + b_2 Z)$ – with $a_1 Y + b_1 Z$ and $a_2 Y + b_2 Z$ not necessarily distinc – and, conversely, every such resolution comes from a fiber of $\mathscr{X} \times_S S_2 \to S_2$.

Continuing in this way, we eventually exhaust S. This may summarized as follows.

Proposition 3.6. *There is a (finite) stratification $S = S_0 \cup S_1 \cup S_{2,1} \cup \cup S_{2,2} \cup \ldots$ and, for each stratum $\mathscr{S}$, there exists a generic minimal free resolution $\mathscr{R}(\mathscr{S})$ such that*

(i) For every (closed) point $s \in \mathscr{S}$, the gradient ideal of the fibers $\mathscr{X}_s$ of $\mathscr{X} \times_S \mathscr{S} \to \mathscr{S}$ over s admits $\mathscr{R}(\mathscr{S}) \otimes_{\mathcal{O}_S} k(s)$ as a minimal free resolution.

(ii) For every stratum $\mathscr{S}$ and every reduction $\mathcal{O}_S \to k$ of coefficients, the corresponding reduced resolution $\mathscr{R}(\mathscr{S}) \otimes_{\mathcal{O}_S} k$ is of the form $\mathscr{R}(\mathscr{S}) \otimes_{\mathcal{O}_S} k(s)$ for a (closed) point $s \in \mathscr{S}$

The first interesting case is $n = 3$. Here S is stratified into two strata S_0 and S_1, the first of which parametrizes a family of elliptic cubics, while the second one parametrizes a family of irreducible cubics having a cusp. For $n = 4$, we find a stratum parametrizing elliptic quartics, and so forth.

Let us remark that Proposition 3.6 is really a result on deformations of free resolutions. As such, however, it is not entirely satisfactory. In fact, one would like to have an answer to the following question.

Question 1. Let $I = \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right)$ be the gradient ideal of a plane curve C defined by the equation $f = 0$. Suppose I admits a minimal free resolution such as (*). Does C belong to the family $\{f_{(\alpha)}(Y, Z) + \alpha_{n+1} X^n\}$ where $n = \deg f$?

This question should perhaps be preceded by another (rather technical) question.

Question 2. Let $I = \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right)$ admit an almost self-dual set of generators. Is the gradient itself almost self-dual?

A great many curious questions impose themselves in this kind of analysis. Let us point out, for instance, that we do not know an example of a triply generated ideal, having projective dimension 2 and type 1, that is not generated by an almost self-dual set.

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