Numerical study of an equation related to wave propagation

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Abstract.

We discuss two algorithms for the computation of approximate solutions of a generalization of the so-called Benjamin-Bona-Mahony equation, which is a model proposed to describe unidirectional propagation of long water waves.

Both schemes discussed are quadratically convergent with respect to Δt in the H^1 − norm. They use the Galerkin method for the space variable in such a way that the global truncation error has the same order as the error for the interpolation with the Galerkin basis.

Estimates are obtained for the study of the discretization that also yield an existence proof for the exact problem.

Results of some numerical experiments are presented.

1. Introduction.

The equation

(1.1) $$u_t + u_x + uu_x - u_{xxt} = 0$$

was suggested in [1] as a model for uni-directional propagation of long water waves. Such an equation was proposed as an alternative for the Korteweg-de Vries equation

(1.2) $$u_t + u_x + uu_x + u_{xxx} = 0$$

which supposedly describes the same phenomenum.

In this paper, we discuss two numerical schemes for the study of

(1.3) $$u_t + \frac{\partial}{\partial x} f(u) - \delta\, u_{xxt} = g(x, t)$$

which generalizes (1.1). These schemes modify the ones proposed in [2] for (1.1), and have the advantage of dealing with matrices that are independent of the time variable, thus achieving great computational savings. Part of the results discussed here were announced in [3, 4].

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We consider the problem of finding a solution of (1.3) subjected to the conditions

$$u(x,0) = u_0(x), \tag{1.4}$$

$$u(0,t) = u(1,t), \tag{1.5}$$

for $0 \le x \le 1$ and $0 \le t \le T$, with $T > 0$ preassigned.

In (1.3), $\delta > 0$ in an arbitrary constant and we assume that f and g are as smooth as needed, the precise assumptions being described below. We also require that

$$g(0,t) = g(1,t). \tag{1.6}$$

We shall stick to the notation used in [5]. By H^k and L^2 we shall mean $H^k(0,1)$ and $L^2(0,1)$, respectively, while H^k_p shall stand for the subspace of H^k formed by the functions v such that

$$\frac{d^j}{dx^j}\,v(0) = \frac{d^j}{dx^j}\,v(1), \tag{1.7}$$

for $j = 0, 1, \ldots, k-1$, with $k \ge 1$.

The rectangle $[0,1] \times [0,T]$ will be denoted by Q_T and $\langle .,. \rangle$ will represent the inner product in $H^0 \equiv L^2$. If $f, g \in H^1$, we shall write

$$(f \mid g) \equiv \langle f, g \rangle + \delta \langle f_x, g_x \rangle. \tag{1.8}$$

The norms in L^2 and H^k will be denoted by $|\cdot|_{L^2}$ and $|\cdot|_{H^k}$, respectively.

We shall consider the weak form of (1.3), which with the above notation appears as

$$(u_t \mid w) = \langle f(u),\, w_x \rangle + \langle g, w \rangle, \tag{1.3'}$$

for any w in H^1_p. A *weak solution* $t \to u(\cdot, t)$ must belong to H^1_p and satisfy (1.3′) a.e. in $[0,T]$, u_t assumed to belong also to H^1.

Note added in proof.

The present article was already accepted for publication when Professor J. Douglas, Jr. kindly pointed to the authors the contents of [7]. In that work, L. Wahlbin considers another generalization of (1.1) and analyzes a numerical scheme which is closely related to one of the algorithms we proposed.

Wahlbin's paper differs from ours essentially in the following points:

i) The behavior of the non-linear term, which in [7] appears in the form $\partial f(x,u)/\partial x$. Of course, different assumptions on f are then required and other techniques employed to deduce the stability of the scheme. (Recall that the "chop-off" technique introduced in (3.2) makes our a priori estimates straightforward.)

ii) The algorithm proposed in [7] is a two-step discretization obtained by simulating the equation at time levels $t_n \equiv n\Delta t$ (and not at $t_{n+\frac{1}{2}}$). The same approximation spaces of piecewise-polynomial functions are employed, leading to error estimates that coincide with the ones we have gotten.

iii) Our computer experiments indicated that the "predictor-corrector" scheme we introduced performs much better than the "two-step" algorithm. We expect that the other "two-step" algorithm would not fare much better.

iv) We present an existence proof for the exact solution of (1.3) based on the numerical schemes we designed.

2. Description of the algorithm.

Let r, ℓ and N be positive integers such that $0 < r < \ell - 1$, and denote $1/N$ by h. Consider the space $S^{\ell, r}_h \subset H^{r+1}_p$ of functions w that coincide with a polynomial of degree less than ℓ in each subinterval $[jh, (j+1)h]$, $0 \le j < N$. Thus any function w in $S^{\ell, r}_h$ is of class C^r and further satisfy (1.5). Take also a positive integer M, denote T/M by Δt, $s\Delta t$ by t_s, $u(\cdot, t_n)$ by u_n and $g(\cdot, t_s)$ by g_s.

For any integer n, $0 \le n \le M$, we seek functions $U^n \equiv U^n_N$ in $S^{\ell, r}_h$ to approximate the solution u at instant t_n.

The first algorithm we propose is of *predictor-corrector* type. Define U^0 by

$$(U^0 \mid w) = (u_0 \mid w) \tag{2.1}$$

and for $n \ge 0$, determine a first approximation $\hat{U}^{n+1}$ *thru*

$$\left(\frac{\hat{U}^{n+1} - U^n}{\Delta t} \,\Big|\, w \right) = \langle f(U^n), w_x \rangle + \langle g_{n+\frac{1}{2}}, w \rangle \tag{2.2}$$

and then obtain the "corrected" approximation U^{n+1} by

$$\left(\frac{U^{n+1} - U^n}{\Delta t} \,\Big|\, w \right) = \left\langle f\left(\frac{\hat{U}^{n+1} + U^n}{2} \right), w_x \right\rangle + \langle g_{n+\frac{1}{2}}, w \rangle, \tag{2.3}$$

where (2.1) – (2.3) must hold for any w in $S^{\ell, r}_h$.

When w runs thru a basis of $S_h^{\ell,r}$, (2.2) and (2.3) give two non-singular systems of linear algebraic equations. As remarked above, the coefficient matrix of the systems thus obtained remains the same for all time levels, so that triangularization is done once for all. To advance each time level, two systems must always be solved. The non-singularity of the linear systems we deal with is a consequence of being the inner-product $(\cdot\,|\,\cdot)$ equivalent to the standard one in H^1.

The second algorithm involves a *two-step discretization* in the time variable:

(2.4) $$V^0 \equiv U^0 \qquad V^1 \equiv U^1$$

with U^0 and U^1 as defined in (2.1) – (2.3), and for $n \geq 1$

(2.5) $$\left(\frac{V^{n+1} - V^n}{\Delta t} \,\Big|\, w\right) = \left\langle f\left(\frac{3}{2}V^n - \frac{1}{2}V^{n-1}\right), w_x\right\rangle + \langle g_{n+\frac{1}{2}}, w\rangle,$$

for any w in $S_h^{\ell,r}$.

This scheme has the same accuracy as the previous one and requires approximately half the calculations, as one solves only one system for each time level. Nevertheless the numerical experiments described in section 4 indicate a sensibly better performance of the first algorithm, as compared with the latter.

Both schemes are unconditionally convergent. This is proved by using a slight change in the arguments presented in [2], and will be done in the next section. Since all results we obtain are valid for both schemes, in the sequel W^n will denote approximations defined by either algorithm.

3. An existence theorem and convergence results.

From now on we shall assume that f is continuously differentiable on the real line and that g, besides fulfilling (1.6), satisfies also

$$\int_0^T |g(\cdot,t)|^2_{H^1}\, dt < \infty,$$

that is, $g \in L^2(0,T;H^1_p)$. Furthermore, u_0 will be taken as an arbitrary function in H^1_p.

First we state a basic fact:

Lemma 1. *There exists at most one solution of* (1.3′) – (1.5) *and it must satisfy*

(3.1) $$|u(\cdot,t)|_{H^1} \leq C, \qquad 0 \leq t \leq T$$

where $C = C(u_0, g, T)$ *independs of* f.

Proof. Let $u \in H^1_p$ satisfy (1.3′) – (1.5) and take $w \equiv u$ in (1.3′). By (1.5),

$$\langle f(u), u_x\rangle = \int_0^{u(x)} f(s)\,ds\,\Big]_{x=0} = 0.$$

Thus (1.3′) may be written in the form

$$\frac{1}{2}\frac{d}{dt}(u\,|\,u) = \langle g, u\rangle \leq \frac{1}{2}\{|g|^2_{L^2} + |u|^2_{L^2}\},$$

from which we obtain by integration

$$|u|^2_{H^1}(t) \leq |u_0|^2_{H^1} + \int_0^t \{|g|^2_{L^2} + |u|^2_{L^2}\}\, d\tilde{t}.$$

An application of Gronwall's lemma to this relation then gives (3.1). We shall not repeat here the argument for uniqueness, which is standard.

Thanks to this lemma, we can assume, with no loss of generality, that f has a bounded support. In fact, consider instead of (1.3′), the equation

(3.2) $$(u_t\,|\,w) = \langle f_C(u), w_x\rangle + \langle g, w\rangle,$$

with f_C defined by $f_C(s) \equiv f(s)\,\theta(s/C)$, where

$$\theta(s) \equiv \begin{cases} 1 & 1 \geq |s| \\ \exp\{\exp(1/1 - |s|)/(|s| - 2)\} & 1 < |s| < 2 \\ 0 & |s| \geq 2. \end{cases}$$

Notice that f_C has compact support and the same regularity as f. Moreover, if u_C satisfies (3.2), (1.4) and (1.5), $f(u_C) = f_C(u_C)$, since the constant C in (3.1) depends only on u_0, g and T. Consequently u_C is also a solution of (1.3′) – (1.5), and by the uniqueness property, $u_C = u$.

We further remark that there is no loss of generality in assuming $f(0) = 0$, as (1.3′) remains unchanged when a constant is added to f.

Under the above hypothesis for f, g, and u_0, we have

Lemma 2. *There exists* $\tau > 0$ *such that*

(3.3) $$|\hat{U}^n|_{H^1} + |W^n|_{H^1} \leq C,$$

for $n = 0, 1, \ldots, M = T/\Delta t$, *with* $C = C(T, f, g, u_0)$ *independent of* M, N *and* n.

To prove this result, the same steps in the proof of Lemma 2.1 in [2] must be followed. The independence of the estimate of $f(U^n)$ on n is a consequence of f being assumed a regular function of bounded support.

In an analogous fashion, we take in (2.3) and (2.5)

$$w \equiv \delta_t W^{n+1/2} \equiv (W^{n+1} - W^n)/\Delta t$$

to get the following a priori estimate:

$$|\delta_t U^{n+1/2}|^2_{H^1} \leq \left\{ \left| f\left(\frac{\hat{U}^{n+1} + U^n}{2}\right) \right|_{L^2} + |g_{n+\frac{1}{2}}|_{L^2} \right\} |\delta_t U^{n+\frac{1}{2}}|_{L^2} \leq$$

$$\leq \frac{1}{2} \{C_1^2 + |\delta_t U^{n+\frac{1}{2}}|^2_{H^1}\},$$

and analogously for $\delta_t V^{n+\frac{1}{2}}$, so that we can state

Lemma 3. *There exist constants* $\tau > 0$ *and* $C = C(f, g, u_0, T)$ *such that*

(3.4) $$|\delta_t W^{n+1/2}|_{H^1} \leq C,$$

for $$n = 0, 1, \ldots, M = T/\Delta t.$$

To obtain (3.3) and (3.4) no relation between N and M needs to be assumed. This means that both numerical schemes are *unconditionally stable.*

We now define for $(x, t) \in Q_T$ the global approximations

(3.5) $$W_{N,M}(x, t) \equiv \sum_{j=0}^{M-1} \theta^j_M(t) \, [W^j(x) + \delta_t W^{j+1/2}(x) \, (t - j\Delta t)]$$

where θ^j_M is the characteristic function of the interval $[j\Delta t, (j+1)\Delta t[$. Notice that

$$\frac{d}{dt} W_{N,M}(x, t) = \sum_{j=0}^{M-1} \theta^j_M(t) \, \delta_t W^{j+1/2}(x), \text{ for } t \neq t_j$$

and thus the above estimates imply:

$$W_{N,M}, \; \dot{W}_{N,M} \in L^2(0, T; H^1).$$

They also indicate that both families $W_{N,M}$ and $\dot{W}_{N,M}$ remain in a bounded set of $L^2(0, T; H^1)$. Consequently, there exist sub-sequences $W'_{N,M}$ and $\dot{W}'_{N,M}$ and functions W in $L^2(0, T; H^1_p)$, W_1 in $L^2(0, T; H^1)$ such that as $N, M \to \infty$

$$\left. \begin{array}{l} W'_{N,M} \to W \\ \dot{W}'_{N,M} \to W_1 \end{array} \right\} \text{weakly in } L^2(0, T; H^1).$$

Of course, by making use of the standard argument from the theory of distributions, $W_1 = \dot{W}$ follows. Moreover, there is also a subsequence, still denoted by $W'_{N,M}$, such that

$$W'_{N,M} \to W \qquad \text{a.e. in } Q_T,$$

as a consequence of Sobolev's Imbedding Theorem. Notice that both sequences

$$\hat{U}_{N,M}(x, t) \equiv \sum_{j=0}^{M-1} \theta^j_M(t) \, [\hat{U}^{j+1}(x) + U^j(x)]/2$$

and

$$\hat{V}_{N,M}(x, t) \equiv \sum_{j=0}^{M-1} \theta^j_M(t) \, [3\, V^{j+1}(x) - V^j(x)]/2$$

have the same pointwise limit as $U'_{N,M}$ and $V'_{N,M}$ respectively. This is clear for the latter sequence. To study the former, take $w \equiv \hat{U}^{n+1} - U^{n+1}$ in both (2.2) and (2.3), then subtract, thus obtaining

$$\hat{U}^{n+1} - U^{n+1}|^2_{H^1} \leq \Delta t \, \{|f(\hat{U}^{n+1}/2 + U^n/2)|_{L^2} |\hat{U}^{n+1} - U^{n+1}|_{H^1}\} \leq$$
$$\leq \Delta t \, \{|f(\hat{U}^{n+1}/2 + U^n/2)|^2_{L^2} + |\hat{U}^{n+1} - U^{n+1}|^2_{H^1}\}/2.$$

We also have that as $M \to \infty$,

$$g_M(x, t) \equiv \sum_{j=0}^{M-1} \theta^j_M(t) \, g_{j+\frac{1}{2}}(x) \to g(x, t)$$

weakly in $L^2(0, T; H^1_p)$.

Now let ω be an arbitrary function in $L^2(0, T; H^1_p)$ and let $\Omega^n \in S^{\ell,r}_h$ yield strong approximations of ω, in the sense that, denoting

$$\Omega_{N,M}(x, t) \equiv \sum_{j=0}^{M-1} \theta^j_M(t) \, \Omega^n(x),$$

we have

$$\int_0^T |\Omega_{N,M} - \omega|^2_{H^1} \, dt \to 0, \text{ as } M, N \to \infty.$$

For $i, j = 1, \ldots, M-1$, take in either (2.3) or (2.5) $w \equiv \theta^j_M(t)$, multiply by $\theta^i_M(t)$ and add to obtain

$$\int_0^T (\delta_t W_{N,M} \mid \Omega_{N,M})\, dt = \int_0^T \langle f(\hat{W}_{N,M}), (\Omega_{N,M})_x \rangle\, dt + \int_0^T \langle g_M, \Omega_{N,M} \rangle\, dt .$$

Passing to the limit as $M, N \to \infty$ we get

$$\int_0^T (\dot{W} \mid \omega)\, dt = \int_0^T \langle f(W), \omega_x \rangle\, dt + \int_0^T \langle g, \omega \rangle\, dt,$$

for any ω in $L^2(0, T, H_p^1)$. Now let $\tau \in (0, T)$ be arbitrary, choose $\delta > 0$ small enough and pick any w in H_p^1. Take ω as equal to the characteristic function of $(\tau - \delta, \tau + \delta)$ multiplied by $w/2\delta$ and thus get

$$\frac{1}{2\delta} \int_{\tau-\delta}^{\tau+\delta} \{(\dot{W} \mid w) - \langle f(W), w_x \rangle - \langle g, w \rangle\}\, dt = 0.$$

When we take the limit as $\delta \to 0$ and use a result of Lebesgue's, we conclude that

$$(\dot{W} \mid w) = \langle f(W), w_x \rangle + \langle g, w \rangle, \text{ a.e. in } (0, T).$$

This means that W equals the solution u of equation (1.3′) we are seeking and further that the proposed algorithms do give approximations for it. Observe that the uniqueness property implies that there is no need for taking subsequences of $W_{N,M}$, the whole sequence being convergent.

To obtain a precise estimate for the error $(u - W_{N,M})$, the argument explained in [2] may be followed with no changes so that we simply state

Theorem 1. *Assume that $f \in C^1(\mathbb{R})$, $u_0 \in H_p^1(0, 1)$ and $g \in L^2(0, T; H_p^1)$. Then there exists a unique solution of (1.3′) – (1.5). This solution may be obtained as the limit of the sequences $U_{N,M}$ or $V_{N,M}$ defined by (3.5) and algorithms (2.1) – (2.3) or (2.4) – (2.5), respectively. Furthermore, if $u \in C^4(Q_T)$, the error in these approximations satisfies*

$$\sup_{0 \le n \le M} |W^n - u(\cdot, t_n)|_{H^1} \le C\,[h^{\ell-1} + (\Delta t)^2],$$

with $C = C(u_0, f, g, T)$ and $M = T/\Delta t$.

Conditions implying the regularity of the solution u were obtained in [6]. We should remark that weaker assumptions upon g would suffice.

4. Numerical experiments.

We have implemented the algorithms described above with cubic splines as the approximating spaces, that is, we took for $S_h^{\ell,r}$ the parameters

$\ell \equiv 4$ and $r \equiv 2$. The value for δ was always picked as 1/6 and the experiments were performed on the IBM-370/145 at CBPF, with double-precision in all runs.

In the first two examples we took $g \equiv 0$, $\Delta t \equiv 1/25$, $h \equiv 1/20$ and used the "predictor-corrector" scheme.

Example 1. The plots in Fig. 1 show the system evolution with $f(s) \equiv s + 3s^2/4 + s^3/20$ and $u_0(x) \equiv (x(x-1))^3$.

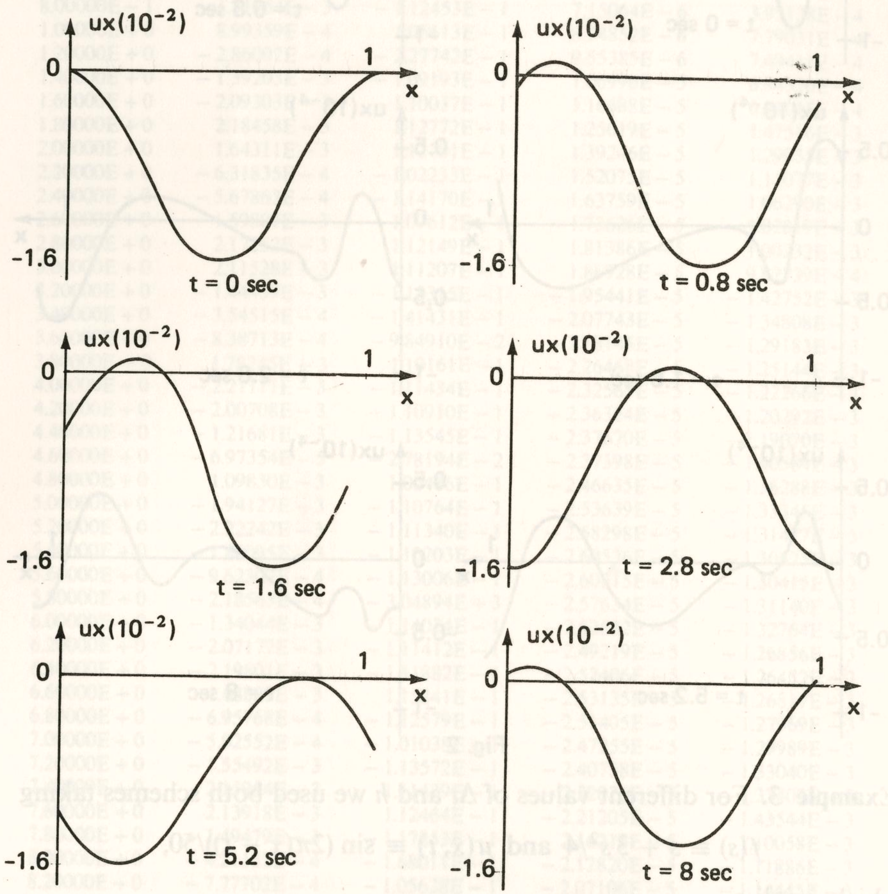

Fig. 1

Example 2. In this example, $f(s) \equiv s + 3s^2/4$ so that the equation assumes the form (1.1). The initial value has a pulse-like shape:

$$u_0(x) \equiv \begin{cases} [8 \times (x - 1/4)]^3/2 & 0 \le x \le 1/4 \\ 0 & 1 \ge x \ge 1/4. \end{cases}$$

The plots obtained are shown in Fig. 2.

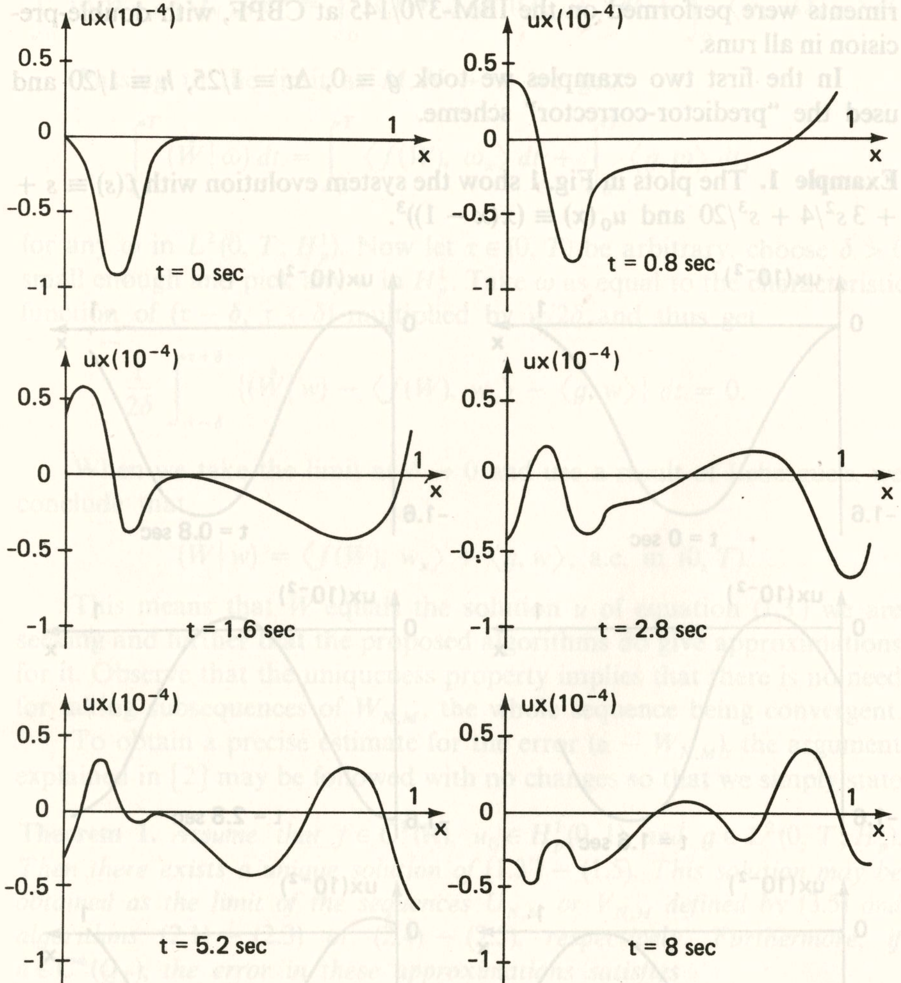

Fig. 2

Example 3. For different values of Δt and h we used both schemes taking

$$f(s) \equiv s + 3s^2/4 \text{ and } u(x, t) \equiv \sin(2\pi(x - t))/50,$$

which implies

$$g(x, t) \equiv \frac{2\pi}{50} \cos(2\pi(x - t)) \left[\frac{3}{100} \sin(2\pi(x - t)) - \frac{2\pi^2}{3} \right].$$

The variation of the error thus obtained is consistent with the estimates deduced (see Table 1) and further indicates a sensibly better performance of the first algorithm, as compared to the second one (see Table 2).

Table 1

Time	$\Delta t \equiv 1/5$, $h \equiv 1/10$ Maximum Absolute error	Maximum Relative error	$\Delta t \equiv 1/50$, $h \equiv 1/10$ Maximum Absolute error	Maximum Relative error
0.00000E+0	-4.51652E-6	-2.37447E-4	-4.51652E-6	-2.37447E-4
2.00000E-1	-1.15600E-3	-1.07870E-1	4.94997E-6	2.51961E-4
4.00000E-1	-1.97411E-3	-1.09087E-1	5.34073E-6	3.46570E-4
6.00000E-1	-2.21629E-3	-1.11033E-1	6.34775E-6	3.75906E-4
8.00000E-1	-1.81954E-3	-1.12453E-1	7.15064E-6	3.95138E-4
1.00000E+0	8.99359E-4	1.05613E-1	8.34852E-6	7.79031E-4
1.20000E+0	-2.86002E-4	-2.27742E-1	9.55385E-6	7.49411E-4
1.40000E+0	-1.39203E-3	-1.09193E-1	1.06996E-5	6.94320E-4
1.60000E+0	-2.09303E-3	-1.10037E-1	1.16688E-5	6.91015E-4
1.80000E+0	2.18458E-3	1.12772E-1	1.25619E-5	1.47516E-3
2.00000E+0	1.64311E-3	1.12701E-1	1.39246E-5	1.29935E-3
2.20000E+0	-6.31835E-4	-1.02233E-1	1.52075E-5	1.11077E-3
2.40000E+0	-5.67863E-4	-1.14170E-1	1.63759E-5	1.06290E-3
2.60000E+0	-1.59807E-3	-1.09612E-1	1.73626E-5	1.02819E-3
2.80000E+0	2.17252E-3	1.12149E-1	1.81386E-5	1.00232E-3
3.00000E+0	2.11528E-3	1.11207E-1	1.86928E-5	9.82739E-4
3.20000E+0	-1.44459E-3	-1.13315E-1	-1.95441E-5	-1.42752E-3
3.40000E+0	-3.54515E-4	-1.41431E-1	-2.07743E-5	-1.34808E-3
3.60000E+0	-8.38713E-4	-9.84910E-2	-2.18145E-5	-1.29183E-3
3.80000E+0	1.78245E-3	1.10161E-1	-2.26468E-5	-1.25144E-3
4.00000E+0	-2.21111E-3	-1.11434E-1	-2.32565E-5	-1.22266E-3
4.20000E+0	-2.00708E-3	-1.10910E-1	-2.36324E-5	-1.20292E-3
4.40000E+0	-1.21681E-3	-1.13545E-1	-2.37670E-5	-1.19070E-3
4.60000E+0	-6.97354E-5	-2.78194E-2	-2.37398E-5	-1.40584E-3
4.80000E+0	1.09830E-3	1.02486E-1	-2.46635E-5	-1.36288E-3
5.00000E+0	-1.94127E-3	-1.10764E-1	-2.53639E-5	-1.33345E-3
5.20000E+0	-2.22242E-3	-1.11340E-1	-2.58298E-5	-1.31477E-3
5.40000E+0	-1.86095E-3	-1.10203E-1	-2.60536E-5	-1.30525E-3
5.60000E+0	-9.62307E-4	-1.13006E-1	-2.60315E-5	-1.30415E-3
5.80000E+0	-2.18563E-4	-3.04894E+3	-2.57634E-5	-1.31140E-3
6.00000E+0	-1.34044E-3	-1.14024E-1	-2.52532E-5	-1.32764E-3
6.20000E+0	-2.07177E-3	-1.11412E-1	-2.49219E-5	-1.26856E-3
6.40000E+0	-2.19801E-3	-1.11882E-1	-2.52406E-5	-1.26452E-3
6.60000E+0	1.68889E-3	1.15841E-1	-2.53135E-5	-1.26817E-3
6.80000E+0	-6.95768E-4	-1.12579E-1	-2.51405E-5	-1.27969E-3
7.00000E+0	-5.02552E-4	-1.01038E-1	-2.47255E-5	-1.29989E-3
7.20000E+0	-1.55492E-3	-1.13572E-1	-2.40758E-5	-1.33040E-3
7.40000E+0	2.15954E-3	1.11479E-1	-2.32026E-5	-1.37402E-3
7.60000E+0	2.13918E-3	1.12464E-1	-2.21205E-5	-1.43544E-3
7.80000E+0	-1.49479E-3	-1.17253E-1	2.16218E-5	1.10058E-3
8.00000E+0	-4.21132E-4	-1.68011E-1	-2.12820E-5	-1.11886E-3
8.20000E+0	-7.77702E-4	-1.05628E-1	-2.07106E-5	-1.14445E-3
8.40000E+0	1.74201E-3	1.13041E-1	-1.99165E-5	-1.17943E-3
8.60000E+0	2.20927E-3	1.11341E-1	-1.89145E-5	-1.22739E-3
8.80000E+0	-2.03705E-3	-1.12566E-1	-1.77222E-5	-1.29445E-3
9.00000E+0	-1.27459E-3	-1.18938E-1	-1.63605E-5	-1.39171E-3
9.20000E+0	-1.38168E-4	-1.10013E-1	1.54129E-5	8.51705E-4
9.40000E+0	1.03990E-3	1.07928E-1	1.46538E-5	8.67780E-4
9.60000E+0	1.90867E-3	1.08904E-1	1.36884E-5	8.88270E-4
9.80000E+0	-2.22473E-3	-1.11236E-1	1.25435E-5	9.83928E-4
1.00000E+1	-1.90102E-3	-1.12576E-1	1.13383E-5	1.05802E-3

Table 2

	Maximum Absolute Error			
	$\Delta t \equiv 1/15$, $h \equiv 1/10$		$\Delta t \equiv 1/25$, $h \equiv 1/20$	
Time	*Algorithm* 1	*Algorithm* 2	*Algorithm* 1	*Algorithm* 2
0.00000E+0	−4.51652E−6	−4.51652E−6	−2.76795E−7	−2.76795E−7
6.66666E−2	4.49580E−5	4.49580E−5	9.40378E−6	9.40378E−6
1.33333E−1	−8.52266E−5	1.73508E−4	−1.86729E−5	−3.81426E−5
2.00000E−1	−1.27154E−4	−3.04737E−4	2.77060E−5	−6.65822E−5
2.66666E−1	−1.58268E−4	−4.16813E−4	−3.64292E−5	−9.40829E−5
3.33333E−1	−1.92236E−4	−5.23510E−4	−4.47531E−5	−1.20707E−4
4.00000E−1	−2.15404E−4	−6.10108E−4	−5.24396E−5	−1.45522E−4
4.66666E−1	−2.28858E−4	−6.69713E−4	−5.97207E−5	−1.69130E−4
5.33333E−1	−2.42922E−4	−7.21818E−4	−6.59036E−5	−1.90063E−4
6.00000E−1	−2.38599E−4	−7.34975E−4	−7.18487E−5	−2.09551E−4
6.66666E−1	−2.35313E−4	−7.33629E−4	−7.62094E−5	−2.25667E−4
7.33333E−1	−2.21221E−4	−7.07349E−4	−8.05566E−5	−2.39946E−4
8.00000E−1	1.95034E−4	−6.49299E−4	−8.30827E−5	−2.50656E−4
8.66666E−1	1.71499E−4	−5.82976E−4	−8.53605E−5	−2.58987E−4
9.33333E−1	1.34516E−4	−4.86409E−4	−8.60595E−5	−2.63813E−4
1.00000E+0	9.81930E−5	−3.79377E−4	−8.61051E−5	−2.65705E−4
1.06666E+0	−5.66962E−5	−2.59177E−4	−8.49471E−5	−2.64462E−4
1.13333E+0	−1.41062E−5	1.27942E−4	−8.26968E−5	−2.59823E−4
1.20000E+0	−3.19312E−5	−8.34261E−6	−7.97302E−5	−2.52568E−4
1.26666E+0	−7.60889E−5	1.40580E−4	−7.53927E−5	−2.41659E−4
1.33333E+0	−1.13227E−4	2.65305E−4	−7.07340E−5	−2.28641E−4
1.40000E+0	−1.52850E−4	3.88821E−4	6.45506E−5	−2.12074E−4
1.46666E+0	−1.83740E−4	4.95509E−4	5.83404E−5	−1.93744E−4
1.53333E+0	−2.06849E−4	−5.81897E−4	5.07851E−5	−1.72463E−4
1.60000E+0	−2.29684E−4	−6.61452E−4	4.31754E−5	−1.49660E−4
1.66666E+0	−2.35057E−4	−7.04752E−4	3.46342E−5	−1.24666E−4
1.73333E+0	2.42746E−4	−7.35451E−4	−2.59700E−5	−9.84673E−5
1.80000E+0	2.38052E−4	−7.40340E−4	−1.68030E−5	−7.09131E−5
1.86666E+0	2.21710E−4	−7.11853E−4	−7.59468E−6	4.27233E−5
1.93333E+0	2.07091E−4	−6.74214E−4	−1.97956E−6	1.39503E−5
2.00000E+0	1.76644E−4	−6.01854E−4	−1.12661E−5	1.51641E−5
2.06666E+0	−1.46643E−4	−5.15661E−4	−2.06105E−5	4.40619E−5
2.13333E+0	−1.09601E−4	−4.12950E−4	−2.94782E−5	7.21867E−5
2.20000E+0	−6.77359E−5	2.91440E−4	−3.82811E−5	9.97091E−5
2.26666E+0	−2.65010E−5	1.68823E−4	−4.63416E−5	1.25778E−4
2.33333E+0	−2.11495E−5	3.41982E−5	−5.40788E−5	1.50522E−4
2.40000E+0	−6.24694E−5	1.01258E−4	−6.10285E−5	1.73395E−4
2.46666E+0	−1.04961E−4	2.33613E−4	−6.72790E−5	1.94067E−4
2.53333E+0	−1.42216E−4	3.55822E−4	−7.28392E−5	−2.12867E−4
2.60000E+0	−1.73351E−4	4.63143E−4	−7.72080E−5	−2.28604E−4
2.66666E+0	−2.03964E−4	−5.65220E−4	−8.11559E−5	−2.42438E−4
2.73333E+0	2.19078E−4	−6.36232E−4	−8.84502E−5	−2.52493E−4
2.80000E+0	2.37388E−4	−6.98082E−4	−8.56284E−5	−2.60517E−4
2.86666E+0	2.42468E−4	−7.32711E−4	−8.58546E−5	−2.64466E−4
2.93333E+0	2.36395E−4	−7.35817E−4	8.60043E−5	−2.66141E−4
3.00000E+0	2.31433E−4	−7.28686E−4	8.43683E−5	−2.63913E−4

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