Complete minimal surfaces with total curvature -2π

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1. Introduction.

In 1964, Osserman [6] showed that the total curvature of each complete, regular and orientable, minimal surface in $\mathbb{R}^3$ is a multiple of -4π; and in the case of total curvature -4π, the only possible surfaces are the Enneper's surface and the Catenoid. And in 1967, Chern and Osserman [3] showed that the total curvature of each complete, regular and orientable, minimal surface in $\mathbb{R}^n$ $(n > 3)$ is multiple of -2π.

In this paper, we would like to discuss some geometric properties of those surfaces with total curvature -2π. All surfaces will be assumed to be regular and orientable. The following theorems will be proved.

Theorem 1. *Let S be a minimal surface in $\mathbb{R}^n$, given by the immersion $\vec{x} : M^2 \to \mathbb{R}^n$. If the total curvature of S, $C(S) = -2\pi$, then*

(i) *M is simply connected and S is parabolic.*
(ii) *the gauss map $g : M \to P^{n-1}(\mathbb{C})$ is injective and degenerate*
(iii) *$\vec{x}(M) \subset \mathbb{R}^4$ and $\vec{x}$ is an embedding*
(iv) *S does not satisfy the Ricci condition, i.e., S is not locally isometric to any minimal surface in $\mathbb{R}^3$.*

Theorem 2. *Any two complete isometric minimal surfaces in euclidean spaces, with total curvature -2π, are congruent.*

Theorem 3. *There exists a natural way to describe the set of all non-congruent complete minimal surfaces in euclidean spaces, with total curvature -2π, as the set of all positive real numbers.*

2. Notations and basic facts.

Let $\vec{x} : M^2 \to \mathbb{R}^n$ be an minimal immersion, where M is an orientable differentiable 2-manifold. In terms of isothermal parameters (ξ_1, ξ_2) the immersion is characterized by the following properties:

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$$\sum_{k=1}^{n} \phi_k^2(\zeta) \equiv 0 \tag{2.1}$$

where $\zeta = \xi_1 + i\xi_2$, $\phi_k(\zeta) = \dfrac{\partial x_k}{\partial \xi_1} - i\dfrac{\partial x_k}{\partial \xi_2}$, $\vec{x} = (x_1, \ldots, x_n)$ and

(2.2) $\phi_k(\zeta)$'s are analytic functions of ζ

$$\sum_{k=1}^{n} |\phi_k(\zeta)|^2 \neq 0 \quad \text{for all } \zeta, \text{ and} \tag{2.3}$$

$$\vec{x}(\zeta) = Re \int_{\zeta_0}^{\zeta} \vec{\phi}(\zeta)\, d\zeta \tag{2.4}$$

integrated along any path with any fixed initial point, where $\vec{\phi} = (\phi_1, \ldots, \phi_n)$.

Geometrically, (2.1) means the parameters (ξ_1, ξ_2) are isothermal, (2.2) means the immersion is harmonic in terms of isothermal parameters, and (2.3) means the regularity of the surface.

The gauss map $g : M \to P^{n-1}(\mathbb{C})$ is defined by

$$g(\zeta) = [\bar{\phi}_1(\zeta), \ldots, \bar{\phi}_n(\zeta)] \tag{2.5}$$

in terms of homogeneous coordinates. In fact, $g(\zeta)$ represents the oriented tangent plane generated by $\dfrac{\partial \vec{x}}{\partial \xi_1}(\zeta)$ and $\dfrac{\partial \vec{x}}{\partial \xi_2}(\zeta)$. For details, see [3].

3. Proof of theorem 1.

(i) Since the surface is orientable, we use isothermal parameters to put a Riemann surface structure on the manifold M. Since S has finite total curvature, it's known [3] that M is conformally equivalent to a compact Riemann surface W punctured at a finite number of points $p_1, \ldots, p_r$, $r \geq 1$; and the differentials $\phi_k(\zeta)d\zeta$ are meromorph at each p_j.

Let γ be the genus of W, X the Euler characteristic of M, then $X = 2 - 2\gamma - r$.

By a theorem of Chern and Osserman [3], we have

$$C(S) \leq 2\pi(X - r) = 2\pi(2 - 2\gamma - 2r) \tag{3.1}$$

with $C(S) = -2\pi$. Thus we can conclude that

$$\gamma = 0 \quad \text{and} \quad r = 1 \tag{3.2}$$

that is M is conformally equivalent to the complex plane $\mathbb{C}$. Therefore M is simply connected and S is parabolic.

(ii) Since $M = \mathbb{C}$ and the analytic functions ϕ_k's are meromorph at ∞, ϕ_k's are polynomials. From $C(S) = -2\pi$, we have

$$\max_k \text{ degree } \phi_k = 1, \tag{3.3}$$

because the total order of intersection with each hyperplane in $P^{n-1}(\mathbb{C})$, which does not contain the image of the extended gauss map

$$\hat{g} : \hat{\mathbb{C}} \to P^{n-1}(\mathbb{C}),$$

has to be 1.[3]. Accordingly, let

$$\phi_k(\zeta) = a_k\zeta + b_k,\ a_k, b_k \in \mathbb{C} \tag{3.4}$$

and

$$\vec{a} = (a_1, \ldots, a_n),\ b = (b_1, \ldots, b_n) \tag{3.5}$$

Then we have

$$\vec{\phi}(\zeta) = \zeta\vec{a} + \vec{b} \tag{3.6}$$

From (2.3), (3.3), (3.6), we can conclude that

(3.7) $\vec{a}$ and $\vec{b}$ are linearly independent in $\mathbb{C}^n$

Therefore, from (2.5), (3.6), (3.7), we see that the gauss map g is injective, and degenerate (i.e., the gauss image lies in some hyperplane of the projective space $P^{n-1}(\mathbb{C})$).

(iii) Let's write $a_k = \alpha_k + i\beta_k$, $b_k = u_k + iv_k$. Then

$$\vec{a} = \vec{\alpha} + i\vec{\beta},\quad \vec{b} = \vec{u} + i\vec{v} \tag{3.8}$$

where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\vec{\beta} = (\beta_1, \ldots, \beta_n)$, $\vec{u} = (u_1, \ldots, u_n)$, $\vec{v} = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$.

From (2.1), (3.4), we have

$$\sum_{k=1}^{n} a_k^2 = 0, \sum_{k=1}^{n} b_k^2 = 0 \quad \text{and} \quad \sum_{k=1}^{n} a_k b_k = 0$$

i.e.,

$$|\vec{\alpha}| = |\vec{\beta}|,\ \vec{\alpha} \perp \vec{\beta};\ |\vec{u}| = |\vec{v}|,\ \vec{u} \perp \vec{v} \quad \text{and}$$

$$\langle \vec{\alpha}, \vec{u} \rangle = \langle \vec{\beta}, \vec{v} \rangle, \langle \vec{\alpha}, \vec{v} \rangle = -\langle \vec{\beta}, \vec{u} \rangle \tag{3.9}$$

From (2.4), (3.6), we have

$$\vec{x}(\zeta) = Re\left(\frac{\zeta^2}{2}\vec{a} + \zeta\vec{b}\right) = \frac{\xi_1^2 - \xi_2^2}{2}\vec{\alpha} - \xi_1\xi_2\vec{\beta} + \xi_1\vec{u} - \xi_2\vec{v}; \tag{3.10}$$

here we assume that $\vec{x}(0) = 0$

From (3.10), we see immediatly that $\vec{x}(M) \subset \mathbb{R}^4$.
Now we want to show that $\vec{\alpha}, \vec{\beta}, \vec{u}, \vec{v}$ are linearly in dependent in $\mathbb{R}^n$.
Let

$$c_1\vec{\alpha} + c_2\vec{\beta} + c_3\vec{u} + c_4\vec{v} = 0, \; c_i \in \mathbb{R} \tag{3.11}$$

be given. By taking inner product with $\vec{\alpha}, \vec{\beta}, \vec{u}, \vec{v}$, respectively, we have

$$\begin{cases} c_1\,|\vec{\alpha}|^2 + c_3\,\langle\vec{\alpha},\vec{u}\rangle + c_4\,\langle\vec{\alpha},\vec{v}\rangle = 0 \\ c_2\,|\vec{\beta}|^2 + c_3\,\langle\vec{\beta},\vec{u}\rangle + c_4\,\langle\vec{\beta},\vec{v}\rangle = 0 \\ c_1\,\langle\vec{\alpha},\vec{u}\rangle + c_2\,\langle\vec{\beta},\vec{u}\rangle + c_3\,|u|^2 = 0 \\ c_1\,\langle\vec{\alpha},\vec{v}\rangle + c_2\,\langle\beta,\vec{v}\rangle + c_4\,|\vec{v}|^2 = 0 \end{cases} \tag{3.12}$$

From (3.9), (3.12), we can show that $-c_2\vec{\alpha} + c_1\vec{\beta} - c_4\vec{u} + c_3\vec{v}$ is orthogonal to $\vec{\alpha}, \vec{\beta}, \vec{u}$ and $\vec{v}$. Hence we have

$$-c_2\vec{\alpha} + c_1\vec{\beta} - c_4\vec{u} + c_3\vec{v} = 0 \tag{3.13}$$

From (3.11) (3.13), we get

$$(c_1 - ic_2)\,(\vec{\alpha} + i\vec{\beta}) + (c_3 - ic_4)\,(\vec{u} + i\vec{v}) = 0 \tag{3.14}$$

From (3.7), (3.8), (3.14), we have $c_1 = c_2 = c_3 = c_4 = 0$, which shows that

(3.15) $\vec{\alpha}, \vec{\beta}, \vec{u}, \vec{v}$ are linearly independent in $\mathbb{R}^n$

From (3.10), (3.15), we see that $\vec{x}(M)$ is homeomorphic linearly with a graph. Therefore $\vec{x}$ is an embedding.

(iv) A riemannian metric ds^2 on a surface is said to satisfy the Ricci condition if its Gauss curvature K satisfies $K < 0$, and if the new metric $\widehat{ds^2} = \sqrt{-K}\,ds^2$ is flat, i.e., its Gauss curvature $\hat{K}$ satisfies $\hat{K} \equiv 0$.

It's that every metric on a minimal surface in $\mathbb{R}^3$ satisfies this condition away from the points where $K = 0$. As a matter of fact, Ricci [1 p. 124] showed that every metric satisfying this condition can be locally realized on a minimal surface in $\mathbb{R}^3$. More details can be found in the paper of Lawson [4].

With respect to isothermal parameters $\zeta = \xi_1 + i\xi_2$, the metric induced is given by

$$ds^2 = \lambda^2\,|d\zeta|^2 \tag{3.16}$$

and the Gauss curvature is given by

$$K = -\frac{\Delta \log \lambda}{\lambda^2} = -\left(\frac{\Delta \log \lambda^4}{4\lambda^2}\right) \tag{3.17}$$

For minimal surface we have [3]

$$\lambda^2 = \frac{1}{2}\,|\phi|^2, \; K = -\frac{4\,|\phi \wedge \phi'|^2}{|\phi|^6}, \tag{3.18}$$

where $|\phi \wedge \phi'|^2 = \sum_{1 \le j < k \le n} |\phi_j\phi'_k - \phi_k\phi'_j|^2$.

Therefore ds^2 satisfies the Ricci condition if and only if

$$\Delta \log(-K\lambda^4) \equiv 0 \Leftrightarrow \Delta \log \frac{|\phi \wedge \phi'|^2}{|\phi|^2} \equiv 0 \Leftrightarrow \tag{3.19}$$

$$\partial\bar{\partial} \log \frac{|\phi \wedge \phi'|^2}{|\phi|^2} \equiv 0,$$

where $\partial = \dfrac{\partial}{\partial\zeta}, \; \bar{\partial} = \dfrac{\partial}{\partial\bar{\zeta}}$.

In our case, from (3.6), (3.7), we have

$$|\phi \wedge \phi'|^2 = \sum_{1 \le j < k \le n} |b_j a_k - a_j b_k| = \text{constant} > 0$$

and

$$\partial\bar{\partial}\; \log|\phi|^2 = \frac{1}{|\phi|^4}\,(|\vec{a}|\;|\vec{b}|^2 - |\langle\vec{a},\vec{b}\rangle|^2)$$

which is positive because of the Schwarz inequality. Therefore S does not satisfy the Ricci condition.

Q.E.D.

Remark. For complete minimal surfaces in $\mathbb{R}^3$, $C(S) = -4\pi$ is equivalent to the $(1-1)$-ness of the gauss map [6]. However, for complete minimal surfaces in $\mathbb{R}^n$, the $(1-1)$-ness of the gauss map is just a necessary condition for the total curvature being -2π. As a matter of fact,

$$\vec{x}(\zeta) = \begin{cases} (\zeta^m, \zeta^2, \zeta), \; (m \ge 3) \\ (e^\zeta, \zeta^2, \zeta) \end{cases} \quad \zeta \in \mathbb{C}$$

are complete minimal surfaces in $\mathbb{C}^3 = \mathbb{R}^6$, with total curvature $-2\pi(m-1)$ and $-\infty$, respectively, and, their gauss maps are injective.

4. Proofs of theorems 2 and 3.

From theorem 1, we may assume all these surfaces lie in $\mathbb{R}^4$. Given $\vec{x}: M \to \mathbb{R}^4$; $y: N \to \mathbb{R}^4$, two complete minimal surfaces with total curvature -2π, let $\theta: M \to N$ be an isometry between them, with respect to the induced metrics.

From (3.10), we see that $\vec{x}$ is isometric to the holomorphic curve $\psi: M \to \mathbb{C}^4$, given by

(4.1) $$\psi(\zeta) = \frac{1}{\sqrt{2}}\left(\frac{\zeta^2}{2}\vec{a} + \zeta\vec{b}\right)$$

which lies fully in a two dimensional complex subspace. After a unitary transformation, we may assume ψ lie in $\mathbb{C}^2$, and

(4.2) $$\vec{a} = (a, 0),\ \vec{b} = (b, c) \text{ with } a \neq 0,\ c \neq 0.$$

Calabi [2] showed that the set $\Gamma(\psi)$ of all non-congruent minimal surfaces which are isometric to ψ, with the same parameter, is naturally described by the set of all 2×2 symmetric complex matrices P with the following properties:

(i) $I_2 - \bar{P}P$ is semi-positive definite
(ii) ${}^t\psi' \cdot P \cdot \psi'' \equiv 0$

From (4.1) and (4.2), it can be easily checked that this set contains only the zero matrix, i.e., there is only one class of non-congruent minimal surfaces isometric to ψ. Therefore $\vec{y} \circ \theta$ and $\vec{x}$ are congruent and theorem 2 is proved.

From the proof of theorem 2, we see that all these non-congruent surfaces are determined by the holomorphic curves whose tangents are "horizontal lines" in $\mathbb{C}^2$ of the form $\psi'(\zeta) = \frac{1}{\sqrt{2}}(a\zeta + b, c)$, with $a \neq 0$, $c \neq 0$. And any two such holomorphic curves ψ_1, ψ_2 are isometric if and only if their tangents ψ'_1, ψ'_2 have the same module at the appopriate corresponding points; which holds if and only if the second coordinates c_1, c_2 have the same module. Therefore theorem 3 is proved.

QED.

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