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SK_1 R[X, Y]/($X^2 + Y^2 - 1$): Remarks on an example of Bass and Milnor

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In [1] pages 338 and 714 it is proved that $SK_1 R[X, Y]/(X^2 + Y^2 - 1) = Z/2Z$ (R = real numbers, Z = integers). This example is mentioned also in [3] and [4]. Here I give a new proof, which uses Quillen's exact sequence of localization and basic facts about the projective line over a field (or ring). Granted these tools it is simpler than the methods in [1] and yields information for other rings besides R, as well as about higher K_i.

1. K-Theoretic preliminaries. A good account of the functors K_0 and K_1 (including some historical notes) can be found in [3]. This paper is chiefly concerned with K_1, which I will now define. Let A be a ring with 1 and let $GL_n(A)$ be the group of $n \times n$ invertible matrices over A. Let $e_{ij}(\lambda)$ ($i \neq j$) be the matrix with l's down the diagonal, λ at the ij position, and 0's elsewhere. Let $E_n(A)$ be the subgroup of $GL_n(A)$ generated by $e_{ij}(\lambda)$, $\lambda \in A$, $1 \leq i \neq j \leq n$. We can think of $GL_n(A)$ as a subgroup of $GL_{n+1}(A)$ by identifying $M \in GL_n(A)$ with

$$\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(A).$$

Let $GL(A) = \bigcup_{n=1}^{\infty} GL_n(A)$. Under the above identification $E_n(A) \subset E_{n+1}(A)$ so we can also form $E(A) = \bigcup_{n=1}^{\infty} E_n(A)$. It turns out that $E(A)$ is a normal subgroup of $GL(A)$ and that the quotient $GL(A)/E(A)$ is an abelian group, which we define to be $K_1(A)$. Now assume that A is commutative, and let $U(A)$ be the units of A. Then the determinant defines a homomorphism det: $K_1(A) \to U(A)$ which is split by sending $\lambda \in U(A)$ to the matrix $\lambda \in GL_1(A)$. The kernel of det is defined to be $SK_1(A)$. Thus we have a direct sum decomposition $K_1(A) = U(A) \oplus SK_1(A)$.

If A is a field (or even a local ring) then elementary row and column operations show that $SK_1(A) = 0$. One of the most easily computed non-trivial SK_1's is $SK_1\ R[X, Y]/(X^2 + Y^2 - 1)$, or more generally

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$SK_1\, k[X, Y]/(X^2 + Y^2 - 1)$, k a field. Even in this simplest case the computation seems to require a fair bit of algebraic geometry, algebraic number theory, or algebraic topology, as well as tools from algebraic K-theory. I will give references for the latter as I need them. Here I will remark only that Milnor in [4] defined groups $K_2(A)$, and more recently Quillen defined groups $K_i(X)$, $i \geq 0$, X a scheme (which agree with the previously defined groups if $i \leq 2$ and X is affine). These groups have certain functorial properties, and the different K_i are related by exact sequences.

Computations of K_1 have found arithmetical and topological applications (see [3] p 359 and 362 respectively).

2. Quillen's exact sequence. Let $A = k[T_0, T_1]/(aT_0^2 + bT_1^2 - 1)$ where k is a field of characteristic $\neq 2$, $a, b \in k$ and $aT_0^2 + bT_1^2 = 0$ has no solutions in k. Suppose that $aT_0^2 + bT_1^2 - T_2^2$ has solutions in k. Any conic in P_k^2 (projective space of dimension 2 over k) that contains a k-rational point is isomorphic to P_k^1 ([6] p6). Thus Spec A has been obtained from P_k^1 by removing a non-rational point $Y =$ Spec K of degree 2 ($K = k(\sqrt{\alpha})$) where $\alpha = -b/a$.

We remark that Y, P_k^1, and Spec A are all regular schemes, hence coherent sheaves and vector bundles yield the same Quillen K-theory, which we denote K_i. By proposition 3.2 p 127 of [5] (applied to P_k^1 and the closed subscheme Y) we have an exact sequence

$$\begin{array}{ccccccccc} \cdots \to & K_i(Y) & \xrightarrow{i_*} & K_i(P_k^1) & \xrightarrow{J^*} & K_i(A) & \to & K_{i-1}(Y) & \to \cdots \\ & \| & & & & & & \| & \\ & K_i(K) & & & & & & K_{i-1}(K) & \end{array}$$

where i_* and J^* are induced by extension of sheaves by zero and restriction to an open subset respectively (both exact functors). It is known that $K_0(P_k^1) = Z \oplus Z$. The most natural choice of basis is the classes of $\mathcal{O}$ and $\mathcal{O} - \mathcal{O}(-1)$ respectively. The first copy of Z gives the rank of a coherent sheaf. If P is a k-rational point of P_k^1 there is an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_P \to 0.$$

Thus the class of $\mathcal{O} - \mathcal{O}(-1)$ corresponds to Spec $k =$ structure sheaf of any closed k-rational point. Furthermore $K_i(P_k^1) \simeq K_i(k) \oplus K_i(k)$. This isomorphism is $K_0(P_k^1) \otimes_{K_0(k)} K_i(k) \xrightarrow{\cong} K_i(P_k^1)$ given by $y \otimes x \to yf^*(x)$ where $f : P_k^1 \to$ Spec k is the structure morphism (Proposition 4.3 p129 of [5]). These observations suggest that $K_i(Y) \xrightarrow{i^*} K_i(P_k^1)$ ought to be

the restriction of scalars map $K_i(K) \to K_i(k)$ followed by inclusion into the second copy of $K_i(k)$. In the next section I show that this is indeed the case.

3. Projection onto direct summands. In this section it is more convenient to take the classes of $\mathcal{O}(-1)$ and $\mathcal{O} - \mathcal{O}(-1)$ as the basis of $K_0(P_k^1)$. Consider the homomorphism $f_* : K_i(P_k^1) \to K_i(k)$. If $i = 0$ this sends F to $\chi(F) \in Z = K_0(k)$. (F a coherent sheaf on X). By the Riemann-Roch Theorem $\chi\mathcal{O}(-1) = 0$ and $\chi(\mathcal{O}) = 1$ so f_* on the $i = 0$ level is projection onto the second copy of Z. By the projection formula (proposition 2.10 p 126 of [5]) f_* is also projection onto the second summand for $i > 0$.

Let $U = A_k^1$ be the affine subscheme obtained by removing any k-rational point, and $j : U \to P_k^1$ be the inclusion. From the definition of K_0-action (p 124 of [5]) we have $j^*(\mathcal{O}(-1)\, f^*(x)) = j^*\mathcal{O}(-1)\, j^*f^*(x) = j^*f^*(x)$ and $j^*((\mathcal{O} - \mathcal{O}(-1))f^*(x)) = j^*(\mathcal{O} - \mathcal{O}(-1))\, j^*f^*(x) = 0$. But j^*f^* is an isomorphism. ($x \in K_i(k)$). Thus $j^* : K_i(^1_k) \to K_i(U) = K_i(k)$ is projection onto the first copy of $K_i(k)$.

We now examine how image i_* behaves under these two projections. There is a commutative diagram of proper morphisms

$$\begin{array}{ccc} Y = \text{Spec } K & \xrightarrow{i} & P_k^1 \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle f} & \\ & \text{Spec } k & \end{array}$$

(π corresponding to the inclusion $k \to K$) so $f_* i_* = \pi_*$ (as maps on K_i).

There is a commutative diagram of schemes

$$\begin{array}{ccccc} V & \xrightarrow{\ell} & U & \xrightarrow{i} & P_k^1 \\ & {\scriptstyle h} \searrow & \downarrow {\scriptstyle g} & \swarrow {\scriptstyle f} & \\ & & \text{Spec } k & & \end{array}$$

where V is the affine scheme $U - Y$, ℓ is the inclusion, and g, h are the structure morphisms. At the coherent sheaf level $j^*, \ell^*, f^*, g^*, h^*$ and i_* are all exact functors so the corresponding maps on the K_i level are induced by these exact functors. As a functor from coherent sheaves on Y to coherent sheaves on $V \ell^* j^* i_* = 0$. The induced maps on K_i then satisfy $\ell^* j^* i_* = 0$. But $\ell^* g^* = h^*$ is an inclusion and g^* is an isomorphism. Thus ℓ^* is an inclusion so $j^* i_* = 0$.

Thus we have determined $j^* i_*$ and $f_* i_*$ (as maps on K_i), and i_* does indeed have the desired form.

4. Completion of the computation. We now have an exact sequence

$$\ldots \to K_i(K) \xrightarrow{i^*} K_i(k) \oplus K_i(k) \xrightarrow{J^*} K_i(A) \to K_{i-1}(K) \xrightarrow{i^*} K_{i-1}(k) \oplus K_{i-1}(k) \to$$

The inclusion $k \to A$ induces a homomorphism $K_i(k) \to K_i(A)$, which can also be thought of as e^* where e: Spec $A \to$ Spec k is the structure morphism (note that we are in this section reverting the usual basis of $K_0(P_k^1)$ – i.e. $\mathcal{O}$ and $\mathcal{O} - \mathcal{O}(-1)$.) This is split by the inclusion of any k-rational point. Let $K_i(A) = K_i(k) \oplus \tilde{K}_i(A)$ be the resulting direct sum decomposition (i.e. $\tilde{K}_i(A) = \ker I^* : K_i(A) \to K_i(k)$ where I : Spec $k \to$ Spec A is the inclusion of some k-rational point.) This direct sum decomposition is not claimed to be independent of the choice of k-rational point (except for K_0, and K_1).

I claim that J^* respects the direct sum decompositions of $K_i(P_k^1)$ and $K_i(A)$ and is an isomorphism on the first summand. This follows from the commutative diagrams

$$\begin{array}{ccc} \text{Spec } A & \xrightarrow{J} & P_k^1 \\ & {\scriptstyle e} \searrow \quad \swarrow {\scriptstyle f} & \\ & \text{Spec } k & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & \text{Spec } k & \\ & {\scriptstyle I} \swarrow \quad \searrow {\scriptstyle I'} & \\ \text{Spec } A & \xrightarrow{J} & P_k^1 \end{array}$$

since image f^* and ker $(I')^*$ are the two copies of $K_i(k)$ in the direct sum decomposition of $K_i(P_k^1)$.

Thus we have exact sequence

$$K_i(K) \underset{\pi *}{\rightrightarrows} K_i(k) \to \tilde{K}_i(A) \to K_{i-1}(K) \underset{\pi *}{\rightrightarrows} K_{i-1}(k)$$

or

$$0 \to K_i(k)/\pi_* K_i(K) \to \tilde{K}_i(A) \to \ker[\pi_* : K_{i-1}(K) \to K_{i-1}(k)] \to 0,$$

where π: Spec $K \to$ Spec k is restriction of scalars (corresponding to the inclusion $k \subset K$).

Let us now examine the case $i = 1$. Here $Z = K_{i-1}(K) \to K_{i-1}(k) = Z$ is just multiplication by 2, so

$$SK_1(A) = \tilde{K}_1(A) = k^*/NK^*$$

where $N : K^* \to k^*$ is the norm map. If $k = R$, $K = \mathbb{C}$, then $N\mathbb{C}^* =$ positive real numbers so $SK_1 R[T_0, T_1]/(T_0^2 + T_1^2 - 1) = Z/2Z$. If $k = Q$ and $K = Q(\sqrt{-1})$ then $a \in Q$ lies in NK^* if and only if $a > 0$ and all primes congruent to 3 mod 4 have even power in the factorization of a. Half the primes are congruent to 3 mod 4 so Q^*/NK^* is the countable direct sum of copies of $Z/2Z$. Thus $SK_1\, Q[T_0, T_1]/(T_0^2 + T_1^2 - 1)$ is the direct sum of a countable number of copies of $Z/2Z$. (*M.P.* Murthy showed me a different proof of this). If $K = Q(\sqrt{d})$ (d square free) the condition for being a norm is somewhat more complicated (see for example [7] 6-6). For $a \in Q^*$ to lie in NK^* it is necessary that all primes at which d is a quadratic non-residue have even power in the factorization of a. Thus $SK_1\, Q[T_0, T_1]/(T_0^2 - dT_1^2 - 1)$ is also the countable direct sum of copies of $Z/2Z$.

For $i = 2$ we get an exact sequence

$$0 \to K_2(k)/\pi_* K_2(K) \to \tilde{K}_2(A) \to \ker(K^* \xrightarrow{N} k^*) \to 0$$

In particular if $k = R$ then $K_2(R) = Z/2Z \oplus D$ where D is divisible. Two times any element of D lies in the image of π_*. Thus $D \subset$ image π_*. But $K_2(\mathbb{C})$ is divisible so cannot map onto $Z/2Z$. Thus $K_2(R)/\pi_* K_2(\mathbb{C}) = Z/2Z$. Also $\ker(\mathbb{C}^* \to R^*)$ is the multiplicative group of complex numbers of norm 1, ie the circle group S^1. Thus we have an exact sequence (for $A = R[T_0, T_1]/T_0^2 + T_1^2 - 1$)

$$0 \to Z/2Z \to \tilde{K}_2(A) \to S^1 \to 0.$$

I was not able to decide myself whether this sequence splits. However *D.* Grayson and *C.* Weibel have been able to show by using the transfer map $K_2(A \otimes_R \mathbb{C}) \to K_2(A)$ that the sequence does not split, and that $\tilde{K}_2(A) = S^1$. Grayson also points out that the non-trivial element of $Z/2Z$ maps to the element $(-1) \begin{bmatrix} T_0 \\ T_1 \end{bmatrix}$ mentioned on page 129 of [4].

5. Other ground rings. If one is sufficiently carefull the ground ring k need not be a field. Define the scheme P_k^1 to be Proj $k[T_0, T_1]$ and let Y be the subscheme defined by $T_0^2 + aT_1^2 = f$. Then as in [2] $P_k^1 - Y =$ Spec A, where $A =$ degree zero part of $k[T_0, T_1]_f \cong k[T_0^2, T_0T_1, T_1^2]/(T_0^2 + aT_1^2 - 1) \cong k[U, V, W]/(U + aV - 1,\ UV - W^2) \cong k[V, W]/(W^2 - V(1 - aV)) = k[V, W]/(W^2 + aV^2 - V)$. If $2a$ is a unit in k, we can change variables so that the polynomial is $W^2 + a(V')^2 - 1/4a$. Otherwise leave A in the form $A = k[V, W]/(W^2 + aV^2 - V)$. The subscheme Y is Proj $k[T_0, T_1]/(T_0^2 + aT_1^2) = D_+(T_1) =$ Spec $k[T_0/T_1]/((T_0/T_1)^2 + a) = k(\sqrt{-a})$. Let $K = k(\sqrt{-a})$. Assume $2 \neq 0$ in k, and that k and K

are both regular. Then the above proof goes through unchanged and we get exact sequences

$$0 \to K_i(k)/i_* K_i(K) \to \tilde{K}_i(A) \to \ker i_* : (K_{i-1}(K) \to K_{i-1}(k)) \to 0$$

where $i_* : K_i(K) \to K_i(k)$ is induced by restriction of scalars, and $\tilde{K}_i(A) = \text{coker}\ (K_i(k) \to K_i(A))$ (The inclusion $k \to A$ is split by $V \to 0, W \to 0$ so $\tilde{K}_i(A)$ is a direct summand of A).

To give a specific example let $k = Z, a = 1, K = Z[i]$. Here $K_0(K) \to K_0(k)$ is multiplication by 2 which is an inclusion. Also $SK_1(K) = SK_1(Z) = 0$ so $K_1(k) = \{\pm 1\}$. The units of $K_1(K)$ all have norm 1 so $\tilde{K}_1(A) = SK_1(A) = Z/2Z$. On the other hand if $a = -2$ then $K = Z[\sqrt{2}]$ and $(1-\sqrt{2})(1+\sqrt{2}) = 1 - 2 = -1$. Some units have norm -1 so $\tilde{K}_1(A) = SK_1(A) = 0$.

Bibliography

[1] H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968.

[2] A. Grothendieck, *Elements de geometrie algebrique* II. *Etude globale elementaire de quelques classes de morphismes*. Inst. Hautes Études Sci. Publ. Math. N.° 8, Paris, France, 1961.

[3] T. Y. Lam and M. Siu, *K_0 and K_1 – An introduction to algebraic K-theory*, Amer. Math. Monthly N.° 82 (1975), 329-364.

[4] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies 72, Princeton University Press, Princeton, N. J. 1971.

[5] D. Quillen, *Higher algebraic K-theory*: I, Lecture Notes in Mathematics 341, Springer-Verlag, New York, N. Y., 1973, 179-198.

[6] I. R. Shafarevich, *Basic algebraic geometry*, Springer-Verlag, New York, Heidelberg, Berlin 1974.

[7] E. Weiss, *Algebraic number theory*, McGraw-Hill, New York, 1963.

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