$SK_1$  R[X, Y]/(X<sup>2</sup> + Y<sup>2</sup> - 1): Remarks on an example of Bass and Milnor

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In [1] pages 338 and 714 it is proved that  $SK_1 R[X, Y]/(X^2 + Y^2 - 1) = Z/2Z$  (R = real numbers, Z = integers). This example is mentioned also in [3] and [4]. Here I give a new proof, which uses Quillen's exact sequence of localization and basic facts about the projective line over a field (or ring). Granted these tools it is simpler than the methods in [1] and yields information for other rings besides R, as well as about higher  $K_i$ .

1. K-Theoretic preliminaries. A good account of the functors  $K_0$  and  $K_1$  (including some historical notes) can be found in [3]. This paper is chiefly concerned with  $K_1$ , which I will now define. Let A be a ring with 1 and let  $GL_n(A)$  be the group of  $n \times n$  invertible matrices over A. Let  $e_{ij}(\lambda)$  ( $i \neq j$ ) be the matrix with 1's down the diagonal,  $\lambda$  at the ij position, and 0's elsewhere. Let  $E_n(A)$  be the subgroup of  $GL_n(A)$  generated by  $e_{ij}(\lambda)$ ,  $\lambda \in A$ ,  $1 \le i \ne j \le n$ . We can think of  $GL_n(A)$  as a subgroup of  $GL_{n+1}(A)$  by identifying  $M \in GL_n(A)$  with

$$\begin{pmatrix} \mathbf{M} & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(A).$$

Let  $GL(A) = \bigcup_{n=1}^{\infty} GL_n(A)$ . Under the above identification  $E_n(A) \subset E_{n+1}(A)$  so we can also form  $E(A) = \bigcup_{n=1}^{\infty} E_n(A)$ . It turns out that E(A) is a

normal subgroup of GL(A) and that the quotient GL(A)/E(A) is an abelian group, which we define to be  $K_1(A)$ . Now assume that A is commutative, and let U(A) be the units of A. Then the determinant defines a homomorphism det:  $K_1(A) \to U(A)$  which is split by sending  $\lambda \in U(A)$  to the matrix  $\lambda \in GL_1(A)$ . The kernel of det is defined to be  $SK_1(A)$ . Thus we have a direct sum decomposition  $K_1(A) = U(A) \oplus SK_1(A)$ .

If A is a field (or even a local ring) then elementary row and column operations show that  $SK_1(A) = 0$ . One of the most easily computed non-trivial  $SK_1$ 's is  $SK_1 R[X, Y]/(X^2 + Y^2 - 1)$ , or more generally

 $SK_1 k[X, Y]/(X^2 + Y^2 - 1)$ , k a field. Even in this simplest case the computation seems to require a fair bit of algebraic geometry, algebraic number theory, or algebraic topology, as well as tools from algebraic K-theory. I will give references for the latter as I need them. Here I will remark only that Milnor in [4] defined groups  $K_2(A)$ , and more recently Quillen defined groups  $K_i(X)$ ,  $i \ge 0$ , X a scheme (which agree with the previously defined groups if  $i \le 2$  and X is affine). These groups have certain functorial properties, and the different K, are related by exact sequences.

Computations of  $K_1$  have found arithmetical and topological applications (see [3] p 359 and 362 respectively).

2. Quillen's exact sequence. Let  $A = k[T_0, T_1]/(aT_0^2 + bT_1^2 - 1)$  where k is a field of characteristic  $\neq 2$ ,  $a, b \in k$  and  $aT_0^2 + bT_1^2 = 0$  has no solutions in k. Suppose that  $aT_0^2 + bT_1^2 - T_2^2$  has solutions in k. Any conic in  $P_k^2$ (projective space of dimension 2 over k) that contains a k-rational point is isomorphic to  $P_k^1$  ([6] p6). Thus Spec A has been obtained from  $P_k^1$ by removing a non-rational point Y = Spec K of degree 2  $(K = k(\sqrt{\alpha}))$ where  $\alpha = -b/a$ .

We remark that  $Y, P_k^1$ , and Spec A are all regular schemes, hence coherent sheaves and vector bundles yield the same Quillen K-theory, which we denote  $K_i$ . By proposition 3.2 p 127 of [5] (applied to  $P_k^1$  and the closed subscheme Y) we have an exact sequence

where  $i_*$  and  $J^*$  are induced by extension of sheaves by zero and restriction to an open subset respectively (both exact functors). It is known that  $K_0(P_k^1) = Z \oplus Z$ . The most natural choice of basis is the classes of  $\mathcal{O}$  and  $\mathcal{O} - \mathcal{O}(-1)$  respectively. The first copy of Z gives the rank of a coherent sheaf. If P is a k-rational point of  $P_k^1$  there is an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_P \to 0.$$

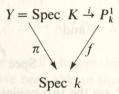
Thus the class of  $\mathcal{O} - \mathcal{O}(-1)$  corresponds to Spec k = structure sheaf of any closed k-rational point. Furthermore  $K_i(P_k^1) \simeq K_i(k) \oplus K_i(k)$ . This isomorphism is  $K_0(P_k^1) \otimes_{K_0(k)} K_i(k) \stackrel{\cong}{\to} K_i(P_k^1)$  given by  $y \otimes x \to yf^*(x)$ where  $f: P_k^1 \to \text{Spec } k$  is the structure morphism (Proposition 4.3 pl29) of [5]). These observations suggest that  $K_i(Y) \xrightarrow{i*} K_i(P_k^1)$  ought to be

the restriction of scalars map  $K_i(K) \to K_i(k)$  followed by inclusion into the second copy of K<sub>1</sub>(k). In the next section I show that this is indeed the case.

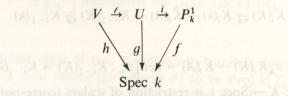
3. Projection onto direct summands. In this section it is more convenient to take the classes of  $\mathcal{O}(-1)$  and  $\mathcal{O}-\mathcal{O}(-1)$  as the basis of  $K_0(P_n^1)$ . Consider the homomorphism  $f_*: K_i(P_k^1) \to K_i(k)$ . If i = 0 this sends F to  $\mathfrak{x}(F) \in Z = K_0(k)$ . (F a coherent sheaf on X). By the Riemann-Roch Theorem  $\mathfrak{x}\mathscr{C}(-1) = 0$  and  $\mathfrak{x}(\mathscr{C}) = 1$  so  $f_*$  on the i = 0 level is projection onto the second copy of Z. By the projection formula (proposition 2.10 p 126 of [5])  $f_*$  is also projection onto the second summand for i > 0.

Let  $U = A_k^1$  be the affine subscheme obtained by removing any k-rational point, and  $j: U \to P_k^1$  be the inclusion. From the definition of  $K_0$ -action (p 124 of [5]) we have  $j^*(\mathcal{C}(-1) f^*(x)) = j^*\mathcal{C}(-1) j^* f^*(x) =$  $= j^*f^*(x)$  and  $j^*((\mathcal{O} - \mathcal{O}(-1))f^*(x)) = j^*(\mathcal{O} - \mathcal{O}(-1))j^*f^*(x) = 0$ . But  $j^*f^*$  is an isomorphism.  $(x \in K_i(k))$ . Thus  $j^*: K_i(k) \to K_i(U) = K_i(k)$  is projection onto the first copy of  $K_i(k)$ .

We now examine how image  $i_*$  behaves under these two projections. There is a commutative diagram of proper morphisms



 $(\pi \text{ corresponding to the inclusion } k \to K) \text{ so } f_*i_* = \pi_* \text{ (as maps on } K_i).$ There is a commutative diagram of schemes of



where V is the affine scheme U - Y,  $\ell$  is the inclusion, and q, h are the structure morphisms. At the coherent sheaf level  $j^*$ ,  $\ell^*$ ,  $f^*$ ,  $g^*$ ,  $h^*$  and  $i_*$ are all exact functors so the corresponding maps on the K, level are induced by these exact functors. As a functor from coherent sheaves on Y to coherent sheaves on  $V\ell^*j^*i_* = 0$ . The induced maps on  $K_i$ , then satisfy  $\ell^*j^*i_* = 0$ . But  $\ell^*g^* = h^*$  is an inclusion and  $g^*$  is an isomorphism. Thus  $\ell^*$  is an inclusion so  $j^*i_* = 0$ .

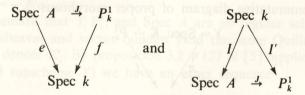
Thus we have determined  $j^*i_*$  and  $f_*i_*$  (as maps on  $K_i$ ), and  $i_*$  does indeed have the desired form.

## 4. Completion of the computation. We now have an exact sequence

$$\dots \to K_i(K) \xrightarrow{i*} K_i(k) \oplus K_i(k) \xrightarrow{J^*} K_i(A) \to K_{i-1}(K) \xrightarrow{i*} K_{i-1}(k) \oplus K_{i-1}(k) \to$$

The inclusion  $k \to A$  induces a homomorphism  $K_i(k) \to K_i(A)$ , which can also be thought of as  $e^*$  where e: Spec  $A \to \operatorname{Spec} k$  is the structure morphism (note that we are in this section reverting the usual basis of  $K_0(P_k^1)$  – i.e.  $\mathscr O$  and  $\mathscr O - \mathscr O(-1)$ .) This is split by the inclusion of any k-rational point. Let  $K_i(A) = K_i(k) \oplus \widetilde K_i(A)$  be the resulting direct sum decomposition (i.e.  $\widetilde K_i(A) = \ker I^* : K_i(A) \to K_i(k)$  where  $I : \operatorname{Spec} k \to \operatorname{Spec} A$  is the inclusion of some k-rational point.) This direct sum decomposition is not claimed to be independent of the choice of k-rational point (except for  $K_0$ , and  $K_1$ ).

I claim that  $J^*$  respects the direct sum decompositions of  $K_i(P_k^1)$  and  $K_i(A)$  and is an isomorphism on the first summand. This follows from the commutative diagrams



since image  $f^*$  and  $\ker(I')^*$  are the two copies of  $K_i(k)$  in the direct sum decomposition of  $K_i(P_k^1)$ .

Thus we have exact sequence

$$K_i(K) \underset{\overrightarrow{\pi *}}{\longrightarrow} K_i(k) \to \widetilde{K}_i(A) \to K_{i-1}(K) \underset{\overrightarrow{\pi *}}{\longrightarrow} K_{i-1}(k)$$

or

$$0 \to K_i(k)/\pi_* K_i(K) \to \widetilde{K}_i(A) \to \ker \left[\pi_* : K_{i-1}(K) \to K_{i-1}(k)\right] \to 0,$$

where  $\pi$ : Spec  $K \to \text{Spec } k$  is restriction of scalars (corresponding to the inclusion  $k \subset K$ ).

Let us now examine the case i = 1. Here  $Z = K_{i-1}(K) \to K_{i-1}(k) = Z$  is just multiplication by 2, so

$$SK_1(A) = \tilde{K}_1(A) = k^*/NK^*$$

where  $N: K^* \to k^*$  is the norm map. If k = R,  $K = \mathbb{C}$ , then  $N\mathbb{C}^* = \text{positive real numbers so } SK_1 R[T_0, T_1]/(T_0^2 + T_1^2 - 1) = \mathbb{Z}/2\mathbb{Z}$ . If  $k = \mathbb{Q}$ 

and  $K = Q(\sqrt{-1})$  then  $a \in Q$  lies in  $NK^*$  if and only if a > 0 and all primes congruent to 3 mod 4 have even power in the factorization of a. Half the primes are congruent to 3 mod 4 so  $Q^*/NK^*$  is the countable direct sum of copies of Z/2Z. Thus  $SK_1 Q[T_0, T_1]/(T_0^2 + T_1^2 - 1)$  is the direct sum of a countable number of copies of Z/2Z. (M.P. Murthy showed me a different proof of this). If  $K = Q(\sqrt{d})$  (d square free) the condition for being a norm is somewhat more complicated (see for example [7] 6-6). For  $a \in Q^*$  to lie in  $NK^*$  it is necessary that all primes at which d is a quadratic non-residue have even power in the factorization of a. Thus  $SK_1 Q[T_0, T_1]/(T_0^2 - dT_1^2 - 1)$  is also the countable direct sum of copies of Z/2Z.

For i = 2 we get an exact sequence

$$0 \to K_2(k)/\pi_* K_2(K) \to \tilde{K}_2(A) \to \ker(K^* \xrightarrow{N} k^*) \to 0$$

In particular if k=R then  $K_2(R)=Z/2Z\oplus D$  where D is divisible. Two times any element of D lies in the image of  $\pi_*$ . Thus  $D\subset \operatorname{image} \pi_*$ . But  $K_2(\mathbb{C})$  is divisible so cannot map onto Z/2Z. Thus  $K_2(R)/\pi_*K_2(\mathbb{C})=Z/2Z$ . Also  $\ker(\mathbb{C}^*\to R^*)$  is the multiplicative group of complex numbers of norm 1, ie the circle group  $S^1$ . Thus we have an exact sequence (for  $A=R[T_0,T_1]/T_0^2+T_1^2-1$ )

$$0 \to Z/2Z \to \tilde{K}_2(A) \to S^1 \to 0.$$

I was not able to decide myself whether this sequence splits. However D. Grayson and C. Weibel have been able to show by using the transfer map  $K_2(A \otimes_R \mathbb{C}) \to K_2(A)$  that the sequence does not split, and that  $\widetilde{K}_2(A) = S^1$ . Grayson also points out that the non-trivial element of  $\mathbb{Z}/2\mathbb{Z}$  maps to the element (-1)  $\begin{bmatrix} T_0 \\ T_1 \end{bmatrix}$  mentioned on page 129 of [4].

5. Other ground rings. If one is sufficiently carefull the ground ring k need not be a field. Define the scheme  $P_k^1$  to be Proj  $k[T_0, T_1]$  and let Y be the subscheme defined by  $T_0^2 + aT_1^2 = f$ . Then as in  $[2]P_k^1 - Y = \operatorname{Spec} A$ , where  $A = \operatorname{degree}$  zero part of  $k[T_0, T_1]_f \cong k[T_0^2, T_0T_1, T_1^2]/(T_0^2 + aT_1^2 - 1) \cong k[U, V, W]/(U + aV - 1, UV - W^2) \cong k[V, W]/(W^2 - V(1 - aV)) = k[V, W]/(W^2 + aV^2 - V)$ . If 2a is a unit in k, we can change variables so that the polynomial is  $W^2 + a(V')^2 - 1/4a$ . Otherwise leave A in the form  $A = k[V, W]/(W^2 + aV^2 - V)$ . The subscheme Y is  $Proj k[T_0, T_1]/(T_0^2 + aT_1^2) = D_+(T_1) = \operatorname{Spec} k[T_0/T_1]/((T_0/T_1)^2 + a) = k(\sqrt{-a})$ . Let  $K = k(\sqrt{-a})$ . Assume  $2 \neq 0$  in k, and that k and K

are both regular. Then the above proof goes through unchanged and we get exact sequences

$$0 \to K_i(k)/i_*K_i(K) \to \tilde{K}_i(A) \to \ker i_* : (K_{i-1}(K) \to K_{i-1}(k)) \to 0$$

where  $i_*: K_i(K) \to K_i(k)$  is induced by restriction of scalars, and  $\widetilde{K}_i(A) =$  coker  $(K_i(k) \to K_i(A))$  (The inclusion  $k \to A$  is split by  $V \to 0$ ,  $W \to 0$  so  $\widetilde{K}_i(A)$  is a direct summand of A).

To give a specific example let k=Z, a=1, K=Z[i]. Here  $K_0(K) \to K_0(k)$  is multiplication by 2 which is an inclusion. Also  $SK_1(K)=SK_1(Z)=0$  so  $K_1(k)=\{\pm 1\}$ . The units of  $K_1(K)$  all have norm 1 so  $\widetilde{K}_1(A)=SK_1(A)=Z/2Z$ : On the other hand if a=-2 then  $K=Z[\sqrt{2}]$  and  $(1-\sqrt{2})(1+\sqrt{2})=1-2=-1$ . Some units have norm -1 so  $\widetilde{K}_1(A)=SK_1(A)=0$ .

## **Bibliography**

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