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On Jacobian extensions of ideals

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0. Introduction

Given an affine algebraic variety V of dimension n the Thom-Boardman numbers will define a stratification. This concept is studied in [7] where it is shown that these strata are nonsingular locally closed sets, each one indicated with a symbol

$$\Sigma(t(1), t(2), \ldots, t(k), \ldots), n \geq t(2) \geq t(2) \geq \ldots \geq t(k) > \ldots \geq 0.$$

A first approach to them are the locally closed sets $\Sigma(\ell)\, 0 \leq \ell \leq n$ which form a partition of the original algebraic set.

Let A be the coordinate ring of V. A maximal prime P of A belongs to $\Sigma(\ell)$ if the tangent space in the sense of Zariski at P (i.e. m/m^2 where m is the maximal ideal of A_p) has rank $n - \ell$ over $A_P/m = A_P/PA_P$.

An affine open set U can be found such that P belongs to it and the set of points Q in U such that the tangent space has the same rank as in P is a closed set given by an ideal which is obtained by the Fitting theory. On that closed set we have a new ring of functions and we repeat the process; $P \in \Sigma(\ell, s)$ if $P \in \Sigma(\ell)$ and the closed set $\Sigma(\ell) \cap U$ has a tangent space of rank $n - s$ at P and so on.

Given a rational regular maximal ideal $P \in \Sigma(t(1), \ldots, t(s))$ Mather [3] constructs an ideal $\beta(I) \subset k[[x_1, \ldots, x_n]]/(x_1, \ldots, x_n)^{s+1}$ such that for some regular system of parameters $\{y_1, y_2, \ldots, y_n\}$ and s natural numbers $n \geq t(1) > \ldots \geq t(s) \geq 0$ we have:

$$I \subset \beta(I) = (y_1, \ldots, y_{n-t(1)}) + (y_1, \ldots, y_{n-t(2)})^2 + \ldots + (y_1, \ldots, y_{n-t(s)})^s$$

and $(t(1), \ldots, t(s))$ is maximal in the lexicographic order in the set of s-tuples $(j(1), \ldots, j(s))$ such that for some regular system of parameters $\{z_1, \ldots, z_n\}$ we have

$$I \subset (z_1, \ldots, z_{n-j(1)}) + (z_1, \ldots, z_{n-j(2)})^2 + \ldots + (z_1, \ldots, z_{n-j(s)})^s.$$

Moreover he shows that the number $\mu(I)$ defined by Boardman in ([10]) is

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$$\mu(I) = \dim(m/\beta(I) + m^{s+1})$$

$$I = (t(1), \dots, t(s)) \quad m = (y_1, \dots, y_n)$$

What we will show is that given P in the above conditions and

$$P \in \Sigma(t(1), \dots, t(s), t(s+1)) \subset \Sigma(t(1), \dots, t(s))$$

then there is an open set U such that

$$P \in U \subset \Sigma(\pm(1), \dots, \pm(s)) \text{ and } U \cap \Sigma(\pm(1), \dots, \pm(s+1))$$

is a Zariski-closed set given by the set of points of $\Sigma(t(1), \dots, t(s))$ where $\mu(t(1), \dots, t(s), t(s+1))$ is constant. That closed set will be defined by an ideal that we will obtain by Fitting theory (2.1) and the modules of higher order differentials [8] (Def. 1.3). In Theorem 2 that ideal will be shown to be the same one that defines $\Sigma(t(1), \dots, t(s), t(s+1))$ in Mather's construction in [3].

Moreover the possibility of computing these ideals from the higher orders differentials is the first step for an extension of Mather's construction to positive characteristic [11].

1. Modules of differentials and higher order derivations [8].

In what follows K will be a field and when we say ring or K-algebra we will understand them as unitary and commutative.

1.1. Given a K-algebra A, let us take the exact sequence

$$0 \to I(A/K) \to A \underset{K}{\otimes} A \xrightarrow{\varphi} A \to 0,$$

$\varphi(a \otimes b) = a.b.$

Give $A \underset{K}{\otimes} A$ the natural structure of a left A-module. The ideal $I(A/K)$ becomes an A-submodule generated by the elements of the form $1 \otimes a - a \otimes 1$, $a \in A$. In fact given $x \in I(A/K)$, $x = \Sigma_i a_i \otimes b_i \varphi(x) = 0 = \Sigma_i a_i b_i$ so $x = \Sigma_i a_i \otimes b_i - \Sigma_i a_i b_i \otimes 1 = \Sigma_i a_i (1 \otimes b_i - b_i \otimes 1)$.

If we define T_K: $A \to I(A/K)$ by $T_K(a) = 1 \otimes a - a \otimes 1$, then T_K has the following properties:

i) $T_K(1) = 0$
ii) T_K is K-linear
iii) $T_K(a \cdot b) = a \cdot T_K(b) + bT_K(a) + T_K(a)\, T_K(b)$.

Given an A-algebra B, a map L: $A \to B$ which satisfies i), ii) and iii) will be called a Taylor K-map.

Property 1.1.

If L: $A \to B$ is a Taylor K-map there is one and only one A-algebra morphism F: $I(A/K) \to B$ such that $F \circ T_K = L$; T_K will be called the universal Taylor K-map.

Lemma 1.2. *Given an A-module M, A a K-algebra and φ: $A \to M$ a K-linear morphism such that $\varphi(1) = 0$, then there is one and only one morphism of A-modules θ: $I(A/K \to M$ such that $\theta.\ T_K = \varphi$.*

Proof. We can show that

$$A \underset{K}{\otimes} A = A(1 \otimes 1) \underset{A}{\oplus} I(A/K)$$

as left A-modules. In fact the map T_K: $A \to I(A/K)$ can be extended to an A-linear map

$$1_A \otimes T_K{:}\ A \underset{K}{\otimes} A \to I(A/K)$$

$$1_A \otimes T_K(a \otimes b) = a\, T_K(b)$$

which is a projection on $I(A/K)$. Then

$$(1 \otimes T_K)(1 \otimes b - b \otimes 1) = 1 \cdot T_K(b) - bT_K(1) = T_K(b) = 1 \otimes b - b \otimes 1.$$

and for every $y \in A \underset{K}{\otimes} A$

$$y = \sum_{i=1}^{n} a_i \otimes b_i = \Sigma a_i(1 \otimes b_i - b_i \otimes 1) + \Sigma a_i b_i \otimes 1$$

$$= \Sigma a_i\, T_K(b_i) + (\Sigma a_i b_i)(1 \otimes 1).$$

Since φ: $A \to M$ is K-linear then we can define $1_A \otimes \varphi$: $A \underset{K}{\otimes} A \to M$ $(1_A \otimes \varphi)(a \varphi b) = a \cdot \varphi(b)$ so that $(1_A \otimes \varphi)(1 \otimes 1) = \varphi(1) = 0$, thus $1 \otimes \varphi$ is an A-linear map that factors through $I(A/K)$.

Given a ring R and a set $\{a_1, \dots, a_n\}$ of elements of R we shall denote by $a_1 \dots \hat{a}_{i(1)} \dots \hat{a}_{i(r)} \dots a_n$ the product $\prod_{k \neq i(1), \dots, i(r)} a_k$.

Definition 1.3. Let L_n: $R \to M$ be a K-linear map from a K-algebra R to an R-module M which satisfies:

i) for any set $\{a_0, \dots, a_n\} \subset R$

$$L_n(a_0 \dots, a_n) = \sum_{i=1}^{n} (-1)^{i+1} \sum_{j_1 < \dots < j_i} a_{j_1} \dots a_{j_i} L_n(a_0 \dots \hat{a}_{j_1} \dots \hat{a}_{j_i} \dots a_n)$$

ii) $L_n(1) = 0$

L_n will be called a derivation of order n.

The ideal $I(R/K)$ of $R \underset{K}{\otimes} R$, defined in 1.1, has a natural projection p

$$R \xrightarrow{T_K} I(R/K) \xrightarrow{p} I(R/K)/I(R/K)^{n+1}.$$

We will write $\Omega^n(R/K) = I(R/K)/I(R/K)^{n+1}$ and $T^n = p \circ T_K$.

Theorem 1.4 [9] *The map $T^n: R \to \Omega^n(R/K)$ of definition 1.3 is a K-linear derivation of order n and given $L: R \to M$, a K-linear derivation of order n from a K-algebra R to an R-module M, there is a unique morphism of R-modules $h: \Omega^n(R/K) \to M$ such that $h\ T^n = L$.*

Conversely if $h: \Omega^n(R/K) \to M$ is an R-linear morphism then $h \circ T^n$: $R \to M$ is a K-linear derivation of order n.

1.5. Suppose that R is a local ring with maximal ideal M and a subfield K. Then the R-module

$$\Omega^n(R/K/\bigcap_{s\in N} M^S\Omega^n(R/K = \hat{\Omega}^n(R/K) \quad ([9])$$

is separated in the M-adic topology.

Let $\theta: \Omega^n(R/K) \to \hat{\Omega}^n(R/K)$ be the natural projection, then $\theta \cdot T^n_K = \hat{T}^n_K$ is obviously a K-linear derivation of order n and the pair $(\hat{T}^n_K, \hat{\Omega}^n(R/K))$ is universal with the properties of Theorem 1.4 for the sub-category of separated R-modules in the M-adic topology [9].

In what follows we will make use of derivations from a noetherian regular local ring R to itself, and since R is separated as R-module with the topology of its maximal ideal we will be interested in the pairs $(\hat{T}^k, \hat{\Omega}^k(R/K))\ k \geq 0$ if R is a K-algebra.

Given a ring R with the conditions stated above which has a regular family of parameters $\{x_1, \ldots, x_n\}$ of its maximal ideal M, and such that $R/M \simeq K$ (or a separable extension of K), it was shown that $\hat{\Omega}^k(R/K)$ is a free R-module that has as a base the set of monomials in $\{Tx_1, \ldots, Tx_n\}$ of degree at most k [8][9].

We put our attention on $\mathrm{Hom}_R(\hat{\Omega}(R/K), R)$ which corresponds by Theorem 1.4 to the derivations from R to R.

Let α be an n-uple of non negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$; we call $|\alpha| = \Sigma\, \alpha_i \geq 0$ and $M_\alpha(Tx) = Tx_1^{\alpha_1} \ldots Tx_n^{\alpha_n}$. If we take the family $\{M_\alpha(Tx); |\alpha| \leq k\}$ as a base of $\hat{\Omega}(R/K)$ and call $\{\Delta_\alpha; |\alpha| \leq k\}$ the corres-

ponding dual base, it is shown in [1] that, if the characteristic of K is zero and $R = K[[x_1, \ldots, x_n]]$, then

$$\Delta_\alpha = \frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_n!} \frac{\partial^{\alpha_1}}{\partial\alpha_{1x_1}} \cdots \frac{\partial^{\alpha_n}}{\partial\alpha_{n x_n}}$$

where $\frac{\partial}{\partial x_j}$ are usual partial derivatives. This result can be extended to any regular local ring R with the conditions given above ([8], [9]).

2. Jacobian extensions of ideals.

2.1. Let M be a finitely generated module over a ring A.

There is an exact sequence

$$0 \to R \to A^n \xrightarrow{\varphi} M \to 0$$

for some $n \geq 0$, where R is the set of n-tuples which belong to ker φ.

We consider the matrix whose rows are these n-tuples. For every natural s, if $0 \leq s \leq n$ we define $f_s(M)$, to be the ideal in A generated by the $(n - s + 1) \times (n - s + 1)$ minors of that matrix, if $s > n$, $f_s(M) = A$. We will call them the Fitting ideals of M.

Fitting in [2] shows that these ideals depend only on M, i.e. they are independent of the exact sequence given above. Given a prime ideal $P \subset A$ the rank over A_P of the module $A_P \underset{A}{\otimes} M$ is s if and only if $f_s(M) \subset P$ and $f_{s+1}(M) \not\subset P$. It is easy to see that $f_s(M) \subset f_t(M)$ if $s < t$.

Note that in the sheaf induced by M over the spectrum of A, Spec (A) with the Zariski topology, the intersection of the open set induced by the ideal $f_{s+1}(M)$ with the closed set induced by $f_s(M)$ is the set of points where $\mathrm{rank}_{A_P}(M_p) = s$ ([4]).

If A is a local ring we will denote by $F(M)$ the biggest proper Fitting ideal.

Given an ideal I in a K-algebra A let $\Delta(I) \subset \hat{\Omega}^1(A/K)$ be the A-submodule generated by the elements $\{\hat{T}^1_K(f)/f \in I\}$ (see 1.5).

The Fitting ideals corresponding to the A-module

$$A/I \underset{A}{\otimes} \hat{\Omega}^1(A/K)/\Delta(I)$$

are called the Jacobian extensions of the ideal I. In what follows we will take $A = K[[x_1, \ldots, x_n]]$ as before with K a field.

Proposition 2.1. *Given an ideal $I \subset A = K[[x_1, \ldots, x_n]]$, and $(\hat{\Omega}^s(A/K)\hat{T}^s)$ as in 1.5, let $\Delta_s(I) \subset \hat{\Omega}^s(A/K)$ be the A-submodule of $\hat{\Omega}^s(A/K)$ generated* by the set $\{Tf/f \in I\}$. Then the natural image of $\Delta s(I)$ in $A/I \otimes_A \Omega^s(A/K)$ is an ideal.

Proof. It is enough to show that given $g \in A, f \in I$ then $\hat{T}f\,\hat{T}g \in \Delta_s(I)$ in fact:

$$\hat{T}g\,\hat{T}f = -\,g\hat{T}f - f\hat{T}g + \hat{T}(f \circ g) \in \Delta_s(I)$$

since $f \circ \hat{T}g = 0$ in $A/I \underset{A}{\otimes} \hat{\Omega}^s(A/K)$ ([8]).

2.2. Using the notation of [3] let $\delta(I)$ be the biggest Jacobian extension of an ideal $I \subset K[[x_1, \dots, x_n]]$. Mather shows that if $\{z_1, \dots, z_t\} \subset I$ is such that $\{z_1, \dots, z_t\}$ is a base of

$$\frac{I + m^2}{m^2};\ m = (x_1, \dots, x_n) \subset K[[x_1, \dots, x_n]]$$

then extending the set $\{z_1, \dots, z_t\}$ to a regular system of parameters $\{z_1, \dots, z_t, \dots, z_n\}$ we have $\delta(I) = I + \left[\frac{\partial f}{\partial z_j} / f \in I;\ j > t\right]$ (the ideal generated by this set).

Definition 2.2. Given an ideal $I \subset A = K[[x_1, \dots, x_n]]$ let:

$$\delta(I) = F(A/I \underset{A}{\otimes} \hat{\Omega}^1(A/K)/\Delta_1(I)) \quad \text{(see 2.1)}$$

Having defined $\delta_{k-1}(I)$, then taking:

$$\beta_k(I) = \Delta_k(I) + (\Delta_k(\delta I))^2 + \dots + (\Delta_k(\delta_{k-1}(I)))^k$$

where

$$\beta_k(I) \subset A/\delta_{k-1}(I) \underset{A}{\otimes} \hat{\Omega}^k(A/K)$$

we define

$$\delta_k(I) = F(A/\delta_{k-1}(I) \underset{A}{\otimes} \hat{\Omega}^k(A/K)/\beta_k(I)).$$

As before $A = K[[x_1, \dots, x_n]]$ is the ring of formal power series on n variables over K with maximal ideal m.

Theorem 2. *If the characteristic of K is zero then $\delta_k(I) = \delta^k(I)$ (where $\delta^k(I) = \delta(\delta^{k-1}(I))$ in the sense of Mather [3]).*

Proof. By induction on k. The case $k = 1$ follows by Definition ([6]).

We now show that $k - 1$ implies k. Given the increasing chain of ideals

$$I \subset \delta(I) \subset \delta^2(I) \subset \dots \delta^k(I),$$

there is a set of parameters $\{x_1, \dots, x_n\}$ of A and numbers $i(1) \le \dots \le i(n)$ such that for every $1 \le j \le n$:

i) $\{x_1, \dots, x_{i(j)}\} \subset \delta^{j-1}(I)$

ii) $\{\bar{x}_1, \dots, x_{i(j)}\}$ is a base of the A-module $\dfrac{\delta^{j-1}(I) + m^2}{m^2}$.

By [3] we have that

$$\delta^k(I) = \delta(\delta^{k-1}(I)) = \delta^{k-1}(I) + \left[\frac{\partial g}{\partial x_t} / g \in \delta^{k-1}(I);\ t > i(k)\right]$$

An easy induction leads us to the formula

$$(*)\quad \delta^k(I) = I + \left[\frac{\partial}{\partial x_{s(1)}} \cdots \frac{\partial}{\partial x_{s(\ell)}} f / f \in I;\ s(j) > i(j)\right.$$
$$\left. j = 1, \dots, \ell \text{ and } 1 \le \ell \le k\right].$$

Let F_K be these base of m/m^{k+1} formed by the monomials in $\{x_1, \dots, x_n\}$ of degrees $\le k$. Then $F_k = F' \cup F''$ (disjoint union) where F' is the set of monomials that belong to the ideal $(x_1, \dots, x_{i(1)}) + (x_1, \dots, x_{i(2)})^2 + \dots + (x_1, \dots, x_{i(k)})^k$ and F'' is the complement in F, i.e. the monomials of the form $x_{s(1)} \cdots x_{s(\ell)}$ such that $\ell \le k$ and $s(j) > i(j)$ for every $1 \le j \le \ell$.

In the same way, if we take the base G of $\hat{\Omega}^k(A/K)$ formed by the monomials of degree $\le k$ in $\{\hat{T}x_1, \dots, \hat{T}x_n\}$ ($T: A \to \hat{\Omega}^k(A/K)$ the canonical derivation [1.5]) we will have $G = G' \cup G''$ where G' is formed by the monomials that belong to the ideal

$$\langle \hat{T}x_1, \dots, \hat{T}x_{i(1)} \rangle + \dots + \langle \hat{T}x_1, \dots, \hat{T}x_{i(k)} \rangle^k \subset \hat{\Omega}^k(A/K)$$

and G'' the complement of G' in G.

Since $\hat{\Omega}^k(A/K)$ is a finitely generated free A-module we consider the exact sequence

$$(**)\qquad 0 \to R \to \hat{\Omega}^k(A/K) \to \hat{\Omega}^k(A/K)/\beta_k(I) \to 0$$

By induction we have that

$$\delta_j(I) = \delta^j(I) \qquad 0 \le j \le k - 1$$

so

$$\beta_k(I) = \Delta_k(I) + \Delta_k(\delta I)^2 + \Delta_k(\delta^2 I)^3 + \dots + \Delta_k(\delta^{k-1}(I))^k$$

and

$$\langle \hat{T}x_1, \dots, \hat{T}x_{i(1)} \rangle + \dots + \langle \hat{T}x_1, \dots, \hat{T}x_{i(k)} \rangle^k \subset \beta_k(I).$$

We may change the exact sequence (**) to

$$0 \to R' \to L' \to \hat{\Omega}^k(A/K)/\beta_k(I) \to 0$$

where L' is the direct summand generated by the monomials $M_\alpha(\hat{T}x_i) \in G''$, and R' is the image of R in L' by the natural projection whose kernel is the direct summand generated by the monomials $M_\alpha(\hat{T}x_i) \in G'$.

Let us call $//a_{\lambda\alpha}//$ the matrix whose rows are the vectors of R' written in the base G''. If we show that every coefficient $a_{\lambda\alpha} \in m$, i.e. $R' \subset mL'$, since A is a local ring then the ideal $\langle a_{\lambda\alpha} \rangle_{\lambda,\alpha}$ is proper and it is the biggest corresponding Fitting ideal, i.e.,

$$F(\hat{\Omega}^k(A/K)/\beta_k(I)) = \langle a_{\lambda\alpha} \rangle_{\lambda,\alpha}.$$

On the other hand, since $\Delta_k(I) \subset \beta_k(I)$, we have coefficients of the form

$$\frac{\partial}{\partial x_{s(1)}} \cdots \frac{\partial}{\partial x_{s(\ell)}} f;\ f \in I \quad s(j) > i(j) \quad j = 1, \ldots, \ell \text{ and } \ell \leq k$$

i.e. $\delta^k(I) \subset \delta_k(I)$ (*). In fact these coefficients are the coordinates of Tf, $f \in I$ in the monomial $Tx_{s(1)} \cdots Tx_{s(\ell)} \in G''$.

We will show now:

$$F(\hat{\Omega}^k(A/K)/\beta_k(I)) \subset \delta^k(I)$$

Let $f_0, \ldots, f_v \in \delta_v(I) = \delta^v(I)$ $1 \leq v \leq k-1$. Then

$$Tf_0 \cdot Tf_1 \cdots\cdots Tf_v \in \Delta_k(\delta_v(I))^{v+1}$$

and we will consider the coordinate of this element in a monomial $\hat{T}x_{s(1)} \cdots \hat{T}x_{s(r)} \in G''$.

So let us take elements of the form

$$\frac{\partial}{\partial x_{s(p,1)}} \cdots \frac{\partial}{\partial x_{s(p,k)}} f_p \quad \text{for } p = 0, \ldots, v \text{ with}$$

$$s(p,i) \leq s(p,j) \text{ where } i < j$$

such that if $Tx^{\alpha(p)} = \prod_{i=1}^{p} Tx_{s(p,i)}$ then $\prod_{p=0}^{v} Tx^{\alpha(p)} = Tx^\alpha \in G''$.

If $Tx^\alpha = Tx_{s(1)} \cdots Tx_{s(r)}$, then $s(j) > i(j)$ $j = 1, \ldots, r$ since $Tx^\alpha \in G''$.

There must be a $p, 0 \leq p \leq v$ such that $s(p,1) \geq s(v+1)$, for in fact there are $v+1$ elements of the form $s(\ ,1)$. Also $s(p,j) \geq s(v+j)$.

Then $s(p,1) > i(v+j)$ and $s(p,j) > i(v+j)$ $j = 1, \ldots, k$ i.e.

$$\frac{\partial}{\partial x_{s(p,1)}} \cdots \frac{\partial}{\partial x_{s(p,k)}} f_p \in \delta^{k-v}(\delta(I))$$

(by 2.2) so

$$\frac{\partial}{\partial x_{s(p,1)}} \cdots \frac{\partial}{\partial x_{s(p,k)}} f_p \in \delta^k(I)$$

as it was to be shown.

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