Non embedding in sphere bundles

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Abstract. In order to get results of non-embedding of flag-manifolds in the total space of sphere bundles we consider algebraic properties of these two families of manifolds.

The main theorem (2.3) gives a general a non-embedding result in terms of algebraic invariants and this theorem is applied to the case of the flag-manifolds.

0. Introduction.

A flag-manifold is defined as the quotient of the group $U(n)$ by the subgroup $U(n_1)\times \ldots \times U(n_k)$ where $n_1 + n_2 \ldots n_k = n$. This family includes the complex projective spaces and the complex Grassmann manifolds.

In this paper we use those manifolds in order to get results of non-embedding in the total space of sphere bundles. The following theorem, concerning the Chern caracter is proved:

1.8. Theorem. *If $\xi = (E, \pi, S^n)$ is a $(p+1)$-vector bundle over S^n with p and n odd numbers $(n > 1)$ and $\mu \in H_{p+n}(S(\xi); Z)$ is the fundamental class of $S(\xi)$ then, for all $\alpha \in K(S(\xi))$ we have that $\langle ch(\alpha), \mu \rangle$ is an integer.*

Applying this theorem and using group representation theory [4] we prove our main theorem:

2.3. Theorem. *Let ξ be a $(p+1)$-vector bundle over $S^n(n > 1)$ with p and n odd numbers. Let M^{2m} be a closed, connected, oriented manifold of even dimension $2m$. If M embeds in $S(\xi)$ with orientable normal bundle ν^{2k} then, for all $\theta \in K(M)$ we have that $\langle 2^{k-1}\, ch(\theta)B(\nu), [M] \rangle$ is an integer number.*

The application of this theorem to the case of the flag-manifolds gives, after long computations, the following results:

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3.8. Let ξ be a $(p+1)$-vector bundle over S^n, with p and n odd numbers and such that $\tau S^n \oplus \xi$ is trivial. If $L = p + n = \dim\, S(\xi)$ then:

(1) $F(2,2)$ does not embed in $S(\xi)$ if $L = 12$
(2) $F(2,3)$ does not embed in $S(\xi)$ if $L = 20$
(3) $F(2,4)$ does not embed in $S(\xi)$ if $L = 28$
(4) $F(2,1,1)$ does not embed in $S(\xi)$ if $L = 12$
(5) $F(2,1,1)$ does not embed in $S(\xi)$ if $L = 20$.

In particular, if ξ is the $(p+1)$-trivial vector bundle over S^n, then $S(\xi) = S^n \times S^p$ and we have results of non-embedding in products of spheres.

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1. Sphere bundles.

1.1. Definition. A differentiable manifold M is a *π-manifold* if $\tau M \oplus 1$ is a trivial bundle.

1.2. Remark. Every π-manifold is orientable.

1.3. Theorem. *If M^m is a π-manifold with even dimension m and Euler class $X(M) = 0$ then M is parallelizable.*

Proof. Let $f: M \to BSO(m)$ classify τM and let $p: BSO(m) \to BSO(m+1)$ be the canonical fibration with S^m as fiber. The map $p \circ f: M \to BSO(m+1)$ classifies $\tau M \oplus 1$ and then is homotopic to a constant map.

By the "homotopy lifting property" we can assume that f takes values in S^m; the restriction of the universal bundle γ^m over $BSO(m)$ to S^m gives τS^m. Then we have a fiber map : $T(f): T(M) \to T(S^m)$.

Therefore, $f^*X(S^m) = X(M) = 0$; but m even implies $X(S^m) = 2\alpha$, where α is the generator of $H^m(M)$. Since M is orientable, $H^m(M)$ is torsion free and then $f^*(\alpha) = 0$, that is $f^* = 0$. Now, since f takes values in S^m we have that f is homotopic to a constant map. Hence, τM is a trivial bundle.

1.4. Remark. Given a vector bundle ξ, we will denote by $S(\xi)$ the associated sphere bundle. We assume metric on bundles whenever we need without explicit mention since our base spaces are always paracompact.

1.5. Corollary. *Let B be a differentiable manifold, orientable and with Euler class $X(B) = 0$. Let ξ be a vector bundle over B such that $\tau B \oplus \xi$ is trivial. If the dimension of the manifold $S(\xi)$ is even then it is parallelizable.*

Proof. Let E be the total space of ξ and π the projection. Then $\tau E = \pi^!(\tau B \oplus \xi)$.

Since $S(\xi)$ embeds in E with trivial normal bundle 1, we have:

$$\tau S(\xi) \oplus 1 = \tau E/S(\xi) = (\pi/S(\xi))^!(\tau B \oplus \xi)$$

This last bundle is trivial and then $S(\xi)$ is π-manifold; the proposition 7.6. in [3] gives:

$$\tau S(\xi) = (\pi/S(\xi))^!\,(\tau B) \oplus \tau_F$$

where τ_F denotes the "tangent bundle along the fiber of $S(\xi)$". This implies that:

$$X(S(\xi)) = [(\pi/S(\xi))^* X(B)]\ X(\tau_F) = 0,$$

and then we are done.

1.6. Examples.

A. Let $B = S^n$ and ξ the $(p+1)$-trivial vector bundle over B, with p and n odd numbers. Then we have the following:

(a) $X(B) = 0$
(b) $S(\xi) = S^n \times S^p$ and $\dim S(\xi) = n + p$, which is even
(c) $\tau S^n \oplus (p+1) = n + p + 1$

Consequently, $M = S^n \times S^p$ is parallelizable.

B. Let $B = S^n$ with n odd and $n \neq 1$ (mod. 8).

Then $X(S^n) = 0$ and $\tau S^n \oplus \xi$ is trivial for all vector bundle ξ over B. If the dimension of $S(\xi)$ is even then it is parallelizable.

C. Let $\lambda(n)$ denote the number of linearly independent vector fields over S^n. If n is odd, $\lambda(n) \geq 1$ and let us consider the manifold $M_\xi = S(\xi)$, where the bundle ξ is such that

$$\tau S^n = \xi \oplus (\lambda(n) - 1).$$

We observe that the dimension of M_ξ is $2n - \lambda(n)$, the Euler class $X(S^n) = 0$ and $\tau S^n \oplus \xi$ is trivial. If $\lambda(n)$ is even, we have as a consequence of 1.5. that the manifold M_ξ is parallelizable.

1.7. Theorem. *Let ξ be a $(p+1)$-vector bundle over S^n with $n > 1$. If $n \neq p+1$, then the cohomology of $S(\xi)$ is given by:*

$$H^*(S(\xi);\ Z) = Z \oplus Z \oplus Z \oplus Z.$$

Proof. For $n > 1$ ξ is orientable and for $n \neq p+1$, $X(\xi) = 0$. The conclusion then follows from the Gysin sequence associated to the bundle ξ.

1.8. Theorem. *If $\xi = (E, \pi, S^n)$ is a $(p+1)$-vector bundle over S^n with p and n odd numbers $(n > 1)$ and $\mu \in H_{p+n}(S(\xi); Z)$ is the fundamental class of $S(\xi)$ then, for all $\alpha \in K(S(\xi))$ we have that $\langle ch(\xi), \mu \rangle$ is an integer.*
Proof. The manifold $S(\xi)$ has the property that $H^i(S(\xi); Q) = 0$ for $i \neq 0, p, n, n+p$. Since p and n are odd it follows that $ch_i(\eta) = 0$ for $0 < i < p+n$. To conclude the proof we will use the following lemma, which is in [1]:

1.9. Lemma. *If $\eta \in K(X)$ is such that $ch_p(\eta) = 0$ for $0 < p < r$ and $H^*(X; Z)$ is torsion free then $ch_r(\eta) \in H^r(X; Z) \subset H^r(X; Q)$.*

1.10. Remark. Theorem 1.8. is a generalization of a Bott's theorem for even dimensional spheres [2]. This theorem is essential in the proof of the non-embedding theorem 2.3.

1.11. Example. If $\lambda(n) \geq 1$ then theorem 1.8. apllies to the manifolds M_ξ of 1.6. A.

2. Non-embedding in sphere bundles.

2.1. Definition. Let us consider the power series $f(t) = \text{senh}\ \sqrt{t}/\sqrt{t} = 1 + \frac{1}{3!}t + \frac{1}{5!}t^2 + \ldots$ This series defines a multiplicative sequence $B = 1 + B_1 + B_2 + B_3 + \ldots$ (see [6]). Given a real oriented vector bundle η, we define the B-class of η, by:

$$B(\eta) = 1 + B_1(p_1) + B_2(p_1, p_2) + \ldots$$

where $p_1, p_2, p_3, \ldots$ are the Pontriagin classes of the bundle η. Then we have:

$$B_1(p_1) = \frac{1}{6}p_1$$

$$B_2(p_1, p_2) = \frac{1}{36}p_2 + \frac{1}{120}(p_1^2 - 2p_1p_2)$$

$$B_3(p_1, p_2, p_3) = \frac{1}{216}p_3 + \frac{1}{720}(p_1p_2 - 3p_3) + \frac{1}{7!}(p_1^3 - 3p_1p_2 + 3p_3)$$

....

To compute $B(\eta)$ it is enough to know the Pontriagin classes of η.

2.2. Definition. Let X and Y be compact, connected, oriented manifolds with fundamental classes $[X]$ and $[Y]$ respectively. Given a map $f: X \to Y$, the *Gysin homomorphism*: $f_\oplus : H^*(X) \to H^*(Y)$ associated to f is defined by $f_\oplus(a) \cap [Y] = f_*(a \cap [X])$.

We have the following commutative diagram:

$$\begin{array}{ccc} H_*(X) & \xrightarrow{f_*} & H_*(Y) \\ \simeq \uparrow \cap [X] & & \simeq \uparrow \cap [Y] \\ H^*(X) & \xrightarrow{f_\oplus} & H^*(Y) \end{array}$$

The Gysin homomorphism has the following properties:

A. $f_\oplus(f^*(z) \cup w) = z \cup f_\oplus(w)$

B. $f_\oplus(z)[Y] = z([X])$

2.3. Theorem. *Let ξ^{p+1} be a $(p+1)$-vector bundle over $S^n (n > 1)$ with p and n odd numbers. Let M^{2m} be a closed, connected oriented manifold of even dimension $2m$. If M embeds in $S(\xi)$ with orientable normal bundle v^{2k} then, for all $\theta \in K(M)$ we have that $\langle 2^{k-1} ch(\theta) B(v), [M] \rangle$ is an integer number.*

To prove this theorem we will make use of the following lemmas:

2.4. Lemma. *Let I be the Kernel of the restriction map*

$$i^! : R(SO(2k)) \to R(SO(2k-1))$$

and let $f: M \to S(\xi)$ be an embedding with orientable normal bundle v^{2k} Then, for all representations $v \in I$, there is a map: $f_v : K(M) \to K(S(\xi))$ such that the following conditions hold:

A. $f\,!\, f_v(\theta) = \theta \cdot \alpha(v)$

B. $ch(f_v(\theta)) = f_\oplus(ch(\theta)\ ch(v)/e)$

where "e" denotes the Euler class of the normal bundle.

Proof. Let us define f_v and show property A. Details, as well as property B are given in [4]. As a simplication we shall assume that f is the inclusion map; the disc bundle associated to the normal bundle v can be realized as a tubular neighborhood D of M in $S(\xi)$. The sphere bundle $A \to M$ associated to the normal bundle v is isomorphic to the bundle $E/SO(2k-1) \xrightarrow{q} M$, where $E \xrightarrow{p} M$ is the SO(2k)-principal bundle over M associated to the normal bundle. For each $v \in T$ it is possible to construct an element $\alpha(v) \in K(M)$ and this can be extended to $\beta \in K(D, A)$ by the difference construction [4]. Define $f_v : K(M) \to K(S(\xi))$ by the following commutative diagram:

$$\begin{array}{ccc} K(D, A) & \xrightarrow{h} & K(S(\xi), S(\xi) - D^0) \\ \uparrow \varnothing & & \downarrow j! \\ K(M) & \xrightarrow{f_v} & K(S(\xi)) \end{array}$$

where $\varnothing$ is defined by $\varnothing(\theta) = \theta \cdot \beta$, the map h is the isomorphism given by excision and $j!$ is induced by inclusion.

To prove property A. we have only to observe that $f_v(\theta)$ is defined by extending $\theta \cdot \alpha(v)$ from M to D and after to $S(\xi)$-this is possible because $\theta \cdot \beta$ is zero over A. Hence, when we restrict $f_v(\theta)$ to M by f we get $\theta \cdot \alpha(v)$, that is:

$$f\,!\,f_v(\theta) = \theta \cdot \alpha(v)$$

2.5. Lemma. *Let M be as in theorem 2.3. embedded in $S(\xi)$. Then for all $\theta \in K(M)$ and all $v \in I$ we have that $\langle ch(\theta)\, ch(v)/e, [M] \rangle$ is an integer.*

Proof. By theorem 1.8. $ch(\theta)[S(\xi)] \in Z$, for all $\theta \in K(S(\xi))$ and by property B in 2.4. we have: $ch(f_v(\theta)) = f_{\oplus}(ch(\theta)\, ch(v)/e)$. By property B of the Gysin homomorphism we get: $ch(f_v(\theta))\ [S(\xi)] = (ch(\theta)\ ch(v)/e)\ [M]$. Since $f_v(\theta) \in K(S(\xi))$ the left side is an integer and the lemma follows.

2.6. Lemma. *If M embeds in $S(\xi)$ with orientable normal bundle v then there is a representation $\mu \in I$ such that $ch(\mu)/e = 2^{k-1} B(v)$.*

The proof given in [4] applies because the Euler class of the normal bundle of an embedding in $S(\xi)$ is zero.

Using now these lemmas we have that

$$\langle 2^{k-1} ch(\theta)\ B(v),\ [M] \rangle = \langle ch(\theta)\ ch(\mu)/e,\ [M] \rangle$$

which is an integer by 2.5. Therefore our main theorem 2.3 is proved.

3. Applications.

3.1. Definition. The complex flag-manifold $F(n_1, n_2, \ldots, n_\ell)$ is the quotient of the group $U(n)$ by the subgroup $U(n_1) \times U(n_2) \times \ldots \times U(n_\ell)$, where $n_1 + n_2 + \ldots + n_\ell = n$. A point in $F(n_1, n_2, \ldots, n_\ell)$ consists of ℓ orthonormal subspaces of C^n with dimensions $n_1, n_2, \ldots, n_\ell$.

This family of manifolds includes the complex projective spaces: $CP^n = F(n, 1) = U(n+1)/U(n) \times U(1)$ and the complex Grassmann manifolds $F(m, n) = U(m+n)/U(m) \times U(n)$.

The dimension of $F(n_1, n_2, \ldots, n_\ell)$ is $n^2 - [n_1^2 + \ldots + n_\ell^2]$.

3.2. Remark. In order to get non-embedding results by applying theorem 2.3. we have to know the *cohomology of M, the K-theory of M,* the *stable normal bundle of M* and the *B-class of the normal bundle.* Let us describe one method to find each one of these elements, specially in the case when $M = F(n_1, n_2, \ldots, n_\ell)$.

3.3. Cohomology of the flag-manifolds. Let us denote by G the Lie group $U(n)$ and by H the subgroup $U(n_1) \times U(n_2) \times \ldots \times U(n_\ell)$. The fibration $G/H \xrightarrow{i} BH \xrightarrow{j} BG$ gives in cohomology the sequence:

$$H^*(BG; Z) \xrightarrow{j^*} H^*(BH; Z) \xrightarrow{i^*} H^*(G/H; Z)$$

The following results are known:

A. $H^*(BG; Z) = Z[c_1, c_2, \ldots, c_n]$, where $c_i \in H^{2i}(BG; Z)$ are the universal Chern classes.

B. $H^*(BH; Z) = Z[a_1^1, \ldots, a_{n_1}^1] \oplus \ldots \oplus Z[a_1^\ell, \ldots, a_{n_\ell}^\ell]$

C. The cohomology of G/H is given by the quotient of $H^*(BH; Z)$ by the ideal generated by:

$j^*(c_p) = \Sigma\, a_r^1 \oplus \ldots \oplus a_t^\ell$, where $r + \ldots + t = p$, $1 \le r \le n_1, \ldots, 1 \le t \le n_\ell$ and $p = 1, 2, \ldots, n$.

The proof of this last result is in [5].

3.4. Examples.

3.4.1. The complex projective space CP^n.

$$H^*(B(U(n) \times U(1); Z) = Z[a_1, \ldots, a_n] \oplus Z[b_1]$$

$$H^*(BU(n+1); Z) = Z[c_1, c_2, \ldots, c_{n+1}]$$

The relations are the following:

$$j^*(c_1) = a_1 + b_1 = 0$$
$$j^*(c_2) = a_2 + a_1 b_1 = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$j^*(c_{n+1}) = a_n b_1 = 0$$

Then we have: $b_1 = -a_1, a_2 = a_1^2, \ldots, a_n = a_1^n$ e $a_1^{n+1} = 0$ and so, the cohomology of CP^n is given by only one generator $x_1 = i^*(a_1) \in H^2(CP^n, Z)$. The monomials $1, x_1, x_1^2, \ldots, x_1^n$ give an additive basis for $H^*(CP^n, Z)$.

3.4.2. The Grassmann manifold $F(2,2)$

$$H^*(B(U(2) \times U(2)); Z) = Z[a_1, a_2] \oplus Z[b_1, b_2]$$
$$H^*(BU(4); Z) = Z[c_1, c_2, c_3, c_4].$$

The relations are the followings:

$$j^*(c_1) = a_1 + b_1 = 0 \qquad j^*(c_2) = a_2 + a_1 b_1 + b_2 = 0$$
$$j^*(c_3) = a_2 b_1 + a_1 b_2 = 0 \qquad j^*(c_4) = a_2 b_2 = 0.$$

Then, the cohomology of $F(2,2)$ is given by the generators $x_1 = i^*(a_1) \in H^2$ and $x_2 = i^*(a_2) \in H^4$. An additive basis for this cohomology is $1, x_1, x_1^2, x_2, x_1 x_2, x_1^2 x_2$.

3.4.3. The flag-manifold $F(2,2,1)$

Let $H^*(B(U(2) \times U(2) \times U(1); Z) = Z[a_1, a_2] \oplus Z[b_1, b_2] \oplus Z[d_1]$ and let $H^*(BU(5); Z) = Z[c_1, c_2, c_3, c_4, c_5]$.

We have the following relations:

$$j^*(c_1) = a_1 + b_1 + c_1 = 0 \qquad j^*(c_2) = a_2 + b_2 + a_1 b_1 + b_1 c_1 + a_1 d_1 = 0$$
$$j^*(c_3) = a_1 b_2 + a_2 b_1 + a_2 d_1 + b_2 d_1 + a_1 b_1 c_1 = 0$$
$$j^*(c_4) = a_2 b_2 + a_1 b_2 d_1 + a_2 b_1 d_1 = 0$$
$$j^*(c_5) = a_2 b_2 d_1 = 0$$

The cohomology is additively generated by the monomials:

$1, x_1, z_1, x_1^2, x_1 z_1, z_1^2, x_2, x_1 x_2, x_2 z_1, x_1^2 z_1,$
$x_1 z_1^2, z_1^3, x_1^2 x_2, x_1^2 z_1^2, x_1 z_1^3, x_1 x_2 z_1, x_2 z_1^2, z_1^4,$
$x_1 z_1^4, x_1^2 x_2 z_1, x_1 x_2 z_1^2, x_2 z_1^3, x_1^2 z_1^3, x_1^2 x_2 z_1^2,$
$x_1 x_2 z_1^3, x_2 z_1^4, x_1^2 z_1^4, x_1^2 x_2 z_1^3, x_1 x_2 z_1^4, x_1^2 x_2 z_1^4,$ where
$x_1 = i^*(a_1) \in H^2$, $z_1 = i^*(d_1) \in H^*$ and $x_2 = i^*(a_2) \in H^4$.

3.5. The B-class of the normal bundle to G/H.

Let $G = U(n)$ and $H = U(n_1) \times \ldots \times U(n_\ell)$ with $n_1 + n_2 + \ldots + n_\ell = n$. Let ξ be a $(p+1)$-vector bundle over S^n such that $\tau S^n \oplus \xi$ is trivial. If p and n are odd then $S(\xi)$ is a parallelizable manifold (by corollary 1.5) and the normal bundle to G/H in $S(\xi)$ is classified by the map:

$$G/H \xrightarrow{i} BH \xrightarrow{f} BSO(\dim H)$$

where f is induced by the adjoint representation: $Ad: H \to SO(\dim H)$.

The proof of this result is given in [4] where we can also find the following:

A. The map f induces in cohomology the homomorphism

$$f^*: H(BSO(n_1^2 + \ldots + n_\ell^2); Q) \to H^*(BH; Q)$$

and $p_k(v) = i^* f^*(p_k)$, where p_k are the universal Pontriagin classes.

B. The map f^* factors through the maps:

$$f_i^*: H^*(BSO(n_i^2); Q) \to H^*(BU(n_i); Q)$$

and $\quad f^*(p_k) = \Sigma f_1^*(p_r^1) \oplus \ldots \oplus f_\ell^*(p_t^\ell), \; r + \ldots + t = k.$

Each one of the factors f_i^*, $1 \le i \le \ell$, which we will denote simply by f^*, reduces to a map

$$f^*: H^*(BSO(n^2 - n + 1); Q) \to H^*(BU(n); Q)$$

and $f^*(p_k)$ is given by the following:

"We take the $k-th$ elementary symmetric function on $(t_i - t_j)^2$, $1 \le i < j \le n$ and express it in terms of the Chern classes – these are elementary symmetric functions on t_i's".

In case $n = 2$ we have:

$$f^*: H^*(BSO(3); Q) \to H^*(BU(2); Q).$$

By 2.15 of [4] we have $H^*(BSO(3); Q) = Q[p_1]$ and we know that $H^*(BU(2); Q) = Q[a_1, a_2]$ then:

$$f^*(p_1) = (t_1 - t_2)^2 = (t_1 + t_2)^2 - 4t_1 t_2 = a_1^2 - 4a_2.$$

3.5.1. For the Grassmann manifold $F(2,2)$ let us denote by $q_1 = f^*(p_1)$ the class which corresponds to the first factor $U(2)$ and by $q_1' = f_2^*(p_1)$ the class corresponding to the second factor then:

$$p_1(v) = i^*(q_1) + i^*(q_1') = -2x_1^2$$
$$p_2(v) = i^*(q_1)\ i^*(q_1') = -6x_1^2 x_2$$
$$p_k(v) = 0 \text{ if } k \geq 3$$
$$B_1(p_1) = -\frac{1}{3} x_1^2 \quad \text{and} \quad B_k(p_1, p_2, \ldots, p_k) = 0 \quad \text{se} \quad k \geq 2$$
$$B(v) = 1 - \frac{1}{3} x_1^2.$$

3.5.2. The manifold $F(2,3)$ has the following classes:

$$p_1(v) = -3x_1^2 + 2x_2$$
$$p_2(v) = 10x_1^2 x_2 - 15x_2^2$$
$$p_3(v) = -30x_2^3$$
$$p_k(v) = 0 \text{ if } K \geq 4$$
$$B_1(p_1) = \frac{1}{2} x_1^2 - \frac{1}{3} x_2$$
$$B_2(p_1, p_2) = \frac{103}{360} x_1^2 x_2 - \frac{5}{24} x_2^2$$
$$B_3(p_1, p_2, p_3) = -\frac{4}{45} x_1^2 x_2^2$$
$$B_k(p_1, p_2, \ldots, p_k) = 0 \quad \text{if} \quad k \geq 4.$$

3.5.3. The manifold $F(2,2,1)$ has dimension 16 and the classes are:

$$p_1(v) = i^*(q_1) + i^*(q_1') = -2x_1^2 - 3z_1^2 - 2x_1 z_1$$
$$p_2(v) = i^*(q_1) i^*(q_1') = -6x_1^2 x_2 - 9x_1 x_2 z_1 - 3x_2 z_1^2 - x_1^2 z_1^2 + $$
$$+ 2x_1 z_1^3 + 15z_1^4$$
$$p_k(v) = 0 \quad \text{if} \quad k \geq 3$$
$$B_1(p_1) = \frac{1}{6} [-2x_1^2 - 3z_1^2 - 2x_1 z_1]$$
$$B_2(p_1, p_2) = \frac{1}{18} x_1^2 z_1^2 + \frac{1}{18} x_1 z_1^3 + \frac{5}{25} z_1^4$$
$$B_3(p_1, p_2, 0) = -\frac{1}{72} x_1^2 z_1^4$$
$$B_k(p_1, p_2, \ldots, p_k) = 0 \quad \text{if} \quad k \geq 4.$$

3.6. K-theory and Chern character for the flag-manifolds.

We give here one short description of the K-theory for the flag-manifolds and one method to compute the Chern character of an additive basis for this K-theory; the details are in [4] and [5].

Given the principal H-bundle: $H \to G \xrightarrow{\pi} G/H$ we have a homomorphism $\alpha_\pi : RH \to K(G/H)$ where RH denote the ring of complex representations of the group H; in our case $G = U(n)$ and $H = U(n_1) \times \ldots \times U(n_k)$. We will consider only the cases $H = U(2) \times U(2)$ and $H = U(2) \times U(2) \times U(1)$. Let z_1, z_2, z_3 and z_4 denote the elementary representations of the torus $T(4) \subset U(4)$ and let:

$$\gamma^i = \sigma_i(z_1 - 1, z_2 - 1, z_3 - 1, z_4 - 1), \; i = 1, 2, 3, 4$$

be the elementary symmetric functions, and:

$$\alpha_i = \sigma_i(z_3 - 1, z_2 - 1) \in RU(2), \; i = 1, 2$$
$$\beta_i = \sigma_i(z_3 - 1, z_4 - 1) \in RU(2), \; i = 1,2$$
$$\xi_i = \alpha_\pi(\alpha_i) \in K(F(2,2)), \; i = 1, 2.$$

By definition, $ch(\xi_i) = ch(\alpha_i) = \sigma_i(e^{t_1} - 1, e^{t_2} - 1)$, and then $ch(\xi_i) = x_i +$ + higher order terms.

The following theorem is proved in [5]:

3.6.1. Theorem. *The K-theory $K(F(2,2))$ has as an additive basis the monomials* $1, \xi_1, \xi_2, \xi_1^2, \xi_1\xi_2, \xi_1^2\xi_2$.

The multiplicative structure is similar to the multiplicative structure for the cohomology and the relations are the same. To compute the Chern characters of an additive basis we compute $ch(\xi_i)$, $ch(\xi_2)$ and then we make use of the multiplicative property of ch. We have:

$$ch(\xi_1) = \sigma_1(e^{t_1} - 1, e^{t_2} - 1) = e^{t_1} - 1 + e^{t_2} - 1 =$$
$$= t_1 + t_2 + \frac{1}{2}(t_1^2 + t_2^2) + \frac{1}{6}(t_1^3 + t_2^3) + \ldots =$$
$$= x_1 + \frac{1}{2}(x_1^2 - 2x_2) - \frac{1}{6} x_1 x_2$$
$$ch(\xi_2) = \sigma_2(e^{t_1} - 1, e^{t_2} - 1) = (e^{t_1} - 1)\ (e^{t_2} - 1) =$$
$$= t_1 t_2 + \frac{1}{2}(t_1 t_2^2 + t_2 t_1^2) + \frac{1}{6}(t_1 t_2^3 + t_2 t_1^3 t_2^2 + \frac{1}{4} t_1^2 t_2^2 + \ldots =$$
$$= x_2 + \frac{1}{2} x_1 x_2 + \frac{1}{12} x_1^2 x_2$$

Let us consider now the flag-manifold $F(2,2,1)$; let z_1, z_2, z_3, z_4, z_5 be the elementary representations of $T(5) \subset U(5)$ and let:

$$\alpha_i = \sigma_i(z_1 - 1, z_2 - 1), \; i = 1, 2$$
$$\beta_i = \sigma_i(z_3 - 1, z_4 - 1), \; i = 1, 2$$
$$\lambda_1 = \sigma_1(z_5 - 1), \; \xi_i = \alpha_\pi(\alpha_i), \; i = 1, 2 \qquad \lambda = \alpha_\pi(\lambda_1)$$

3.6.2. Theorem. *The monomials*

$$1,\ \xi_1,\ \lambda,\ \xi_1^2,\ \xi_1\lambda,\ \lambda^2,\ \xi_2,\ \xi_1\xi_2,\ \lambda\xi_2,\ \xi_1^2\lambda,$$
$$\xi_1\lambda^2,\ \lambda^3,\ \xi_1^2\xi_2,\ \xi_1^2\lambda^2,\ \xi_1\lambda^3,\ \xi_1\xi_2\lambda,\ \xi_2\lambda^2,\ \lambda^4,$$
$$\xi_1\lambda^4,\ \xi_1^2\xi_2\lambda,\ \xi_1\xi_2\lambda^2,\ \xi_2\lambda^3,\ \xi_1^2\lambda^3,\ \xi_1^2\xi_2\lambda^2,$$
$$\xi_1\xi_2\lambda^3,\ \xi_2\lambda^4,\ \xi_1^2\lambda^4,\ \xi_1^2\xi_2\lambda^3,\ \xi_1\xi_2\lambda^4,\ \xi_1^2\xi_2\lambda^4$$

give an additive basis for $K(F(2,2,1))$; the multiplicative structure and the relations are similar to those for the cohomology.

The proof of this theorem is in [5] and, to compute the Chern character for the basis we make use of the following formulas:

$$ch(\xi_1) = \Sigma\ \frac{S_n(x_1, x_2, 0, \ldots 0)}{n!},\ n = 1, 2, 3$$

$$ch(\xi_2) = e^{x_1} - 1 - ch(\xi_1) \qquad ch(\lambda) = e^{z_1} - 1.$$

It is possible to show that $S_i(x_1, x_2, 0, \ldots, 0) = 0$ if $i \geq 5$, and then:

$$ch(\xi_1) = S_1(x_1) + \frac{1}{2}S_2(x_1, x_2) + \frac{1}{6}S_3(x_1, x_2, 0) +$$

$$+ \frac{1}{4!}S_4(x_1, x_2, 0{,}0) = x_1 + \frac{1}{2}(x_1^2 - 2x_2) + \frac{1}{6}(-x_1x_2 +$$

$$x_2z_1 - x_1^2z_1 - x_1z_1^2 - z_1^3) + \frac{1}{4!}(x_1x_2z_1 + x_2z_1^2 - z_1^4).$$

$$ch(\xi_2) = x_2 + \frac{x_1x_2}{2} + \frac{1}{12}x_1^2x_2 - \frac{1}{12}x_1x_2z_1 - \frac{1}{12}x_2z_1^2 +$$

$$+ \frac{1}{12}z_1^4 + \frac{1}{5!}x_1^5 + \frac{1}{6!}x_1^6.$$

$$ch(\lambda) = z_1 + \frac{1}{2}z_1^2 + \frac{1}{6}z_1^3 + \frac{1}{4!}z_1^4.$$

3.7. Computations.

3.7.1. Let us denote by $[F(2,2)]$ the fundamental class of $F(2,2)$ and by A_i the number $\langle ch(\xi)B(v), [(F(2,2)]\rangle$ where ξ is an element of the additive basis for $K(F(2,2))$ in the order given in theorem 3.6.1. The results are the following:

$$A_1 = A_2 = A_5 = 0 \qquad A_3 = \frac{1}{2^2} \qquad A_4 = -\frac{1}{2} \qquad A_6 = 1$$

Then, the best result is given by $A_3 = -\frac{1}{2^2}$; this shows that $\langle 2^1 ch(\xi_2)B(v), [F(2,2)]\rangle \notin Z$ and then we take $K - 1 = 1$ to conclude that $F(2,2)$ does not embed in $S(\xi)$ if $L = \dim S(\xi) = 12$ and ξ is a $(p+1)$-vector bundle over S^n with p and n odd numbers and $\tau S^n \oplus \xi$ trivial.

3.7.2. Let A_i denote $\langle ch(\xi)B(v), [F(2,2,1]\rangle$ where ξ is an element of the additive basis for $K(F(2,2,1))$ in the order given in theorem 3.6.2., so that $i = 1, 2, \ldots, 30$. The computation shows the following:

$$A_1 = A_2 = \ldots = A_{20} = A_{29} = 0$$

$$A_{21} = \frac{1}{4} \qquad A_{22} = A_{25} = A_{26} = -\frac{1}{4}$$

$$A_{23} = A_{24} = A_{27} = \frac{1}{2} \qquad A_{28} = \frac{1}{2} \qquad A_{30} = 1$$

The best result is given by $A_{21} = \frac{1}{4}$ and then we take $K - 1 = 1$ to conclude that $F(2,2,1)$ does not embed in $S(\xi)$ if $L = \dim S(\xi) = 16 + 4$ and ξ as before.

3.7.3. A similar computation for the manifold $F(2,1,1)$ gives the following result: this manifold does not embed in $S(\xi)$ if $\dim S(\xi) = 12$ and ξ as before.

3.7.4. The computations in [4] for the Grassmann manifolds allow us to conclude that:

$F(2,3)$ does not embed in $S(\xi)$, if $\dim\ S(\xi) = 20$
$F(2,4)$ does not embed in $S(\xi)$, if $\dim\ S(\xi) = 28$

Finally, let us group these results together the following may:

3.8. Let ξ be a $(p+1)$-vector bundle over S^n with p and n odd numbers and such that $\tau S^n \oplus \xi$ is trivial (see 1.6. *B*); let $L = p + n = \dim S(\xi)$. Then:

(1) $F(2,2)$ does not embed in $S(\xi)$ if $L = 12$
(2) $F(2,3)$ does not embed in $S(\xi)$ if $L = 20$
(3) $F(2,4)$ does not embed in $S(\xi)$ if $L = 28$
(4) $F(2,1,1)$ does not embed in $S(\xi)$ if $L = 12$
(5) $F(2,2,1)$ does not embed in $S(\xi)$ if $L = 20$

3.8.1. Remark. If $\xi = \theta^{p+1}$ is the trivial $(p+1)$-vector bundle over S^n, p and n odd numbers, then $S(\xi) = S^n \times S^p$ and we have non-embedding results in products of spheres of odd dimensions.

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