The connectivity of a finite H-space

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All spaces considered here are simply connected.

In [3] A. Clark proved the following: Let X be a finite loop space, then $\pi_i(X) \neq 0$ for some $i \leq 3$, otherwise X is contractible. Several people have considered the similar question under the weak hypotheses that X is either a finite H-space or a homotopy associative finite H-space.

Let us consider a certain class of spaces. Call the higher connectivity of the class the smallest integer $n-1$ such that if X belongs to the class and $\pi_i(X) = 0$ $i \leq n$ then X is contractible. If such n does not exist then we say that the higher connectivity is infinite. For example the higher connectivity of the class of finite mod-p H-spaces where p is an odd prime is infinite. See [1]. The purpose of this note is to prove the following two theorems:

Theorem 1. *The higher connectivity of the class of finite H-spaces is the same as the higher connectivity of the class of finite mod-2 H-spaces.*

Theorem 2. *The higher connectivity of the class of finite homotopy associative H-spaces is the same as the higher connectivity of the class of finite homotopy associative mod-(2, 3) H-spaces.*

I would like to think the referee for given a much simpler proof of proposition 2, altough less elementary. We describe his proof in page 3.

The general reference for localization is [4] and for H-spaces is [7].

Proof of Theorems 1 and 2.

Proposition 1. *Let (M, d) be a connected positive graded differential finite Hopf Algebra over $\mathbb{Z}_p$ where dim M (as a $\mathbb{Z}_p$-vector space) is even, and p is any prime. Assume that $d(\overline{M}) \subset \overline{M}$ where $\overline{M}$ is the augmented Hopf algebra. Then $H(\overline{M}, \bar{d}) \neq 0$, where $\bar{d} = d \mid \overline{M}$.*

Proof: Since $\bar{d}$ is also a differential we have $\mathrm{im}\bar{d} \subset \mathrm{Ker}\,\bar{d}$. We also have $\dim(\overline{M}) = \dim(\mathrm{Ker}\,\bar{d}) + \dim(\mathrm{im}\bar{d})$. But $\dim(\overline{M})$ is odd so $\dim(\mathrm{im}\bar{d}) \neq$ $\neq \dim(\mathrm{Ker}\,\bar{d})$. Therefore $\mathrm{im}\bar{d} \not\subset \mathrm{Ker}\,\bar{d}$ and $H(\overline{M}, \bar{d}) \neq 0$.

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Proposition 2. *Let X be a finite H-space. Then X is contractible if and only if X_0 (X localized at the rational) is contractible.*

Proof: ($\Rightarrow$) It is obvious.

($\Leftarrow$) To show that X is contractible, it suffices to show that $H^*(X, \mathbb{Z}_p) \simeq \bar{\mathbb{Z}}_p$ for all prime p. (Remember that X is a simply connected finite complex.) Let p be any prime. Let us consider the Bockstein Spectral Sequence on cohomology associated with the prime p. See [2] for more details. Let (E_r, β_r) be the r-th term of the spectral sequence. We know that $E_r = \mathbb{Z}_p \oplus \bar{E}_r$, $\beta_r(\bar{E}_r) \subset \bar{E}_r$ and $\beta_r(1) = 0$. So $E_{r+1} \simeq \mathbb{Z}_p \oplus H(\bar{E}_r, \bar{\beta}_r)$ where $\bar{\beta}_r = \beta_r \mid \bar{E}_r$.

Because X_0 is contractible we have $E_\infty \simeq \mathbb{Z}_p$. Suppose $E_1 \neq \mathbb{Z}_p$. Let r be the smallest integer such that $E_r \simeq \mathbb{Z}_p$. But $E_r \simeq \mathbb{Z}_p \oplus H(\bar{E}_{r-1}, \bar{\beta}_{r-1})$.

If $E_{r-1} \neq \mathbb{Z}_p$ we have $Q^{odd} E_{r-1} \neq 0$, otherwise $E_r \neq \mathbb{Z}_p$. So by the Borel decomposition theorem dim E_{r-1} is even then (E_{r-1}, β_{r-1}) satisfies the hypothesis of proposition 1. But this means $E_r \neq \mathbb{Z}_p$ which is a contradicton. So $E_1 \simeq H^*(X, \mathbb{Z}_p) \approx \mathbb{Z}_p$ and X is contractible. Q.E.D.

Referee's prof:

Browder proved in [3] the following: "Every connected finite H-space X is contractible or else there is an interger $m > 0$ such that ξ generates $H_m(X, Z) \simeq Z$ and $\Lambda\xi : H^q(X, Z) \to H_{m-q}(X, Z)$ is an isomorphism for all q". So if X_0 is contractible $\Rightarrow \tilde{H}_*(X_0, Z) \simeq \tilde{H}_*(X, Z) = 0$.

By Browder's theorem X is contractible.

Proof of Theorem 1. Let $n-1$, respectively $m-1$ be the higher connectivity of the class of finite mod-2 H-spaces respectively the class of finite H-spaces. Let me show first that $m \leq n$. Let X be any finite H-space. Then $X_{\{2\}}$ is certainly a mod-2 H-space. If $\pi_i(X) = 0$ $i \leq n$ then $\pi_i(X_{\{2\}}) = = 0, i \leq n$. So $X_{\{2\}}$ is contractible. Therefore X_0 is also contractible. Now by Proposition 2 follows that X is contractible. So $m \leq n$.

Now let me show that $n \leq m$. Let Y be any finite mod-2 H-space. It is well-known that $Y_0 = \prod_{i=1}^{s} K(Q, 2n_i + 1)$. Now let us consider the space $(\prod_{i=1}^{s} S^{2n_i+1})_{\mathbb{P}-\{2\}}$ where $\mathbb{P}$ is the set of all primes. By [1], this is an H-space. Now by Proposition 4.7.2 of [7] there is a finite X space such that $X_{\{2\}} \simeq Y_{\{2\}}$ and $X_{\mathbb{P}-\{2\}} \simeq (\prod_{i=1}^{s} S^{2n_i+1})_{\mathbb{P}-\{2\}}$. But the connectivity of X is the same as the connectivity of $X_{\{2\}}$ and greater or equal to the connectivity of Y. So $m \geqslant n$.

Proof of Theorem 2. The proof is quite similar. Let $n-1$ resp $m-1$ be the higher connectivity of the class of finite homotopy associative mod-$\{2, 3\}$ H-spaces resp. the class of finite homotopy associative H-spaces. The reason that $n \leq m$ is analogous to the previous case. So let me prove that $m \leq n$. Let Y be any finite homotopy associative mod-(2, 3) H-space. We know that $Y_0 \simeq \prod_{i=1}^{s} K(Q, 2n_i + 1)$. Now let us consider the space $(\prod_{i=1}^{s} S^{2n_i+1})_{\mathbb{P}-\{2,3\}}$. By [6] this is a homotopy associative H-space. So by the main theorem part 2 of [5] there is a finite homotopy associative H-space X such that $X_{\{2,3\}} \simeq Y_{\{2,3\}}$ and $X_{\mathbb{P}-\{2,3\}} \approx (\prod_{i=1}^{s} S^{2n_i+1})_{\mathbb{P}-\{2,3\}}$. Since the connectivity of X is the same as the connectivity of $X_{\{2,3\}}$ and greater or equal to the connectivity of Y we have $m \geq n$.

Bibliography

[1] Adams, J. F., *The Sphere, considered as an H-space mod* p, Quart. J. Math. Oxford (2) 12 (1961), 52-60.

[2] Browder, W., *Torsion in H-spaces*, Ann. of Math. 74(1961).

[3] Clark, A., *On π_3 of finite dimensional H-spaces*. Ann. of Math. 78(1963), 193-196.

[4] Hilton, P., Mislin, G., Roitberg, J., *Localization of Nilpotent Groups and Spaces*, North Holland Mathematics Studies 15 (1975).

[5] Mislin, G., *H-spaces mod* p (I), *H*-spaces Nenchatel (Suisse) Aout 1970. Lecture Notes in Mathematics 196 Springer-Verlag, 5-10.

[6] Stassheff, J., *Homotopy associativity of H-spaces* II, Trans. Amer. Math Soc. 108 (1963), 293-312.

[7] Zabrodsky, A., *Hopf Spaces*, North Holland Mathematics Series 22.

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