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Semi-free S^1-action and embedding of four manifolds

Janey A. Daccach

1. Introduction.

A very well known theorem of Whitney asserts that all n-dimensional manifolds can be embedded in $2n$ dimensional Euclidean space.

A natural question concerning this matter is: what is the minimum dimension of an Euclidean space in which a given manifold M can be embedded?

Massey and Peterson in a joint work (3), among other results obtained the following:

Theorem. *A compact, orientable differentiable n-manifold, can be embedded differentiably in $2n-1$ dimensional Euclidean space (with the possible exception of the case $n=4$).*

In dimension four Boéchat and Haefliger (1) proved the following:

Theorem. *A compact orientable differentiable four manifold M^4 can be embedded smoothly in R^7 if and only if there is an integral class $W \in H^2(M; Z)$ satisfying:*

(1) $W \cdot X \equiv X^2$ *mod 2, for all* $X \in H^2(M; Z)/\mathrm{Tor}$

(2) $\langle W^2, [M] \rangle = \tau$, *where τ denotes the index of M.*

Condition (1) is equivalent to saying that $w_2(M)$ is the mod 2 reduction of an integral class.

It is the purpose of this paper to prove the

Theorem 3.4. *A compact orientable differentiable four manifold admitting a non-trivial semi-free S^1-action can be smoothly embedded in R^7.*

2. Semi-free S^1-action.

An action $\Phi : S^1 \times M^n \to M^n$ is called semi-free if for all $x \in M - M^{S^1}$ its isotropy group is $\{1\}$.

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We denote by (M^n, Φ) a manifold M, together with a semi-free S^1-action.

Let M be a closed, compact, oriented manifold, then we say that (M^n, Φ) bords if there is a compact differentiable, oriented manifold W^{n+1} and a semi-free S^1-action $\psi : S^1 \times W \to W$, such that $(\partial W, \psi \mid \partial W)$ is equivariantly diffeomorphic to (M, Φ).

The pair (M_1^n, Φ_1) is cobordant to (M_2^n, Φ_2) if $(M_1^n, \Phi_1) \cup (-M_2^n, \Phi_2)$ bords. This relation turns out to be an equivalence relation. The class of (M, Φ) is denoted by $[M^n, \Phi]$. A group structure is imposed on the set of these cobordism class, and the group is denoted by $SF_n(S^1)$.

The normal bundle of each component of the fixed point set in M^n has a natural complex structure, and denoting by F_{n-2k} the union of the $(n-2k)$-dimensional components, the normal bundle ν^k of F_{n-2k} in M^n is classified by a map $f_k : F_{n-2k} \to BU(k)$. Defining

$$M_n(U) = \sum_{k=0}^{[\frac{n}{2}]} \Omega_{n-2k}\, BU(k)$$

we have a well defined homomorphism

$$\beta : SF_n(S^1) \to M_n(U)$$

Another homomorphism

$$\partial : M_n(U) \to \Omega_{n-2}(BU(1))$$

is defined by: $\partial[\nu^k]$ is the class of the principal S^1-fibration $\pi : S(\nu^k) \to CP(\nu^k)$ where $S(\nu^k)$ is the sphere-bundle and $CP(\nu^k)$ the complex projective bundle associated to ν^k respectively.

It was proved (4) that the following short sequence

$$0 \to SF_n(S^1) \xrightarrow{\beta} M_n(U) \xrightarrow{\partial} \Omega_{n-2}(BU(1)) \to 0$$

is exact.

If M^n is an oriented closed manifold and Φ a semi-free S^1-action then

$$\tau(M) = \Sigma\, \tau(F_\gamma)$$

where F_γ is the γ-dimensional fixed point set, τ is the index; and if $n = 4p$ and $\tau(M) \neq 0$ then some component of the fixed point set is of dimension $\geq 2p$. See (2).

3. The proof of the theorem.

Let M^4 be a compact closed, orientable four manifold in which S^1 acts semi-freely and non trivially. Then the fixed point set is $M^{S^1} = F_0 \cup F_2$ where F_0 is a disjoint union of isolated points P_i and F_2 is the disjoint union of closed two dimensional manifolds N_j^2. Each component of the fixed point set is canonically oriented since the normal bundle ν_j^γ admits a U-structure, $\gamma = 4{,}2$.

A fixed point P_j will have orientation $+1$ if the induced S^1-action on $S^3 = \partial(\nu_i^4)$ preserves the orientation on S^3 induced from M, otherwise it will have orientation -1.

Denoting by $J_i : N_i^2 \to M$ the inclusion we can define a homology class $\overline{W} \in H_2(M; Z)$ by

$$\overline{W} = \Sigma\, J_{i*}\, [N_i^2]$$

where $[N_i^2]$ is the orientation class of N_i^2 and k is the number of 2 dimensional components of M^{S^1}, and $W \in H^2(M; Z)$ is its Poincaré-Dual.

Proposition 3.1. *The modulo 2 reduction of W is $w_2(M)$ the second Stiefel-Whitney class of M.*

Proof. First of all let us show that $w_2(M - M^{S^1}) = 0$.

The semi-free S^1-action restricts to a free S^1-action on $M - M^{S^1}$. Then the orbit space $M - M^{S^1}/S^1 = K^3$ is a 3-dimensional manifold. Let π be the orbit map $\pi : M - M^{S^1} \to K^3$, so

$$T(M - M^{S^1}) \simeq \pi^*(T(K^3)) + \eta$$

where T is tangent bundle and η is the tangent bundle to the fibers. From this bundle equation we have

$$w_1(M - M^{S^1}) = \pi^*(w_1(K^3)) + w_1(K^3)) + w_1(\eta) = 0$$

because $M - M^{S^1}$ is orientable.

Now

$$w_2(M - M^{S^1}) = \pi^*(w_2(K^3)) + w_1(\eta) \cdot \pi^*(w_1(K^3))$$

and from the Wu's relation

$$w_2(K^3) = w_1(K^3) \cdot w_1(K^3).$$

Then

$$w_2(M - M^{S^1}) = \pi^*(w_1(K^3)^2) + w_1(\eta) \cdot \pi^*(w_1(K^3)) = \\ = \pi^*(w_1(K^3)) \cdot (\pi^*(w_1(K^3)) + w_1(\eta) = 0$$

Let U be a closed invariant tubular neighborhood of the 2-dimensional component N_i^2. Since $w_2(M - M^{S^1}) = 0$, $w_2(M)$ lies in the image of

$$H^2(M, \overline{M - U};\ Z_2) \to H^2(M, Z_2).$$

Now just as $W \in H^2(M; Z)$ is defined by Poincaré duality, also a class $W' \in H^2(M, \overline{M - U}; Z)$ can be defined by Lefschetz duality so that W' restricts to W.

Now let us observe that

$$H^2(M, \overline{M - U}; Z) \approx H^2(U, \partial U; Z) \approx \bigoplus_{i=1}^{k} H^2(U_i, \partial U_i; Z)$$

since no contribution is given by isolated fixed points.

Next, the component of W' in $H^2(U_i, \partial U_i; Z)$ under the decomposition

$$H^2(M, \overline{M - U}; Z) \approx \bigoplus_{i=1}^{k} H^2(U_i, \partial U_i; Z)$$

is the Lefschetz dual of $N_i^2 \subset \mathrm{int}(U_i)$ and is a generator

$$W_i' \in H^2(U_i, \partial U_i; Z) \simeq Z$$

namely the Thom class for the normal bundle to N_i^2.

Let us view $\{W_i'\}$ as a basis for $H^2(M, \overline{M - U}; Z)$; and the modulo 2 reductions $\{\rho(W_i')\}$ as a basis for $H^2(M, \overline{M - U}; Z_2)$. A class w' restricting to $w_2 \in H^2(M; Z_2)$ must have the form

$$w' = \Sigma \lambda_i (W_i'),\ \lambda_i \in Z_2$$

and since $w_2 = v_2$ we must have

$$\langle \rho(W_i)^2|_M, [M] \rangle = \langle (W' U \rho(W_i))|_M, [M] \rangle$$

for each i. Also

$$\langle (\rho(W') U \rho(W_i))|_M, [M] \rangle = \langle \rho(W_i)^2|_M, [M] \rangle$$

This implies that $\rho(W') = w'$ and so by restricting to M we get $\rho(W) = w_2(M)$ as desired.

The proof given above was kindly suggested by Professor Peter Landweber.

In dimension four we have the exact sequence

$$0 \to SF_4 \xrightarrow{\beta} M_4(U) \xrightarrow{\partial} \Omega_2(CP(\infty)) \to 0$$

where

$$M_4(U) = \Omega_4(BU(0)) + \Omega_2(BU(1)) + \Omega_0(BU(2))$$

An element of $\Omega_2(BU(1))$ is represented by a pair (N^2, f) where $f: N^2 \to BU(1)$ is a map.

Let us define a homomorphism

$$e: \Omega_2(BU(1)) \to Z$$

in the following way:

if α, is the generator of $H^2(BU(1)); Z) \cong Z$, then

$$e[N, f] = \langle f^*(\alpha), [N] \rangle$$

Let

$$\pi: M_4(U) \to \Omega_2(BU(1))$$

be the projection on the second factor, and define the homomorphism

$$h: M_4(U) \to Z$$

by $h = e \circ \pi$

The index of a manifold extends obviously to a homomorphism

$$\tau: SF_4 \to Z$$

Proposition 3.2. *The homomorphism $h \circ \beta$ and τ are equal in SF_4.*

Proof. For each class $\theta = [M, \Phi]$ we have that $\tau(\theta) = P_+ - P_-$ where P_- is the number of isolated fixed ponts with positive orientation and P_- are the ones with negative orientation. This follows from the the fact $\tau(M) = \Sigma\, \tau(F_k)$.

Then

$$\tau(M_4) = \tau(F_2) + \tau(F_0) = \tau(F_0).$$

Now

$$\beta[M, \Phi] = \sum_{i=1}^{(P_+ + P_-)} [v_i^4] + \sum_{j=1}^{k} [v_j^2]$$

Since $CP(1) \to CP(2)$ has on its normal bundle a complex structure conjugate equivalent to the associated line bundle to the principal fibration $S^3 \xrightarrow{\pi} S^2$, it follows that

$$\partial[v_i^4] = [S^2, g_i]$$

where

$$g_i: S^2 \to BU(1)$$

with

$$g_i^*(\alpha)\ [S^2] = \begin{cases} -1 & \text{if } P_i \text{ has orientation } 1 \\ +1 & \text{if } P_i \text{ has orientation } -1 \end{cases}$$

Then

$$\partial\left(\sum_{i=0}^{(P_+ + P_-)} [v^4]\right) = -\sum_{i=1}^{(P_+ + P_-)} [S^2, g_i]$$

and $\partial[v^2] = [N_j^2, f_j]$ where f_j is the classifying map for v_j^2, because $S(v_j^2)$ is a principal S^1-bundle. Since $\partial \circ \beta = 0$ we have that

$$\partial\left(\sum_{i=1}^{(P_+ + P_-)} [v^4] + \sum_{j=1}^{k} [v_j^2]\right) = 0$$

and so

$$\sum_{i=1}^{(P_+ + P_-)} [S^2, g_i] + \sum_{j=1}^{k} [N_j^2, f_j] = 0$$

In terms of characteristic numbers we have

$$\sum_{i=1}^{(P_+ + P_-)} g_i^*(\alpha)[S^2] + f_j^*(\alpha)[N_j^2] = 0$$

$$(-P_+ + P_-) + \sum_{j=1}^{k} f_j^*(\alpha)[N_j^2 L = 0$$

$$-\tau(\theta) + h\beta(\theta) = 0.$$

Then $h\beta(\theta) = \tau(\theta)$ as desired.

Proposition 3.3. *The self intersection number of $\overline{W} \in H_2(M, Z)$ is equal to the index of M.*

Proof.

The Poincaré dual of $\overline{W}$, W, can be written as $W = \sum_{r=1}^{k} W_r$ where $J_{r*}[N_\gamma^2] = W_\gamma \cap [M]$. Since $N_j^2 \cap N_i^2 = \Phi$ then

$$\langle w_j, J_{i*}[N_i^2]\rangle = 0 \quad \text{if} \quad i \neq j$$

So

$$\langle W \cup W, [M]\rangle = \langle W, W \cap [M]\rangle = \langle W, \sum_{i=1}^{k} J_{i*}[N_i^2]\rangle =$$

$$= \left\langle \sum_{i=1}^{k} W_\gamma, \sum_{i=1}^{k} J_{i*}[N_i^2]\right\rangle = \sum_{r=1}^{k} \langle W_\gamma, J_{\gamma*}[N_\gamma^2]\rangle =$$

$$= \sum_{\gamma=1}^{k} \langle J_\gamma^* W_\gamma, [N^2]\rangle.$$

It is known that $J_\gamma^*(W_\gamma)$ is the Euler class of v_γ^2; then $J_\gamma^*(W_\gamma) = f_\gamma{}^*(\alpha)$ where f_γ is the classifying map for v_γ^2. Hence we conclude that

$$W \cup W = \sum_{j=1}^{k} f_j{}^*(\alpha)\,[N_j^2] = h\beta[M, \Phi] = \tau(M).$$

Theorem 3.4. *A compact orientable differentiable four manifold M^4 admitting a non trivial semi-free S^1-action can be embedded smoothly in R^7.*
Proof. If $\tau(M) = 0$ then M embeds in R^7, since indefinite forms satisfy the Boéchat-Haefliger conditions.

Let us suppose that $\tau(M) \neq 0$, then $F_2 \neq \phi$. Considering $\overline{W} = J_*[F_2]$ and W its Poincaré dual we have:

(1) By Proposition 3.1 the reduction mod 2 of W is $w_2(M)$;

(2) By Proposition 3.3 $\langle W^2, [M]\rangle = \tau(M)$;

So the result follows from the theorem of Boéchat-Haefliger.

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Universidade Federal de São Carlos
Departamento de Matemática
CP – 384
13560 – São Carlos – SP
Brasil