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Invariant polynomials and zonal spherical functions

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0. Introduction.

The main result of this paper is a derivation of the Plancherel measure for zonal spherical functions on a rank one semi-simple group. Of course this is an old result of Harish-Chandra [WII, p 338]; the point here is that the techniques used are entirely elementary. We apply the formulas for inversion of the Abel transform to the inverse Fourier transform of $\phi_\nu(f)$. The operations for inverting the Abel transform can be carried out under the integral for the Fourier inversion ($f \in C_c^\infty(G//K)$) and then we obtain the Plancherel measures as the Taylor coefficients of certain hypergeometric functions.

The first section is devoted to deriving an analogue of a theorem of Mather and Schwartz for semi-simple Lie groups (of any rank). We find that the space of linear combinations of zonal spherical functions which occur as matrix entries of finite dimensional representations forms a free polynomial algebra. Then if $\phi_1, \dots, \phi_\ell$ are generators, any $f \in C_c^\infty(G//K)$ can be written in the form $f(x) = F(\phi_1(x), \dots, \phi_\ell(x))$ where $F \in C_c^\infty(\mathbb{R}^\ell)$.

In the second and third sections we derive again some results of Nolan Wallach (with some modest simplifications). In the second section we give his proof of a simple lemma (lemma 10) which is the basis for the inversion of the Abel transform. To add a little spice, we cast the lemma in the setting of Euclidean motion groups: this gives us the Paley-Wiener theorem for certain Bessel functions ($J_{n/2}, n \in \mathbb{Z}$). The third section gives Wallach's proof of the inversion formula for the Abel transform and of the zonal Paley-Wiener theorem for rank one semi-simple groups [$H2, G$]. I have included these because they are not readily available. The fourth section is devoted to finding the Plancherel measure in rank one, class one.

The techniques of this paper are quite elementary; the most difficult facts used are Mather's theorem and the parameterization of zonal spherical function [H, ch 10]. (Actually one can give a proof of Mather's theorem in the case used here which is considerably simpler). Aside from this we use only some standard structure theory for semi-simple groups, regular singular point theory for ordinary differential equations, and a contour integral.

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1. Mather's Theorem and Some Consequences.

Let K be a compact Lie group and let $\pi : K \to L(V)$ be a smooth representation, where V is a finite dimensional real vector space. Let $P(V)$ be the algebra of real valued polynomial functions on V and $I(V)$ be the subalgebra of K-invariant polynomials ($f(\pi(k)\,v) = f(v)$ for all $k \in K$, $v \in V$). By a theorem of Hilbert, we can choose homogeneous $\sigma_1, \ldots, \sigma_k \in I(V)$ which (with 1) generate $I(V)$ as an algebra and no proper subset generates. Define $\sigma : V \to \mathbb{R}^k$ by $\sigma(v) = (\sigma_1(v), \ldots, \sigma_k(v))$. Then the fact that the σ_i generate may be expressed by saying that

$$F \to F \circ \sigma : P(\mathbb{R}^k) \to I(V)$$

is surjective.

Let $C_c^\infty(V)$ be the space of infinitely differentiable complex valued functions with compact support, endowed with the Schwartz topology, and let $I(V)$ be the subspace of K-invariant functions.

Mather's Theorem. Every $f \in I(V)$ can be expressed in the form

$$f(x) = F(\sigma_1(x), \ldots, \sigma_k(x)) = (\sigma^* F)(x)$$

for some $F \in C_c^\infty(\mathbb{R}^k)$. Moreover $\sigma^* : C_c^\infty(\mathbb{R}^k) \to I(V)$ is a continuous split surjection.

For the proof see [M] and [S]. (Actually Mather does not have the compact support condition, but he shows that σ is a proper map which easily gives the present version of his theorem).

Let G be a connected, non-compact semi-simple Lie group with maximal compact subgroup K. Let $I(G)$ be the space of K-biinvariant C_c^∞ functions on $G(f(k_1 x k_2) = f(x)$ for all $k_1, k_2 \in K$, $x \in G)$. Our first objective is to obtain a result analogous to Mather's theorem for $I(G)$, where $\sigma_1, \ldots, \sigma_k$ are replaced by certain zonal spherical functions $\phi_1, \ldots, \phi_\ell$.

Let $g = k + p$ be the Cartan decomposition of the Lie algebra g of G and let a_p be a maximal abelian subalgebra of p. Let Σ be the set of restricted roots and let Σ_2 be the subset $\lambda \in \Sigma$ such that $2\lambda \notin \Sigma$. Then Σ_2 is a root system; let $\lambda_1, \ldots, \lambda_\ell$ be a simple system of roots for Σ_2. Let $a = a_k + a_p$ be a Cartan subalgebra containing a_p and let ϕ be the set of roots of the complexification g_c with respect to a_c. Recall that a representation $\pi : G \to L(V)$ is called class one if it is irreducible and there is a vector $v_0 \in V$ such that $\pi(k) \cdot v_0 = v_0$ for all $k \in K$. Let $(|)$ denote the Killing form on $a_c^\vee$.

Lemma 1. (*Helgason*). *A linear function $\Lambda : a_c \to \mathbb{C}$ is the highest weight of a finite dimensional class one representation if and only if*

(1) $\Lambda | a_k \equiv 0$

(2) $(\Lambda | \lambda_i)/(\lambda_i | \lambda_i)$ *is a non-negative integer for each* $i = 1, \ldots, \ell$.

Proof. A proof of Helgason's theorem appears in [WI, p 210], but, since it relies on facts which are rather deeply embedded in Helgason's book, I think it is worthwhile to give a simplified proof. First, since the center of G lies in K, there is no loss of generality in assuming that G is a connected subgroup of the simply connected group G_c with Lie algebra g_c. Let π_Λ be the finite dimensional holomorphic representation with highest weight Λ.

Let M be the centralizer of a_p in K and let M° be its identity component. Choose a root vector $X_i \in g$ such that

$$[H, X_i] = \lambda_i(H) X_i \qquad (H \in a_p)$$

$$\langle \lambda_i | H_i \rangle = 2 \quad \text{where} \quad H_i = [\theta X_i, X_i] \in a_p.$$

We may define $\exp(\pi\sqrt{-1}\,H)$ in G_c. The essence of Helgason's proof is the following structural fact:

$$M = M^\circ\, C \quad \text{where}$$

$$C = \{\exp(\pi\sqrt{-1}\,H) \mid H = \Sigma n_i H_i \quad \text{with} \quad n_i \in \mathbb{Z}\}.$$

First it is clear that if $C \subseteq G$ then $M^\circ \cdot C \subseteq M$. To show that $C \subseteq G$ it suffices to show that $\exp(\pi\sqrt{-1}\,H_i) \in G$ for each $i = 1, \ldots, \ell$. Since $S\ell(2, \mathbb{C})$ is simply connected, there is an analytic homomorphism $\phi_i : S\ell(2, \mathbb{R}) \to G$ such that the differential ϕ_i of ϕ_i satisfies

$$\phi_i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = H_i,\ \phi_i\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = X_i,\ \phi_i\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix} = \theta X_i.$$

Then

$$\exp(\pi\sqrt{-1}\,H_i) = \phi_i\left(\exp \pi\sqrt{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) = \phi_i\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \in G.$$

Conversely let $Z(a_{pc})$ be the centralizer of the complexification a_{pc} in G_c. Then $Z(a_{pc})$ is connected (the centralizer of a torus in a complex group), and it is clear that $M \subseteq U \cap Z(a_{pc})$, where U is a compact real form of G_c for which $K = U \cap G$. Now any element $z \in U \cap Z(a_{pc})$ can be written in the form

$$z = m \exp(\pi\sqrt{-1}\,H), \quad m \in M^\circ \quad \text{and} \quad H \in a_p$$

and we may as well assume that $m = 1$. Let $\sigma : G_c \to G_c$ be the real analytic automorphism generated by the conjugation of g with respect to g. If

$z \in G$ then $\sigma(z) = z$; thus

$$\exp(\pi\sqrt{-1}\,H) = \sigma(\exp(\pi\sqrt{-1}\,H)) = \exp(-\pi\sqrt{-1}\,H),$$

that is $\exp(2\pi\sqrt{-1}\,H) = 1$. Obviously this implies that $\langle\Omega|H\rangle \in \mathbb{Z}$ for all integral Ω on a_c, so to show that $\exp(\pi\sqrt{-1}\,H) \in C$ it suffices to find integral Ω_i such that $\langle\Omega_i|H_j\rangle = \delta_{ij}$.

Using the Killing form, we shall identify $a_{pc}^{\vee}$ with a subspace of $a_c^{\vee}$. Then the H_i may also be defined by $H_i \in a_c$ and

$$\langle\beta|H_i\rangle = 2(\beta|\lambda_i)/(\lambda_i|\lambda_i) \qquad (\text{for all } \beta \in a_c^{\vee}).$$

Let $\alpha_1, \dots, \alpha_p$ be a simple system of roots for ϕ with the property that $\lambda_i = \alpha_i|a_p$ or $\lambda_i = 2\alpha_i|a_p$ for $i = 1, \dots, \ell$ [WI, p 23]. Define $h_i \in a_c$ by

$$\langle\beta|h_i\rangle = 2(\beta|\alpha_i)/(\alpha_i|\alpha_i) \qquad (i = 1, \dots, p).$$

To compute H_i in terms of h_i there are three cases to consider.

(1) If $\lambda_i = \alpha_i|a_p$ with $\sigma\alpha_i = \alpha_i$ then $H_i = h_i$ (Identifying $a_{pc}^{\vee} \subseteq a_c^{\vee}$, we have $\lambda_i = \alpha_i$ so $H_i = h_i$ is clear).

(2) If $\lambda_i = \alpha_i|a_p$ and $\sigma\alpha_i \neq \alpha_i$ then $H_i = h_i + \sigma h_i$ (In this case we have $(\alpha_i|\sigma\alpha_i) = 0$ and $(\alpha_i|\alpha_i) = 2(\lambda_i|\lambda_i)$, because $2\lambda_i \in \Sigma$ [WI, p 33, 21]. Now $\lambda_i = \frac{1}{2}(\alpha_i + \sigma\alpha_i)$ and $\langle\beta|h_i + \sigma h_i\rangle = 2(\beta|\alpha_i + \sigma\alpha_i)/(\alpha_i|\alpha_i) = 2(\beta|\frac{1}{2}(\alpha_i + \sigma\alpha_i))/(\lambda_i|\lambda_i)$).

(3) If $\lambda_i = 2\alpha_i|a_p$ then $H_i = h_i + \sigma h_i$ (Here $(\alpha_i|\sigma\alpha_i) < 0$ and $(\alpha_i|\alpha_i) = (\lambda_i|\lambda_i)$ and $\lambda_i = \alpha_i + \sigma\alpha_i$ [WI, p 33, 21]. Thus $\langle\beta|h_i + \sigma h_i\rangle = 2(\beta|\alpha_i + \sigma\alpha_i)/(\alpha_i|\alpha_i) = \langle\beta|H_i\rangle$).

Finally if $\Omega_1, \dots, \Omega_p$ are the fundamental weights corresponding to $\alpha_i, \dots, \alpha_p$ then for $i, j = 1, \dots, \ell$ we have [WI, p 23]

$$\langle\Omega_j|H_i\rangle = \langle\Omega_j|h_i + \sigma h_i\rangle = \langle\Omega_j|h_i\rangle = \delta_{ij}$$

in case 2 and 3, and similarly $\langle\Omega_j|H_i\rangle = \delta_{ij}$ in case 1.

Now let us return to the proof of Helgason's theorem. Actually Helgason proves his theorem under the additional assumption that Λ is dominant integral, but this follows from condition 2 in the statement of the lemma. Condition 1 says that $\Lambda = \sigma\Lambda$ and condition 2 is the same as $\langle\Lambda|H_i\rangle \in 2\mathbb{Z}$. Thus

$$\langle\Lambda|h_i\rangle = \tfrac{1}{2}\langle\Lambda|h_i + \sigma h_i\rangle = \tfrac{1}{2}\langle\Lambda|h_i\rangle \in \mathbb{Z}$$

or

$$\langle\Lambda|h_i\rangle = \langle\Lambda|h_i\rangle \in 2\mathbb{Z} \qquad (\text{if } \sigma\alpha_i = h_i).$$

Let $\pi_\Lambda : G \to L(V)$ be a finite dimensional representation with highest weight Λ, and let v_Λ be a highest weight vector. We may assume that V carries a positive definite Hermitian inner product $(\,,)$ such that the adjoint of $\pi_\Lambda(x)$ is $\pi_\Lambda(x)^* = \pi_\Lambda(\theta x^{-1})$.

Following Helgason [WI, p 210], let $G = KAN$ be the Iwasawa decomposition and write $x = \kappa(x)\exp H(x)\,n(x)$ for the corresponding decomposition of $x \in G$. Normalize the Haar measures on K and M so as to have total volume one, and normalize the Haar measure on N so that the following integration formula holds [WII, p 73]:

$$\int_K f(k)\,dk = \int_N \int_M f(\kappa(\bar{n})\,m)\,e^{-2\rho H(\bar{n})}\,dm\,dn \qquad (\bar{n} = \theta n^{-1}).$$

Since $\pi_\Lambda(AN) \cdot v_\Lambda \subseteq \mathbb{C}v_\Lambda$ we see that V is spanned by the $\pi_\Lambda(k) \cdot v_\Lambda (k \in K)$. Thus π_Λ is a class one representation if and only if

$$\bar{v}_0 = \int_K \pi_\Lambda(k) \cdot v_\Lambda\,dk \neq 0.$$

Assume (as we may) that v_Λ is a unit vector and set $c(\Lambda) = (\bar{v}_0, \bar{v}_0)$, so π_Λ is class one if and only if $c(\Lambda) \neq 0$. For convenience set $\bar{n} = \theta n^{-1}$. Now

$$(\pi_\Lambda(\kappa(\bar{n})\,m)\,v_\Lambda, v_\Lambda) = (\pi_\Lambda(\bar{n}\,n(\bar{n})^{-1}\exp(-H(\bar{n}))\,m)\,v_\Lambda, v_\Lambda) =$$

$$= (\pi_\Lambda(m)\,v_\Lambda, \pi_\Lambda(n)\,v_\Lambda)\,e^{-\Lambda H(\bar{n})} = (\pi_\Lambda(m)\,v_\Lambda, v_\Lambda)\,e^{-\Lambda H(\bar{n})}.$$

Thus we have

$$c(\Lambda) = \int_{K \times K} (\pi_\Lambda(k_1)\,v_\Lambda, \pi_\Lambda(k_2)\,v_\Lambda)\,dk_1\,dk_2 =$$

$$= \int_N \int_M (\pi_\Lambda(\kappa(\bar{n})\,m)\,v_\Lambda, v_\Lambda) e^{-2\rho H(\bar{n})}\,dm\,dn =$$

$$= \int_M (\pi_\Lambda(m)\,v_\Lambda, v_\Lambda)\,dm \cdot \int_N e^{-(\Lambda + 2\rho)H(\bar{n})}\,dn.$$

Now the second integral (over N) is always convergent and positive (the integrand is positive) [WII, p 73 and WI, p 215]. Thus $c(\Lambda) \neq 0$ if and only if the first integral (over M) is non-zero. But by Schur orthogonality, this integral is non-zero if and only if $\pi_\Lambda(m) \cdot v_\Lambda = v_\Lambda$ for all $m \in M$, and the hypotheses of the lemma are precisely the necessary and sufficient conditions that this hold. Namely $\Lambda|a_k \equiv 0$ if and only if v_Λ is fixed by M^0 and $\langle\Lambda|H_i\rangle \in 2\mathbb{Z}$ if and only if v_Λ is fixed by C.

Let L denote the set of linear functions $\Lambda : a_p \to \mathbb{R}$ satisfying the second condition of lemma 1. Define $\Lambda_1, \ldots, \Lambda_\ell \in L$ by

$$(\Lambda_i|\lambda_j)/(\lambda_j|\lambda_j) = \delta_{ij},$$

so L is the set of non-negative integral linear combinations of the $\Lambda_1, \ldots, \Lambda_\ell$. If $\Lambda \in L$ we shall denote by the same symbol the complex linear extension $\Lambda : a_c \to \mathbb{C}$ with $\Lambda|a_k \equiv 0$. Let $\pi_\Lambda : G \to L(V)$ be the finite dimensional class one representation with highest weight $\Lambda \in L$, and let $v_0 \in V$ be a K-fixed unit vector. The corresponding zonal spherical function is

$$\phi_\Lambda(x) = (\pi_\Lambda(x)\, v_0, v_0).$$

In particular we have the zonal spherical functions $\phi_1, \ldots, \phi_\ell$ corresponding to the fundamental spherical weights $\Lambda_1, \ldots, \Lambda_\ell$.

We shall need some information on the shape of the functions $\phi_\Lambda, \Lambda \in L$. First we define a partial ordering on L: if $\Lambda_1, \Lambda_2 \in L$ write $\Lambda_1 < \Lambda_2$ if $\Lambda_2 - \Lambda_1 \neq 0$ and is a non-negative integral linear combination of the positive roots in Σ (not just Σ_2). As usual, for $\Lambda \in L$, there are just a finite number of $\Lambda' \in L$ satisfying $\Lambda' < \Lambda$ [Hu, p 70]. Let W_s denote the Weyl group of g with respect to a_p [H, p 244]. For $\Lambda \in L$ define $\sigma_\Lambda : A \to \mathbb{R}$ $(A = \exp a_p)$ by

$$\text{(2)} \qquad \sigma_\Lambda(a) = [W(\Lambda)]^{-1} \Sigma \{e^{\langle w \cdot \Lambda|\log a\rangle} | \omega \in W_s\}$$

where $W(\Lambda) = \{w \in W | w \cdot \Lambda = \Lambda\}$ and $[X] =$ card X. Since $G = KAK$, we see that ϕ_Λ is uniquely determined by its restriction to A.

Lemma 3. *Let $\Lambda \in L$. Then there are constants $c(\Lambda) > 0$ and $b(\Lambda') \geq 0$ $(\Lambda' \in L, \Lambda' < \Lambda)$ such that*

$$\phi_\Lambda(a) = c(\Lambda)\, \sigma_\Lambda(a) + \Sigma_{\Lambda' \in L, \Lambda' < \Lambda}\, b(\Lambda')\, \sigma_{\Lambda'}(a)$$

for all $a \in A$.

Proof. Let $\pi_\Lambda : G \to L(V)$ be the finite dimensional class one representation with highest weight Λ, and suppose V is equipped with a positive definite Hermitian inner produt (,) such that the adjoint of $\pi_\Lambda(x)$ is $\pi_\Lambda(x)^* = \pi_\Lambda(\theta x^{-1})$. Let $\mu_1, \ldots, \mu_q$ be the distinct dominant a_p-weights in V and let $P(\mu_i)$ be the orthogonal projection onto the eigenspace in V corresponding to μ_i. Since M centralizes a_p, $P(\mu_i)$ commutes with the $\pi_\Lambda(m)$, $m \in M$; thus the expression

$$\pi_\Lambda(w)\, P(\mu_i)\, \pi_\Lambda(w^{-1}) = P(w \cdot \mu_i)$$

makes sense ($w \in W_s = M^*/M$ where M^* is the normalizer of a_p in K). Now the other weights of a_p in V are of the form $w \cdot \mu_i$ ($w \in W_s$, $i = 1, \ldots, q$) and it is easy to see that $P(w \cdot \mu_i)$ is the corresponding orthogonal projection.

Now let $v_0 \in V$ be a K-invariant unit vector. Then for each $i = 1, \ldots, q$, $P(\mu_i) \cdot v_0$ is an M-invariant eigenvector of $\exp(a_{pc})$. Thus we see that if $P(\mu_i) \cdot v_0 \neq 0$ then μ_i must satisfy the integrality condition of lemma 1, that is $\mu_i \in L$. On the other hand, it follows as usual that each $\Lambda - \mu_i$ is a non-negative integral linear combination of the positive roots of Σ^+ [Hu, p 108].

Now for $a \in A = \exp a_p$ we have

$$\phi_\Lambda(a) = (\pi_\Lambda(a)\, v_0, v_0) = \Sigma_{w \cdot \mu_i} (P(w \cdot \mu_i)\, v_0, v_0)\, e^{\langle w \cdot \mu_i|\log a\rangle} =$$

$$= \Sigma_{\Lambda'} \Sigma_{w \in W_s} [W(\Lambda')]^{-1} (\pi_\Lambda(w)\, P(\Lambda')\, \pi_\Lambda(w^{-1})\, v_0, v_0)\, e^{\langle w \cdot \Lambda'|\log a\rangle} =$$

$$= \Sigma_{\Lambda' \in L, \Lambda' \leq \Lambda} (P(\Lambda')\, v_0, v_0)\, \sigma_{\Lambda'}(a).$$

If $\Lambda' < \Lambda$ set

$$b(\Lambda') = (P(\Lambda')\, v_0, v_0) = (P(\Lambda')\, v_0, P(\Lambda')\, v_0) \geq 0.$$

On the other hand for $\Lambda' = \Lambda$, let v_Λ be a highest weight vector of unit length and set

$$c(\Lambda) = (P(\Lambda)\, v_0, P(\Lambda)\, v_0) = (v_0, v_\Lambda)^2 = \left\{\int_K (v_0, \pi_\Lambda(k)\, v_\Lambda)\, dk\right\}^2 =$$

$$= \int_{K \times K} (\pi_\Lambda(k_1)\, v_\Lambda, \pi_\Lambda(k_2)\, v_\Lambda)\, dk_1\, dk_2 = \int_N e^{-(\Lambda + 2\rho) H(\theta n^{-1})}\, dn > 0$$

(Here we have used the fact that the space of K-invariant vectors in V is one dimensional [WI, p 210]; the last equality was proved in the preceeding lemma.).

Let us note that the constant $c(\Lambda)$ is explicitly computable [WII, p 326]. Also the sums σ_Λ have another significance which is of some interest. Let A denote the semi-direct product of A and W_s (with W_s acting on A as usual). For any $\Lambda : a_p \to \mathbb{C}$ define σ_Λ by the same formula 2. Then the functions $\sigma_\Lambda(1)^{-1} \cdot \sigma_\Lambda$ are precisely the zonal spherical functions for A (with respect to the maximal compact subgroup W_s).

Let $\sigma_1, \ldots, \sigma_\ell$ be the sums (2) corresponding to the fundamental spherical weights $\Lambda_1, \ldots, \Lambda_\ell$.

Lemma 4. *The space of finite linear combinations of the $\sigma_\Lambda (\Lambda \in L)$ is a free polynomial algebra on the generators $\sigma_1, \ldots, \sigma_\ell$ (pointwise multiplication).*

Proof. (see [St, p 62]). If $\Lambda, \Lambda' \in L$ then

$$\sigma_\Lambda \cdot \sigma_{\Lambda'} = [W(\Lambda)]^{-1} [W(\Lambda')]^{-1} \Sigma_{s,\, w \in W} \exp w(\Lambda + s\Lambda').$$

Choose $w_1 \in W_s$ so that $M_s = w_1(\Lambda + s\Lambda')$ is dominant. Then $\Lambda + \Lambda' - M_s = (\Lambda - w_1\Lambda) + (\Lambda' - w_1 s\Lambda')$ is a sum of positive roots, and it is a non-trivial sum unless $w_1 \in W(\Lambda)$ and $w_1 s \in W(\Lambda')$ [Hu, p 68]. Similarly we see that $W(\Lambda + \Lambda') = W(\Lambda) \cap W(\Lambda')$. Now $M_s = \Lambda + \Lambda'$ if and only if $s = w_1^{-1}t$ for some $w_1 \in W(\Lambda)$ and $t \in W(\Lambda')$ and any decomposition of s like this is of form $s = (uw_1)^{-1}(ut)$ with $u \in W(\Lambda + \Lambda')$. Thus

$$\text{card}\,\{M_s | M_s = \Lambda + \Lambda'\} = [W(\Lambda)]\,[W(\Lambda')]/[W(\Lambda + \Lambda')].$$

From this we see that

$$\sigma_\Lambda \cdot \sigma_{\Lambda'} = \sigma_{\Lambda + \Lambda'} + \Sigma_{M < \Lambda + \Lambda'}\, c_M\, \sigma_M.$$

Now if $\Lambda \in L$ write $\Lambda = n_1\Lambda_1 + \ldots + n_\ell \Lambda_\ell$ so $\sigma_\Lambda = \sigma_1^{na}, \ldots, \sigma^{n_\ell}$ + sum of smaller ($<$) terms, and the result follows by induction on $<$.

Let us write $P(A/W_s)$ for the polynomial algebra generated by $\sigma_1, \ldots, \sigma_\ell$ of lemma 4. Also let $P(G//K)$ denote the vector space over $\mathbb{R}$ or $\mathbb{C}$ spanned by the zonal spherical functions ϕ_Λ, $\Lambda \in L$. The field was left vague in lemma 4 because it holds over any field (the c_M are integers). But for $P(G//K)$ we need the fact that $\mathbb{R}$ or $\mathbb{C}$ are of characteristic zero (the $c(\Lambda)$ and $b(\Lambda')$ of lemma 3 are rational with nontrivial denominators).

Corollary 5. *The space of finite linear combinations over $\mathbb{R}$ or $\mathbb{C}$ of the zonal spherical functions $\phi_\Lambda (\Lambda \in L)$ is a free polynomial algebra in the generators $\phi_1, \ldots, \phi_\ell$ (pointwise multiplication).*

Proof. [St, p 63]. Let $\beta : P(G//K) \to P(A/W_s)$ be the linear extension of $\beta(\phi_\Lambda) = \phi_\Lambda | A$. In lemma 3 we observed that β is injective and actually does take its values in $P(A/W_s)$. Thus to finish the proof we have only to show that β takes $\phi_1, \ldots, \phi_\ell$ onto generators of $P(A/W_s)$, or, what is the same thing, that each σ_i can be written as a polynomial in the $\beta(\phi_1), \ldots, \beta(\phi_\ell)$. But lemma 3 says that

$$\sigma_i = c(\Lambda_i)^{-1}\,\beta(\phi_i) - \Sigma_{\Lambda' < \Lambda}\, c(\Lambda_i)^{-1}\, b(\Lambda')\, \sigma_{\Lambda'}.$$

By induction on $<$, we may assume that $\sigma_{\Lambda'} (\Lambda' < \Lambda_i)$ can be written as a polynomial in the $\beta(\phi_j)$ with $\Lambda_j < \Lambda_i$.

Now we have a real analytic map $\phi : G \to \mathbb{R}^\ell$ defined by

$$\phi(x) = (\phi_1(x), \ldots, \phi_\ell(x)).$$

This map is far from surjective; since the coeficients of lemma 3 are non-negative and each $\sigma_\Lambda \geq 1$ on a positive Weyl chamber and $\phi_i(1) = 1$, we see that each $\phi_i(x) \geq 1$. Obviously $\phi(x)$ depends only on the K-double coset containing x, but also.

Corollary 6. *If $KxK \neq KyK$ $(x, y \in G)$ then $\phi(x) \neq \phi(y)$.*

Proof. The proof is identical to that of lemma 6.20 in [H, p 433]. (We may assume that $G \subseteq G_c$, a complex simply connected group, and then the matrix entries of finite dimensional representations of G are dense in the Schwartz topology on $C^\infty(G)$.

Next we will need to know about the Jacobian of the mapping ϕ.

Lemma 7. *Let $H_1, \ldots, H_\ell$ be a basis for a_p and define*

$$J(a) = \det(H_i \cdot \phi_j(a)) \qquad (a \in A).$$

Then

$$J(a) = c\,\Pi\,\{\sinh\langle\lambda|\log a\rangle | \lambda \in \Sigma_2^+\}$$

(no multiplicities in the product) where

$$c = 2^n\,\Pi\,\{c(\Lambda_j) | j = 1, \ldots, \ell\} \cdot \det\langle\Lambda_j|H_i\rangle \neq 0$$

$(n = [\Sigma_2^+])$.

The proof of this lemma is identical to that of lemma 8.2 in [St, p 70].

Let $g = k + p$ be the Cartan decomposition, so K acts on p via the adjoint representation of G. Let $I(p)$ be the algebra of K-invariant real valued polynomial functions on p, and let $I(a_p)$ be the real W_s-invariant polynomials on a_p. To apply Mather's theorem in this context we need.

Lemma 8. (*Chevalley*). (A) *The operation of restriction $p \to p|a_p$ maps $I(p)$ isomorphically onto $I(a_p)$* [H, p 430].

(B) *$I(a_p)$ is a free polynomial algebra in ℓ generators* [Wl, p 134].

(C) *$p_1, \ldots p_\ell \in I(a_p)$ are a (minimal) set of homogeneous generators if and only if*

$$\det(H_i \cdot p_j(H)) = c\Pi_{\lambda \in \Sigma_2^+} \langle\lambda|H\rangle \qquad (c \neq 0)$$

[C, p 134].

Finally we come to the promised version of Mather's theorem for G. Let $I(G)$ be the space of C_c^∞ functions f on G satisfying $f(k_1xk_2) = f(x)$ $(k_1, k_2 \in K, x \in G)$, equipped with the Schawartz topology.

Every $f \in I(G)$ can be expressed in the form

$$f(x) = F(\phi_1(x), \ldots, \phi_\ell(x)) = \phi^* F(x) \qquad (x \notin G)$$

for some $F \in C_c^\infty(\mathbb{R}^\ell)$. Moreover the map $\phi^* : C_c^\infty(\mathbb{R}^\ell) \to I(G)$ is a continuous split surjection.

Proof. Define the map $f \to \tilde{f} : I(G) \to I(a_p)$ (the space of W_s-invariant C_c^∞ functions on a_p) by $\tilde{f}(H) = f(\exp H)$. Let $p_1, \ldots, p_\ell$ be the minimal set of homogeneous generators of $I(a_p)$ in lemma 8, and define $p : a_p \to \mathbb{R}^\ell$ by $p(H) = (p_1(H), \ldots, p_\ell(H))$. Applying Mather's theorem in this context, we obtain a split surjection

$$F \mapsto F \circ p : C_c^\infty(\mathbb{R}^\ell) \to I(a_p).$$

We can also use Mather's theorem (the original version without the compact support condition) to write $\tilde{\phi}_i = \phi_i \circ p$ (where ϕ_i are the fundamental zonal spherical functions). Thus we obtain a C^∞ map $\phi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ with $\tilde{\phi} = \phi \circ p$. However, since the $\tilde{\phi}_i$ extend to entire functions on a_{pc}, it is easier simply to construct ϕ directly.

First note that lemma 8 and [H, p 429] imply that $p : a_{pc} \to \mathbb{C}^\ell$ is a (surjective, branched) convering with finite fibers. If $\Lambda \in L$ and ϕ_Λ is the corresponding zonal spherical function, let $\phi_{\Lambda i}$ denote the homogeneous component of degree i in the Taylor series of $\tilde{\phi}_\Lambda$. Then it is clear that each $\phi_{\Lambda i}$ is a W_s-invariant polynomial and $\Sigma\phi_{\Lambda i}$ converges to $\tilde{\phi}_\Lambda$ uniformly on compacta. Let $\phi_{\Lambda i}$ be the (unique) polynomial on $\mathbb{C}^\ell$ such that $\phi_{\Lambda i} = \phi_{\Lambda i} \circ p$. Then the series $\Sigma\phi_{\Lambda i}$ converges uniformly on compacta to give an entire function $\phi_\Lambda : \mathbb{C}^\ell \to \mathbb{C}$ such that $\tilde{\phi}_\Lambda = \phi_\Lambda \circ p$. Applying to $\phi_1, \ldots, \phi_e$, we obtain a holomorphic map $\phi : \mathbb{C}^\ell \to \mathbb{C}^\ell$ such that $\tilde{\phi} = \phi \circ p$.

Next we note that the Jacobian determinant of ϕ is given by

$$J\phi = c\,\Pi\,\{\sinh \lambda/\lambda | \lambda \in \Sigma_2^+\} \qquad (c \neq 0).$$

In particular $J\phi$ does not vanish on a neighborhood of the image R of $p : a_p \to \mathbb{R}^\ell$, so ϕ is locally one-to-one near R. Corollary 6 shows that ϕ is one-to-one on R, and lemma 3 implies that ϕ is proper on R. We wish to find a neighborhood of R on which these properties still hold.

Let $d(x, y) = |x - y|$ on $\mathbb{R}^\ell$ and set

$$U_{r,n} = \left\{ x ; |x| \leq r \text{ and } d(x, R) \leq \frac{1}{n} \right\}.$$

Fix $r = r_1$ and suppose ϕ is not one-to-one on $R \cup U_{r,n}$ for all n. Then we choose $x_n \in U_{r,n}$ and $y_n \in R \cup U_{r,n}$ such that $\phi(x_n) = \phi(y_n)$ and $x_n \neq y_n$ for all n. Now clearly ϕ is proper on $R \cup U_{r,n}$ so the y_n lie in the compact set $\phi^{-1}(\phi(U_{r,n})) \cap (R \cup U_{r,n})$. Thus we may as well assume that x_n converges to x and y_n converges to y in R. But then $\phi(x) = \phi(y)$ so $x = y$. Since $J\phi(x) \neq 0$ we can choose a neighborhood of x on which ϕ is one-to-one. But then $x_n = y_n$ for n sufficiently large, contrary to hypothesis. Thus we can choose $U_r = U_{r,n}$ (n large) so that ϕ is one-to-one and proper on $R \cup U_r$. Replace R by $R \cup U_r$ and repeat the argument with $r_2 > r_1$, etc. Using induction and a sequence $r_i \to \infty$, we obtain a neighborhood $U = \cup U_{r_i}$ of R such that ϕ is one-to-one on U.

Now $\phi(U)$ is a neighborhood of the image $\phi(a_p)$, and $\phi^{-1} : \phi(U) \to U$ is defined and analytic. Thus we see that

$$F \mapsto F \circ \phi \circ p = F \circ \tilde{\phi} : C_c^\infty(\mathbb{R}^\ell) \to I(a_p)$$

is a split surjection. In particular for $f \in I(G) = C_c^\infty(G//K)$ we have

$$f(\exp H) = F \circ \phi(\exp H) \qquad (\text{for all } H \in a_p)$$

which implies that $f(x) = F \circ \phi(x)$ for all $x \in G$. The splitting of the surjection

$$F \mapsto F \circ \phi : C_c^\infty(\mathbb{R}^\ell) \to I(G)$$

is given by $f \to \eta(\tilde{f})$ where $\eta : I(a_p) \to C_c^\infty(\mathbb{R}^\ell)$ is the splitting for $\tilde{\phi}^*$ (i.e. $\tilde{\phi}^* \circ \eta = id$).

Note that the proof above also shows that any W_s-invariant C_c^∞ function on $A = \exp a_p$ can be extended to a (unique) function in $I(G)$.

2. The Proof of Lemma 34.

Of course, there is no lemma 34, but *I* thought this was a very catchy title. The object of this section is to obtain a simple lemma about an integral transform; it appears in [Wa 2]. It is amusing to set the lemma in the context of Euclidean motion groups.

Recall that the Euclidean motion group H is the group of isometries of $\mathbb{R}^n$ with respect to the standard distance $|a - b|$; we shall consider the identity component. Fixing an origin in $\mathbb{R}^n$, we can realize H as the semi-direct product of the compact group $K = SO(n)$ of rotations around the origin and the normal subgroup $P \simeq \mathbb{R}^n$ of translations; the multiplication is

$$(k, x) \cdot (k_1, x_1) = (kk_1, k_1^{-1} \cdot x + x_1) \qquad (k, k_1 \in K, x, x_1 \in P).$$

Let A be a one dimensional vector subspace of P, let $H : \mathrm{P} \to A$ be the orthogonal projection, let N be the kernel of H, and extend H, to the group $H = K \times P$ by $H(k, x) = H(x)$.

Recall that a zonal spherical function ϕ on H is a function of the form $\phi(x) = (\pi(x)\, v_0, v_0)$ $(x \in H)$ where $\pi : H \to L(V)$ is an irreducible (TCI) representation (not necessarily unitary) on a Hilbert space V, and v_0 is a unit vector which spans the space of K-invariant vectors in V. In the present context we can also characterize them as functions satisfying $\phi_\Lambda(kxk_1) = \phi_\Lambda(x)$ $(k, k_1 \in K, x \in H)$, $\phi_\Lambda(1) = 1$ and $\phi_\Lambda|P$ is a rotation invariant eigenfunction of the Laplace operator with eigenvalue $-\Lambda^2$ [H, p 401]. Thus ϕ_Λ is some sort of Bessel function [V, p 553]:

$$\phi_\Lambda(x) = \Gamma(n/2)\,(2/\Lambda r)^{(n-2)/2} J_{(n-2)/2}(\Lambda r)$$

where $x \in P$, $r^2 = |x|^2 = x_1^2 + \ldots + x_n^2$ and $\Lambda \in \mathbb{C}$. Again, if $h \in A$ is a unit vector and if we identify $\Lambda \in \mathbb{C}$ with the linear function $\Lambda : A \to \mathbb{C}$, $th \to \Lambda t$, then we have the formula

$$\phi_\Lambda(x) = \int_K e^{i\Lambda H(xk)}\, dk.$$

Now the ring of K-invariant polynomials on P is $\mathbb{R}[r^2]$ where $r^2(x) = |x|^2 = x_1^2 + \ldots + x_n^2$. Mather's theorem the tells us that we can write any $f \in I(H) = C_c^\infty(H//K)$ in the form $f(x) = F(r^2)$ for some $F \in C_c^\infty(\mathbb{R})$. Now the spherical transform of f is given by

$$\phi_\Lambda(f) = \int_H \phi_\Lambda(x) f(x)\, dx =$$

$$= (2\pi)^{n/2}\, \Lambda^{1-n/2} \int_0^\infty J_{(n-2)/2}(\Lambda r)\, F(r^2)\, r^{n/2}\, dr =$$

$$= \int f(x)\, e^{i\Lambda H(x)}\, dx =$$

$$= \int_{-\infty}^{\infty} e^{i\Lambda t} \cdot vol\,(S^{n-2}) \int_0^\infty F(t^2 + u^2)\, u^{n-2}\, du\, dt.$$

(Here we have used integration in polar coordinates twice, on the second line for $P \simeq \mathbb{R}^n$ and on the fourth line for $N \simeq \mathbb{R}^{n-1}$. Also u represents the radius in N and t is a parameter for A as above). Motivated by the last line, we now make the.

Definition. If $f \in C_c^\infty(\mathbb{R}^+)$, $\mathbb{R}^+ = [0, \infty)$, define

$$T_n(f)(t) = vol\,(S^{n-1}) \int_0^\infty f(t + u^2)\, u^{n-1}\, du.$$

The following lemma appears in [Wa 2].

Lemma 10. *$T_n : C_c^\infty(\mathbb{R}^+) \to C_c^\infty(\mathbb{R}^+)$ and $T_n \circ D = D \circ T_n$ $(D = d/dt)$. Moreover*

(1) $-1/\pi D \circ T_n = T_{n-2}$,

(2) *$T_{2n} : C_c^\infty(\mathbb{R}^+) \to C_c^\infty(\mathbb{R}^+)$ is bijective with inverse $(-1/\pi D)^n$,*

(3) *T_{2n+1} is bijective with inverse*

$$(-1/\pi D)^{n+1} \circ T_1 = T_1 \circ (-1/\pi D)^{n+1}.$$

Proof. The first statement is clear; in fact if supt $f \subseteq [0, a]$ then supt $T_n(f) \subseteq [0, a]$. Also $T_n \circ D = D \circ T_n$ says merely that it is permissible to differentiate under the integral sign (and $D_t(t + u^2) = 1$). The assertion (1) is just integration by parts, once you recall the formula for the volume of the sphere, $vol\,(S^{n-1}) = 2\pi^{n/2}\Gamma(n/2)^{-1}$. For (2) we use (1) and

$$(-1/\pi)\, DT_2(f)(t) = 2\int_0^\infty -f'(t + u^2)\, u\, du = f(t).$$

For (3) we use (1) and $T_1 \circ T_1 = T_2$ and then (2):

$$T_1^2(f)(x) = 4\int_0^\infty \int_0^\infty f(x + u^2 + v^2)\, du\, dv = T_2(f)(x).$$

As a corollary we have the following Paley-Weiner theorem for Bessel functions.

Corollary 11. *Let ψ be a complex valued function on $\mathbb{C} = \{\Lambda | \Lambda : A \to \mathbb{C}$ is real linear$\}$. Then ψ is the spherical transform of some $f \in I(H)$ if and only if*

(1) *ψ is holomorphic and $\psi(\Lambda) = \psi(-\Lambda)$.*

(2) *For each $m = 0, 1, 2, \ldots$ there is a constant $C_m > 0$ such that*

$$|\psi(\Lambda)| \leq C_m(1 + |\Lambda|)^{-m} \exp\,(2\pi R\, |Im\, \Lambda|).$$

Proof. As we have seen, the spherical transform of $f \in I(H)$ is given by

$$\phi_\Lambda(f) = \int_{-\infty}^{\infty} e^{i\Lambda t}\, T_{n-1}(F)(t^2)\, dt.$$

Now by lemma 10, $T_{n-1}(F)(t^2)$ is an arbitrary C_c^∞ function of t^2, and by Mather's theorem applied to the group $t \to \pm t$ acting on the line $\mathbb{R}$, we see that $t \to T_{n-1}(F)(t^2)$ is an arbitrary even C_c^∞ function. Now the result follows from the classical Paley-Weiner theorem for the line.

Note that by the first line of the proof of lemma 10, the R which occurs in corollary 11 can be taken to be

$$R = \inf\{r \mid f(x) = 0 \text{ whenecer } |x| \geq r\}.$$

It is also possible to obtain the Fourier inversion formula for these Bessel transforms (by methods similar to those of the next section) and to study other function spaces (by extending the domain of T_n); however I choose not to do so.

3. The Paley-Weiner theorem for rank one semi-simple groups.

The results of this section (as well as the previous one) are due to Nolan Wallach [Wa 2]. Let G be a semi-simple Lie group and fix an Iwasawa decomposition $G = KAN$ and write $x = \kappa(x) \exp H(x)\, n(x)$ accordingly ($x \in G$, $\kappa(x) \in K$, $H(x) \in a_p$, $n(x) \in N$). We assume that G is connected and $\dim A = 1$. We also have the Cartan decomposition $G = KAK$, so define $A(x)$ by $x = k_1 \exp A(x)\, k_2$ ($x \in G$, k_1, $k_2 \in K$, $A(x) \in a_p$; note $A(x)$ is only determined up to the action of W_s, up to $a \pm$ sing).

As usual, for $H \in a_p$ set $2\rho(H) = tr(ad\, H|\eta)$. Then the zonal spherical functions are given by

$$\phi_\Lambda(x) = \int_K e^{(i\Lambda - \rho)H(xk)}\, dk$$

[H, p 432]. Using the integral formula associated with the Iwasawa decomposition [H, p 373], for $f \in I(G) = C_c^\infty(G//K)$, we have

$$\phi_\Lambda(f) = \int \phi_\Lambda(x)\, f(x)\, dx =$$

$$= \int_{-\infty}^{\infty} e^{i<\Lambda|H>t}\, e^{<\rho|H>t} \int_\eta f(\exp tH \exp X)\, dX\, dt.$$

Thus define [H, p 378]

$$F_f(t) = e^{<\rho, H>t} \int_\eta f(\exp tH \exp X)\, dX.$$

We wish to study the Abel transform $f \mapsto F_f : I(G) \to I(a_p)$ in some detail. (In the case where $G = SO\ (n; 1)$ it can be shown [V, p 524] that

$$\phi_\Lambda(\exp tH) = (\tfrac{1}{2} \sinh t)^{1-n/2}\, \Gamma(n/2)\, \mathscr{P}^{1-n/2}_{i\Lambda - 1/2}(\cosh t)$$

where $\langle \lambda | H \rangle = 1$ and $\mathscr{P}$ is the associated Legendre function and we have identified $i\Lambda$ with $\langle i\Lambda | H \rangle \in \mathbb{C}$. If we write $f(\exp tH)\ U\ F(\cosh t)$ and make use of the integral formula associated with the KAK decomposition [H, p. 382] we find that

$$\phi_\Lambda(f) = 2^{n-1/2}(n-1)^{n/2 - 1/2}\pi^{n/2} \int_1^\infty (x^2 - 1)^{(n-2)/4}\, \mathscr{P}^{1-n/2}_{i\Lambda - 1/2}(x)\, F(x)\, dx.)$$

In order to compute F_f, we will apply Mather's theorem. Referring to the proofs of corollary 5 and theorem 9, we see that any $f \in I(G)$ can be written in the form $f = F \circ \sigma$ where $F \in C_c^\infty(\mathbb{R})$ and σ is the K-biinvariant extension to G of the function

$$\sigma(a) = \cosh \langle \Lambda_1 | \log a \rangle \quad (a \in A)$$

(where Λ_1 is the fundamental zonal weight). Now let $\Sigma^+ = \{\lambda, \lambda/2\}$ be the set of positive restricted roots (so $\lambda/2$ may not be present). Define $p = m(\lambda/2) = \dim \eta^{\lambda/2}$ and $q = m(\lambda)\ U \dim \eta^\lambda$. Define the quadratic form Q on the Lie algebra g of G by

$$Q(X) = -(2p + 8q)^{-1} B(X, \theta X)$$

(where B is the Cartan Killing form; note that p may be zero but $q > 0$). Note that $\Lambda_1 = \lambda$. It is convenient to parameterize A by $\exp tH$ where $\langle \Lambda_1 | H \rangle = 2$.

Lemma 12. (I) *If $p = 0$ and $X \in \eta^\lambda$ then*

$$\sigma(\exp tH \exp X) = \cosh 2t + e^{2t} Q(X).$$

(II) *If $p \neq 0$, $X \in \eta^{\lambda/2}$ and $Y \in \eta^\lambda$ then*

$$\sigma(\exp tH \exp(X + Y)) = 2\left(\cosh t + \frac{1}{4} Q(X)\, e^t\right)^2 - 1 + Q(Y)\, e^{2t}.$$

Proof. (I) Fix $X \in \eta^\lambda$. Then X, θX and H span a Lie subalgebra of g which is isomorphic to $sl(2, \mathbb{R})$. Now we can assume that G is embedded in a simply connected complex group. Then there is a homomorphism $\beta : S\ell(2, \mathbb{R}) \to G$ such that $\beta \circ \theta = \theta \circ \beta$ and

$$\beta\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \exp tH \ (a = e^t) \text{ and } \beta\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \exp X$$

where q is determined (almost) by

$$Q(X) = Q_1 \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} = \frac{1}{2} q^2.$$

(Note that $Q\circ\beta$ is a multiple of the corresponding function Q_1 on $sl(2,\mathbb{R})$ and $Q(H) = Q_1\begin{pmatrix}1&0\\0&-1\end{pmatrix} = 1$ so the constant is one.) Now $\sigma\circ\beta$ is a K-biinvariant function on $S\ell(2,\mathbb{R})$ (K_1 the maximal compact subgroup) which agrees on A with

$$\cosh\langle\Lambda|\log a\rangle = \frac{1}{2}(a^2+a^{-2}) = \frac{1}{2}\left\|\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}\right\|^2_{HS}$$

Thus

$$\sigma(\exp tH\ \exp X) = \frac{1}{2}\left\|\begin{pmatrix}a&aq\\0&a^{-1}\end{pmatrix}\right\|^2_{HS} =$$

$$= \cosh\langle\Lambda|\log a\rangle + \frac{1}{2}e^{\langle\Lambda|\log a\rangle}q^2 = \cosh 2t + e^{2t}Q(X).$$

(II) Now suppose $p\neq 0$ and fix $X\in\eta^{\lambda/2}$ and $Y\in\eta^{\lambda}$. Then like above [Wa 1, p 257] there is a homomorphism $\beta: SU(2,1)\to G$ such that $\beta\circ\theta = \theta\circ\beta$ and

$$\beta(\exp tH_1) = \exp tH,\quad \beta(\exp rA) = \exp X,\quad \beta(\exp isB) = \exp Y$$

where

$$H_1 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad A = \begin{pmatrix}0&-1&1\\1&0&0\\1&0&0\end{pmatrix},\quad B = \begin{pmatrix}0&0&0\\0&-1&1\\0&-1&1\end{pmatrix}.$$

Similarly to the above we find that r and s are determined by

$$2r^2 = Q(X) \text{ and } 2s^2 = Q(Y).$$

Now again $\sigma\circ\beta$ is biinvariant with respect to the maximal compact subgroup of $SU(2,1)$ and agrees on A with

$$\cos h\langle\Lambda|tH_1\rangle = \cos h2t = \frac{1}{2}\left\|\begin{pmatrix}1&0&0\\0&\cosh t&\sinh t\\0&\sinh t&\cosh t\end{pmatrix}\right\|^2_{HS} - \frac{1}{2}.$$

We find that $\exp(rA+isB) = 1 + rA + \left(\frac{1}{2}r^2 + is\right)B$ and then, after a straightforward but slightly arduous computation, that

$$\sigma(\exp tH\exp(X+Y)) = \frac{1}{2}\left\|\exp tH_1\left(1 + rA + \left(\frac{1}{2}r^2 + is\right)B\right)\right\|^2_{HS} - \frac{1}{2} =$$

$$= \cosh 2t + r^2(e^{2t}+1) + \frac{1}{2}r^4e^{2t} + 2s^2e^{2t} =$$

$$= 2\left(\cosh t + \frac{1}{4}Q(X)e^t\right)^2 - 1 + Q(Y)e^{2t}.$$

(Hint: where $a = \exp tH_1$, first show that a, $a\cdot A$ and $a\cdot B$ are orthogonal.)

Now we can compute $F_f(t)$ (defined at the beginning of this section). If $f\in I(G)$ write $f = F\circ\sigma$ ($F\in C_c^\infty(\mathbb{R})$). We normalize the Lebesgue measure on η^λ and $\eta^{\lambda/2}$ to be the Euclidean measures corresponding to the Euclidean structures defined by Q.

Lemma 13. (i) *If* $p = 0$ *then*

$$F_f(t) = T_q(F)(\cosh 2t)$$

(ii) *If* $p\neq 0$ *then*

$$F_f(t) = 2^pT_p((T_qF)\circ U)(\cosh t),\ \textit{where}\ U(x) = 2x^2-1.$$

Proof. (i) First note that $\langle\rho|H\rangle = q$. We compute the integral over η in polar coordinates. Thus

$$F_f(t) = e^{qt}\int_\eta F(\cosh 2t + e^{2t}Q(X))\,dX =$$

$$= e^{qt}\,vol(S^{q-1})\int_0^\infty F(\cosh 2t + e^{2t}r^2)\,r^{q-1}\,dr =$$

$$= vol(S^{q-1})\int_0^\infty F(\cosh 2t + u^2)\,u^{q-1}\,du = T_q(F)(\cosh 2t)$$

(where $r = Q(X)$ and $u = e^tr$).

(ii) Now suppose $p\neq 0$, so $\langle\rho|H\rangle = \frac{1}{2}p + q$. Let $r^2 = Q(X)$ and $s^2 = Q(Y)$. Then

$$F_f(t) = e^{(\frac{1}{2}p+q)t}\int_\eta F\left(2\left(\cosh t + \frac{1}{4}e^tQ(X)\right)^2 - 1 + Q(Y)e^{2t}\right)dX\,dY =$$

$$= e^{(\frac{1}{2}p+q)t}\,vol(S^{p-1})\,vol(S^{q-1})\times$$

$$\times \int_0^\infty \int_0^\infty F(2(\cosh t + \tfrac{1}{4} e^t r^2)^2 - 1 + e^{2t} s^2)\, r^{p-1} s^{q-1}\, dr\, ds =$$

$$= 2^p\, vol\,(S^{p-1})\, vol\,(S^{q-1}) \times$$

$$\times \int_0^\infty \int_0^\infty F(2(\cosh t + r^2)^2 - 1 + s^2)\, r^{p-1} s^{q-1}\, dr\, ds =$$

$$= 2^p\, T_p((T_q F) \circ U)\, (\cosh t).$$

Corollary 14. *A complex valued function* ψ *on* $\{\Lambda | \Lambda : a_p \to \mathbb{C}$ *is real linear*$\}$ *is the sphericall transform of some* $f \in I(G)$ *if and only if*

(1) ψ *is entire and* $\psi(\Lambda) = \psi(-\Lambda)$.

(2) *For all* $m \geq 0$ *in* $\mathbb{Z}$ *there is a* $C_m > 0$ *such that*

$$|\psi(\Lambda)| \leq C_m (1 + |\Lambda|)^{-m} \exp(2\pi R\, |Im\, \Lambda|),$$

for all Λ.

Proof. When $p = 0$ one merely has to observe that $T_q(F)(\cosh 2t)$ is an arbitrary C_c^∞ function of $\cosh 2t$, and by Mather's theorem (applied to W_s acting on a_p) this is an arbitrary even C_c^∞ function of t. When $p \neq 0$. Note first that $\cosh t + r^2 \geq 1$ always, and $U : x \mapsto 2x^2 - 1$ is a diffeomorphism on a neighborhood, say $(\frac{1}{2}, \infty)$, of this domain. Thus $(T_q F) \circ U$ is a typical function in $C_c^\infty([\frac{1}{2}, \infty))$ and then, as above, $F_f(t)$ is an arbitrary even function of $t \in \mathbb{R}$. Now the result follows from the classical Paley-Wiener theorem upon observing that

$$\phi_\Lambda(f) = \int_{-\infty}^{\infty} e^{i\langle\Lambda|H\rangle} F_f(t)\, dt.$$

If one exercises more care in keeping track of the supports, it can be shown that the R which figures in the corollary is given by

$$R = \sup\{t |\, f(\exp tH) \neq 0\}.$$

The proof given above is due to Wallach [Wa 2] and the result is originally due to Helgasson [H 2] and Gangoli [G].

4. The zonal Plancherel theorem in rank one.

The objective of this section is to obtain the Plancherel measure for inversion of the spherical transform on a rank one semi-simple group. The result, of course, is due to Harish-Chandra [WII, p 338, 303]; the point here is that we obtain it by entirely elementary methods.

Let G be a connected rank one semi-simple group, and, without loss of generality, assume that G is a real subgroup of a simply connected complexification. Let $\mathcal{F} = \{\Lambda : a_p \to \mathbb{R} | \Lambda$ is real linear$\}$. Recall that the Plancherel measure is a positive measure $\mu(\nu)\, d\nu$ on $\mathcal{F}$ such that (with $\Lambda = \nu\Lambda_1$)

$$f(x) = \int_{\mathcal{F}} \phi_\nu(f)\, \overline{\phi_\nu(x)}\, \mu(\nu)\, d\nu$$

for all $f \in I(G)$. Note that it suffices to obtain the formula for $x = 1$. Then for any other x consider the function

$$f_x(y) = \int_K f(xky)\, dk.$$

We have $f(x) = f_x(1)$ and $\phi_\nu(f_x) = \phi_\nu(f)\, \phi_\nu(x^{-1}) = \phi_\nu(f)\, \overline{\phi_\nu(x)}$, so the result for $y = 1$ implies the formula for all x.

Now since F_f is an even function we change the formula for $\phi_\nu(f)$ into a Fourier cosine transform:

$$\phi_\nu(f) = 2 \int_0^\infty F_f(t) \cos(2\nu t)\, dt$$

Then we may invert the cosine transform to obtain

$$F_f(t) = \frac{2}{\pi} \int_0^\infty \phi_\nu(f) \cos(2\nu t)\, d\nu.$$

The idea is to apply the inversion formulas of lemma 10 to the formulas of lemma 13, and apply these operations on t under the integral sign in the proceeding expression. This is permissible since $\phi_\nu(f)$ is rapidly decreasing in ν (corollary 14). The problem splits into four cases according to the parity of q and $\frac{1}{2}p$.

For the first case, assume that $p = 0$ and q is even. Define ϕ_0 on $\mathcal{F} \times \{z \in \mathbb{R} | z \geq 1\}$ by

$$\phi_0(\nu, \cosh 2t) = \cos(2\nu t)$$

Then we have

$$F_f(t) = T_q(F)\,(\cosh 2t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_\nu(f)\, \phi_0(\nu, \cosh 2t)\, d\nu$$

and with $q = 2n$

$$f(1) = F(\cosh 0) = \left(-\frac{1}{\pi}\frac{d}{dz}\right)^n T_{2n}(F)(z)|_{z=1} =$$

$$= \frac{1}{\pi}\left(-\frac{1}{\pi}\right)^n \int_{-\infty}^{\infty} \phi_\nu(f)\, D_1^n \cdot \phi_0(\nu, z)\, d\nu$$

(where $D_1^n \cdot g(z) = ((d/dz)^n \cdot g)(1)$). Thus to compute the Plancherel measure in this case we must find the Taylor series at $z = 1$ of $\phi_0(\nu, z) = \cos(\nu \cosh^{-1} z)$.

To find the Taylor series of ϕ_0 we use.

Lemma. 16. *$\phi_0(\nu, z)$ satisfies the ordinary differential equation*

$$(z^2 - 1)\phi_0'' + z\phi_0' + \nu^2\phi_0 = 0$$

$$\phi_0(\nu, 1) = 1,\ \phi_0'(\nu, 1) = -\nu^2.$$

Moreover if $\phi_0^{(k)}$ denotes the kth derivative of ϕ_0 with respect to z, then there is a constant $C_k > 0$ such that

$$|\phi_0^{(k)}(\nu, z)| \le C_k(1 + |\nu|)^{3k}(1 + z)^{-k}$$

for all $\nu \in \mathscr{F}$ and all $z \ge 1$.

Proof. We have

$$\cos 2\nu t = \phi_0(\nu, \cosh 2t) - 2\nu \sin 2\nu t = \phi_0'(\nu, \cosh 2t)\, 2\sinh 2t - 4\nu^2 \cos 2\nu t =$$

$$= -4\nu^2\phi_0(\nu, \cosh 2t) = \phi_0''(\nu, \cosh 2t)\, 4\sinh^2 2t + \phi_0'(\nu, \cosh 2t)\, 4\cosh 2t.$$

Now the differential equation follows upos replacing $\cosh 2t$ by z in the fourth equality above, and the initial condition results from applying l'Hospital's rule to the second equality.

As for the inequality of the lemma, the cases $k = 0$ and $k = 1$ follow easily from the first two equations above. For general k, assume first that $z \ge a > 1$; in this case we can obtain the stronger inequality

$$(17) \qquad |\phi_0^{(k)}(\nu, z)| \le \left(\frac{a+1}{a-1}\right)^k C_k(1 + |\nu|)^k (1 + z)^{-k}$$

(where $C_k > 0$ depends only on k). To get this, we change the differential equation into a recursion relation by differentiating with respect to z, k times:

$$(18) \qquad (z^2 - 1)\phi_0^{(k+2)} + (2k + 1)z\phi_0^{(k+1)} + (\nu^2 + k^2)\phi_0^{(k)} = 0.$$

Now for $z \ge a > 1$ we have

$$1/(z^2 - 1) \le (a + 1)/(a - 1)(1 + z)^2.$$

Assuming the inequality (17) for k and $k + 1$ we have

$$|\phi_0^{(k+2)}(\nu, z)| \le (z^2 - 1)^{-1}\{(2k + 1)z|\phi_0^{(k+1)}(\nu, z)| + (\nu^2 + k^2)|\phi_0^{(k)}(\nu, z)|\} \le$$

$$\le \frac{a+1}{a-1}(1 + z)^{-2}\left\{(2k + 1)(1 + z)C_{k+1}\left(\frac{a+1}{a-1}\cdot\frac{1+|\nu|}{1+z}\right)^{k+1} + \right.$$

$$\left. + (\nu^2 + k^2)C_k\left(\frac{a+1}{a-1}\cdot\frac{1+|\nu|}{1+z}\right)^k\right\} \le$$

$$\le \left(\frac{a+1}{a-1}\right)^{k+2}\{(2k + 1)C_{k+1}(1 + |\nu|)^{k+1} +$$

$$+ (\nu^2 + k^2)C_k(1 + |\nu|)^k\}(1 + z)^{-k-2}$$

The inequality (17) now follows by induction. To obtain the inequality of the lemma near $z = 1$ we must first solve the differential equation.

Lemma 19. *ϕ_0 has the Taylor expansion, valid for $|z - 1| < 2$, given by*

$$\phi_0(\nu, z) = \sum_{n=0}^{\infty} a_n(z - 1)^n$$

where

$$a_n = \frac{\Gamma(n + i\nu)\,\Gamma(n - i\nu)\,(-1)^n\sqrt{\pi}}{\Gamma(i\nu)\,\Gamma(-i\nu)\,n!\,\Gamma\left(n + \frac{1}{2}\right)2^n}.$$

Proof. One merely applies the standard theory of regular singular points to the differential equation in lemma 16 at $z = 1$ [WW, p 197]. The indicial equation reads $(2\alpha - 1)\alpha = 0$ and we must take $\alpha = 0$ to satisfy the initial conditions. Then the recursion relation is

$$a_{n+1} = -a_n \cdot (\nu^2 + n^2)/(2n + 1)(n + 1)$$

which gives the result. (Alternatively, note that the substitution $w = \frac{1}{2}(1 - z)$ changes the differential equation of lemma 16 into the hypergeometric differential equation, and the the initial conditions imply that

$$\phi_0(v, z) = {}_2F_1(iv, -iv, \tfrac{1}{2}; \tfrac{1}{2}(1 - z)).$$

See [WW, p 283].)

Completion of the proof of lemma 16. Now we note that for $z > 1$ and sufficiently near one, the Taylor series for $\phi_0^{(k)}$ is an alternating series. Explicitly, the ratio of successive terms is given by

$$(n+1)(z-1)a_{n+1}/(n+1-k)a_n = (-1)\tfrac{1}{2}(z-1)(n^2+v^2)/(n+1-k)(n+\tfrac{1}{2})$$

Now it is not hard to see that

$$(n^2+v^2)/(n+1-k)\left(n+\frac{1}{2}\right) \le (k^2+v^2)\Big/\left(k+\frac{1}{2}\right) \qquad (n \ge k)$$

for $k \ge 2$ (graph it). Thus we see that when

$$0 \le (z-1) \le |(k+1/2)/(k^2+v^2)|$$

then the Taylor series for $\phi_0^{(k)}$ is an alternating series. In particular $|\phi_0^{(k)}(v,z)|$ is dominated by the first non-zero term (at $n = k$), that is

$$|\phi_0^{(k)}(v,z)| \le \frac{\Gamma(k+iv)\,\Gamma(k-iv)\,\sqrt{\pi}}{\Gamma(iv)\,\Gamma(-iv)\,\Gamma\left(k+\frac{1}{2}\right)2^k} \le$$

$$\le (k^2+v^2)^k \le k^{2k}(1+|v|)^{2k}$$

for $k \ge 2$, $1 \le z \le a = |(k+1/2)/(k^2+v^2)| + 1$.

This gives the inequality of lemma 16 when $1 \le z \le a$. To get it for all other values of z we substitute this value of a into the inequality (17). Note that

$$(a+1)/(a-1) = \left(k+\frac{1}{2}+2k^2+v^2\right)\Big/\left(k+\frac{1}{2}\right)$$

and this, together with (17), gives the result of lemma 16.

Note that the inequality in lemma 16 justifies performing the differentiation under the integral sign in (15). Before we state the Plancherel theorem in this case, $p = 0$ and q even, let us note that we have implicitly chosen a Haar measure for G. Namely $dx = e^{2\rho}\,dk\,da\,dn$ where dk has total volume 1 and da and dn are the transport of the Euclidean measure determined by Q.

Lemma 20. *For $G = SO(2n+1, 1)$ the zonal Plancherel measure is given by*

$$f(1) = \int_{\mathcal{F}} \phi_v(f)\,\mu(v)\,dv$$

where

$$\mu(v) = \pi^{-n-\frac{1}{2}} \frac{\Gamma(n+iv)\,\Gamma(n-iv)}{\Gamma(iv)\,\Gamma(-iv)\,\Gamma\left(n+\frac{1}{2}\right)2^n}.$$

Proof. Use the formula for a_n of lemma 19 in formula 15.

Now let us turn to the next case, $p = 0$ and $q = 2n+1$ is odd. The formula analogous to (15) is

$$(21) \qquad f(1) = \left(-\frac{1}{\pi}\frac{d}{dz}\right)^{n+1} T_1\, T_{2n+1}(F)(z)|_{z=1} =$$

$$= 2(-1)^{n+1}\left(\frac{1}{\pi}\right)^{n+2}\int_{-\infty}^{\infty}\phi_v(f)\left(\frac{d}{dz}\right)^n \int_0^{\infty} \phi_0'(v, z+u^2)\,du|_{z=1}\,dv.$$

The inequality in lemma 16 implies that it is permissible to exchange the order of integration with respect to v and u and to differentiate under the integral sign. Explicitly, set

$$\phi_1(v,z) = \int_0^{\infty} \phi_0'(v, z+u^2)\,du.$$

Then lemma 16 gives us

$$(2) \qquad \left|\left(\frac{d}{dz}\right)^n \phi_1(v,z)\right| \le \int_0^{\infty} |\phi_0^{(n+1)}(v, z+u^2)|\,du \le$$

$$\le C_{n+1}(1+|v|)^{3n+3}\int_0^{\infty} du/(1+z+u^2)^{n+1} \le$$

$$\le C_n'(1+|v|)^{3n+3}/(1+z)^{n+\frac{1}{2}}$$

(where, in the first inequality, we have interchanged the order of integration with respect to u and differentiation with respect to z, which is again justified by the inequality of lemma 16.) Thus again we can find the Plancherel measure in this case by computing the Taylor series for ϕ_1.

Lemma 23. *The first Taylor coefficient for* ϕ_1 *is*

$$\phi_1(v, 1) = -\frac{1}{\sqrt{2}} \int_0^\infty \frac{v \sin 2vt}{\sinh t}\, dt = -\frac{\pi}{2\sqrt{2}}\, v \tanh \pi v.$$

Proof. We have

$$\phi_1(v, 1) = \int_0^\infty \phi_0'(v, 1 + u^2)\, du$$

and

$$\phi_0'(v, \cosh 2t) = -\, v \sin(2vt)/\sinh 2t.$$

The first equality follows upon making the substitution

$$u = \sqrt{2} \sinh t,\ 1 + u^2 = \cosh 2t.$$

As for the evaluation of the integral, we shall not stop to do it here, but will compute a similar integral later in lemma 29 (see [WII, p 340]).

Lemma 24. $\phi_1(v, z)$ *satisfies the ordinary differential equation*

$$(z^2 - 1)\phi_1'' + 2z\, \phi_1' + \left(\frac{1}{4} + v^2\right)\phi_1 = 0$$

$$\phi_1(v, 1) = -\,(\pi/2\sqrt{2})\, v \tanh \pi v.$$

Thus ϕ_1 *has the Taylor series expansion*

$$\phi_1(v, z) = \sum_{n=0}^{\infty} a_n (z - 1)^n$$

where

$$a_n = \frac{\Gamma\left(n + \frac{1}{2} + iv\right)\Gamma\left(n + \frac{1}{2} - iv\right)(-1)^{n+1}\pi}{\Gamma\left(\frac{1}{2} + iv\right)\Gamma\left(\frac{1}{2} - iv\right)2^n (n!)^2\, 2\sqrt{2}}\, v \tanh \pi v \quad (|z - 1| < 2).$$

Proof. From formula 18 we see that ϕ_0' satisfies the differential equation

$$(z^2 - 1)\, A'' + (az + b)\, A' + cA = 0$$

with $a = 3$, $b = 0$ and $c = 1 + v^2$. Moreover ϕ_0 satisfies the inequality $|A^{(k)}(z)| \le C_k(1 + z)^{-k-1}\ (z \ge 1)$. Now $\phi_1 = B$ when $A = \phi_0'$ where

$$B(z) = \int_0^\infty A(z + u^2)\, du.$$

Because of the inequality, the formulas of lemma 10 remain valid, and in particular

$$B(z) = -\,2 \int_0^\infty A'(z + u^2)\, u^2 du = 4/3 \int_0^\infty A''(z + u^2)\, u^4 du.$$

Thus we obtain

$$(z^2 - 1)B''(z) = \int_0^\infty ((z + u^2)^2 - 1)A''(z + u^2) +$$

$$+ (u^4 - 2u^2(z + u^2))\, A''(z + u^2)\, du =$$

$$= \int_0^\infty -\,[(a - 1)(z + u^2) + b]\, A'(z + u^2) - \left(\frac{1}{4} + c\right) A(z + u^2)\, du$$

and

$$[(a - 1)\, z + b]\, B'(z) = \int_0^\infty [(a - 1)(z + u^2) + b]\, A'(z + u^2) +$$

$$+ \frac{1}{2}(a - 1)\, A(z + u^2)\, du.$$

Thus B satisfies

$$(z^2 - 1)\, B''(z) + ((a - 1)\, z + b)\, B' + \left(\frac{3}{4} + c - \frac{1}{2} a\right) B = 0.$$

This gives the differential equation of the lemma when $a = 3$, $b = 0$ and $c = 1 + v^2$.

To solve the differential equation we apply the theory of regular singular points at $z = 1$. The indicial equation reads $2\alpha^2 = 0$ so one solution is holomorphic and the other has a logarithmic divergente at $z = 1$; since ϕ_1 is clearly bounded, we must take the holomorphic solution. Then the recursion relation reads

$$a_{n+1} = -a_n\left(\left(n+\frac{1}{2}\right)^2 + v^2\right)/2(n+1)^2.$$

This, together with the value for a_0 given in lemma 23, implies the result.

Corollary 25. *For $G = SO(2n+2, 1)$ the zonal Plancherel measure is given by*

$$\mu(v) = \frac{\Gamma\left(n+\frac{1}{2}+iv\right)\Gamma\left(n+\frac{1}{2}-iv\right)}{\Gamma\left(\frac{1}{2}+iv\right)\Gamma\left(\frac{1}{2}-iv\right)2^n\sqrt{2}\,n!\,\pi^{n+1}}\, v\tanh \pi v.$$

Now we turn to the case $p \neq 0$. Then $p = 2k$ is even and $q = 2n+1$ is odd [WI, p 33]. (These computations break into two cases later according to the parity of k.) Define $A_k(v, w)$ and $B_k(v, w)$ $(w \geq 1)$ by

$$A_k(v, 2z^2-1) = (4z)^{-1}\phi_0^{(k+1)}(2v, z) \qquad (z \geq 1)$$

$$B_k(v, w) = \int_0^\infty A_k(v, w+u^2)\, du.$$

Now with $w = 2z^2 - 1$ or $z = \sqrt{\tfrac{1}{2}(w+1)}$, lemma 16 tells us that

$$|A_k(v, w)| \leq C_{k+1}(1+|2v|)^{3k+3}(1+z)^{-k-1}z^{-1} \leq$$

$$\leq C'_k(1+|v|)^{3k+3}(1+w)^{-\frac{1}{2}k-1}$$

In particular the integral defining B_k converges. We shall also need estimates for the ℓ th derivatives (with respect to w) $A_k^{(\ell)}$ and $B_k^{(\ell)}$.

For this, note that if $w = 2z^2 - 1$ then $d/dw = (4z)^{-1}d/dz$. Then by induction

$$\left(\frac{d}{dw}\right)^{\ell+1} = \sum_{m=0}^{\ell-1} c_m\, z^{-\ell-1-m}\left(\frac{d}{dz}\right)^{\ell+1-m}$$

Applying this to $\phi_0^{(k)}$ we have, by lemma 16,

$$|A_k^{(\ell)}(v, 2z^2-1)| = \left|\left(\frac{1}{4z}\cdot\frac{d}{dz}\right)^{\ell+1}\phi_0^{(k)}(v, z)\right| \leq$$

$$\leq \Sigma\,|c_m|\, z^{-\ell-1-m}C_{k+\ell+1-m}(1+|v|)^{3k+3\ell+3-3m}(1+z)^{-k-\ell-1+m} \leq$$

$$\leq C'_{k\ell}(1+|v|)^{3k+3\ell+3}\, z^{-k-2\ell-2}.$$

Substituting $z = \sqrt{\frac{1}{2}(w+1)}$ we have

$$|A_k^{(\ell)}(v, w)| \leq C_{k\ell}(1+|v|)^{3k+3\ell+3}(1+w)^{-\frac{1}{2}k-\ell-1} \qquad (w \geq 1)$$

This inequality allows us to differentiate under the integral in computing $B_k^{(\ell)}$. Thus

$$|B_k^{(\ell)}(v, w)| \leq \int_0^\infty |A_k^{(\ell)}(v, w+u^2)|\, du \leq$$

$$\leq C_{k\ell}(1+|v|)^{3k+3\ell+3}\int_0^\infty (1+w+u^2)^{-\frac{1}{2}k-\ell-1}\, du \leq$$

$$\leq C'_{k\ell}(1+|v|)^{3k+3\ell+3}(1+w)^{-\frac{1}{2}k-\frac{1}{2}-\ell}.$$

Recall the formula for F_f derived in lemma 13. Thus if we know F_f then we can compute $f(1)$ U $F(1)$as follows: First set $G_1(\cosh t) = F_f(t)$ (by Mather's theorem, since F_f is even). Next define $G_2(2z^2-1) = (d/dz)^k G_1(z)$ $(p = 2k)$. Then

$$f(1) = 2^{-p}(-\pi)^{-k-n-1}\,2\int_0^\infty \left(\frac{d}{dw}\right)^{n+1} G_2(w+u^2)\, du\Big|_{w=1}$$

(where $q = 2n+1$). Now with

$$F_f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi_v(f)\cos(2vt)\, dv$$

the above inequalities imply that we can perform the operations on F_f under the integral sign, on $\cos(2vt)$, to obtain

$$f(1) = 4^{-k}(-\pi)^{-k-n-1}\frac{2}{\pi}\int_{-\infty}^{\infty}\phi_v(f)\, B_k^{(n)}(v, 1)\, dv. \tag{26}$$

(Note that $\phi_0(2v, \cosh t) = \cos(2vt)$.) Thus we see that to obtain the Plancherel measure in the present case, we need the Taylor coefficients of B_k at $w = 1$.

To obtain the Taylor series for B_k we use the differential equations.

Lemma 27. *A_k satisfies the differential equation*

$$(w^2-1)A_k'' + ((k+3)w + k)A_k' + \left(v^2 + \left(\frac{1}{2}k+1\right)^2\right)A_k = 0$$

and the recursion relation

$$8(w+1)A_k'' + 12A_k' = A_{k+2}$$

B_k *satisfies the differential equation*

$$(w^2-1)B_k'' + ((k+2)w+k)B_k' + \left(v^2 + \frac{1}{4}(k+1)^2\right)B_k = 0$$

and the recursion relation

$$8(w+1)B_k'' + 8B_k' = B_{k+2}.$$

Proof. Let us abreviate the notation and write $\phi_0^{(k)}$ for $\phi_0^{(k)}(2v,z)$, $A_k^{(\ell)}$ for $A_k^{(\ell)}(v, 2z^2-1)$, and $w = 2z^2-1$ so $d/dw = (4z)^{-1}\,d/dz$. Then we have

$$\phi_0^{(k+1)} = 4zA_k$$
$$\phi_0^{(k+2)} = 4A_k + 4^2z^2R_k'$$
$$\phi_0^{(k+3)} = 4^2 \times 3zA_k' + 4^3z^3A_k''.$$

Applying formula 18, we get

$$0 = \frac{1}{4z}\left[(z^2-1)\phi_0^{(k+3)} + (2k+3)z\,\phi_0^{(k+2)} + (4v^2+(k+1)^2)\phi_0^{(k+1)}\right] =$$
$$= 4(2z^2-1+1)(2z^2-1-1)A_k'' + 6(2z^2-1-1)A_k' +$$
$$+ 2(2k+3)(2z^2-1+1)A_k' + (4v^2+(k+1)^2+2k+3)A_k.$$

The differential equation now follows upon substituting $w = 2z^2-1$ and simplifying. As for the recursion relation, we have

$$A_{k+2} = (4z)^{-1}\phi_0^{(k+3)} = 12A_k' + 16z^2A_k'' = 8(w+1)A_k'' + 12A_k'.$$

The differential equation for B_k follows from that for A_k and the proof of lemma 24. As for the recursion relation,

$$B_{k+2}(v,w) = \int_0^\infty 8(w+u^2+1)A_k''(v,w+u^2) + 12A_k'(v,w+u^2)\,du =$$
$$= 8(w+1)B_k'' + 12B_k' - 4\int_0^\infty -2u^2A_k''(v,w+u^2)\,du.$$

Now proceed as in the proof of lemma 24 using the formulas of lemma 10.

The next step is to solve the differential equation for B_k.

Lemma 28. B_k *has the Taylor expansion, valid for* $|w-1|<2$,

$$B_k(v,w) = \sum_{n=0}^{\infty} b_{k,n}(w-1)^n$$

where

$$b_{k,n} = \frac{(-1)^n 4\pi\Gamma(n+\frac{1}{2}+\frac{1}{2}k+iv)\,\Gamma(n+\frac{1}{2}+\frac{1}{2}k-iv)\,\Gamma(k+2iv)\,\Gamma(k-2iv)}{2^{n+k}n!(n+k)!\,\Gamma(\frac{1}{2}k+iv)\,\Gamma(\frac{1}{2}k-iv)}\cdot\beta_k$$

and

$$\beta_k = \begin{cases} \Gamma\left(\frac{1}{2}+iv\right)^{-2}\Gamma\left(\frac{1}{2}-iv\right)^{-2}b_{0,0} & (k \text{ even}) \\ \Gamma(1+iv)^{-2}\,\Gamma(1-iv)^{-2}\,\frac{1}{2}b_{1,0} & (k \text{ odd}) \end{cases}$$

(We have yet to compute $b_{0,0}$ *and* $b_{1,0}$.*)*

Proof. The indicial equation at the regular singular point $w = 1$ reads $2\alpha(\alpha+k)=0$. Thus the bounded solution B_k must be a constant multiple of the solution corresponding to $\alpha = 0$. The corresponding recursion relation is

$$b_{k,n+1} = -b_{k,n}\left[\left(n+\frac{1}{2}+\frac{1}{2}k\right)^2 + v^2\right]/2(n+1)(n+k+1)$$

Solving this recursion relation we obtain

$$b_{k,n} = \frac{\Gamma\left(n+\frac{1}{2}+\frac{1}{2}k+iv\right)\Gamma\left(n+\frac{1}{2}+\frac{1}{2}k-iv\right)(-1)^n k!}{\Gamma\left(\frac{1}{2}+\frac{1}{2}k+iv\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}k-iv\right)2^n n!\,(n+k)!}\,b_{k,0}$$

Now we evaluate the recursion relation satisfied by B_k at $w=1$.

$$b_{k+2,0} = B_{k+2}(v,1) = 16B_k''(v,1) + 8B_k'(v,1) = 32b_{k,2} + 8b_{k,1}.$$

Applying the preceeding recursion relation to $b_{k,2}$ and $b_{k,1}$ we get

$$b_{k+2,0} = 8\left(-\frac{\left(1+\frac{1}{2}+\frac{1}{2}k\right)^2+v^2}{k+2}+1\right)\cdot b_{k,1} =$$

$$= 4\,\frac{v^2+\frac{1}{4}(k+1)^2}{k+2}\cdot\frac{v^2+\frac{1}{4}(k+1)^2}{k+1}\,b_{k,0}$$

Solving this recursion relation, we obtain for $b_{k,0}$ the expression

$$\begin{cases} \dfrac{2^k}{k!}\,\dfrac{\Gamma\left(\frac{1}{2}+\frac{1}{2}k+iv\right)^2\Gamma\left(\frac{1}{2}+\frac{1}{2}k-iv\right)^2}{\Gamma\left(\frac{1}{2}+iv\right)^2\Gamma\left(\frac{1}{2}-iv\right)^2}\,b_{0,0} & (k\ \text{even}) \\[2ex] \dfrac{2^{k-1}}{k!}\,\dfrac{\Gamma\left(\frac{1}{2}+\frac{1}{2}k+iv\right)^2\Gamma\left(\frac{1}{2}+\frac{1}{2}k-iv\right)^2}{\Gamma(1+iv)^2\,\Gamma(1-iv)^2}\,b_{1,0} & (k\ \text{odd}) \end{cases}$$

Now the lemma follows when this value for $b_{k,0}$ is substituted back into the expression for $b_{k,n}$ and the result is simplified using the Legendre duplication formula

$$(\Gamma(z+1/2) = \sqrt{\pi}\,\Gamma(2z)/2^{2z-1}\,\Gamma(z)).$$

So now we come to the computation of $b_{0,0}$ and $b_{1,0}$. Substitute $u = \sqrt{2}\sinh x$ so $1+u^2 = 1+2\sinh^2 x = 2\cosh^2 x - 1$ and

$$A_k(1+u^2) = (4\cosh x)^{-1}\,\phi_0^{(k+1)}(\cosh x).$$

This yields

$$b_{k,0} = B_k(1) = \int_0^\infty A_k(1+u^2)\,du = \frac{\sqrt{2}}{4}\int_0^\infty \phi_0^{(k+1)}(2v,\cosh x)\,dx =$$

$$= \frac{\sqrt{2}}{4}\int_0^\infty \left(\frac{1}{\sinh x}\,\frac{d}{dx}\right)^{k+1}\cos 2vx\,dx.$$

Lemma 29.

$$b_{0,0} = -\frac{\sqrt{2}}{4}\int_0^\infty 2v\sin(2vx)/\sinh x\,dx = -\frac{\sqrt{2}}{4}\,\pi v\tanh\pi v$$

$$b_{1,0} = -\frac{\sqrt{2}}{4}\int_0^\infty [4v^2\cos 2vx\cdot\sinh x - 2v\sin 2vx\cdot\cosh x]/\sinh^3 x\,dx =$$

$$= +\frac{\sqrt{2}}{2}\,\pi v^3\coth\pi v.$$

Proof. [WW, p 118] We shall only prove the last equality; the derivation of the formula for $b_{0,0}$ is similar, but much easier. Substitute $e^x = t$ so $dx = 1/t\,dt$. Then

$$\left(\frac{-4}{\sqrt{2}\,v}\right)\quad b_{1,0} = \int_{-\infty}^{\infty} 2v\cos\ldots =$$

$$= \int_0^\infty [4v(t^{2vi}+t^{-2vi})(t^2-1)\,t + 2i(t^{2vi}-t^{-2vi})(t^2+1)\,t\,](t^2-1)^{-3}\,dt.$$

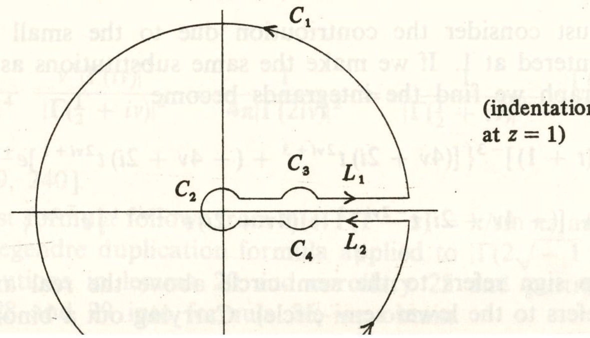
For $z \in \mathbb{C}$ let

$$(-z)^\alpha = \exp\alpha\{\log|z| + i\arg(-z)\}\quad\text{with}\quad -\pi < \arg -z < \pi.$$

We consider the contour integral

$$\int_C [4v((-z)^{2vi} - (-z)^{-2vi})(z^2-1)\,z + 2i((-z)^{2vi} + (-z)^{-2vi})(z^2+1)\,z\,](z^2-1)^{-3}\,dz$$

where C is the contour pictured below



In fact, instead of assuming that v is real, we can make the weaker assumption that $v = \alpha + i\beta$ with $|\beta| < 1$. On the small circle C_2 centered at the origin (of radius δ, say) the absolute value of the integrand is $\leq \text{const} \cdot \delta^{1-2|\beta|}$ so the contribution from C_2 tends to zero as $\delta \to 0$ provided $|\beta| < 1$. On the large circle C_1 (of radius R, say), the absolute value of the integrand is $\leq \text{const} \cdot R^{-3+2|\beta|}$ so again the contribution from the large circle tends to zero as $R \to \infty$ when $|\beta| < 1$.

On the horizontal line L_1, from δ to R just above the positive real axis with an indentation at $z = 1$, we make the substitution

$$-z = t\,e^{-\pi i} \quad \text{with} \quad 0 < \arg t < \pi.$$

Then

$$(-z)^{\alpha} = t^{\alpha} e^{-\pi i \alpha} \quad \text{and} \quad dz = dt.$$

Similary on L_2 (just below the real axis) we substitute

$$-z = t\,e^{+\pi i} \quad \text{with} \quad -\pi < \arg t < 0$$

$$(-z)^{\alpha} = t^{\alpha} e^{+\pi i \alpha} \quad \text{and} \quad dz = dt.$$

Then a one line computation shows that the contribution to the contour integral from the two horizontal lines converges to

$$(-4/\sqrt{2}\,v)\, b_{1,0} \cdot (e^{2\pi v} - e^{-2\pi v}).$$

Next we compute the residue at $z = -1$ of the integrand. The result is

$$2v^2 i = \operatorname{Res}_{z=-1} \left\{ \frac{4v((-z)^{2vi} - (-z)^{-2vi})z}{(z^2-1)^2} + \right.$$

$$\left. + \frac{2i((-z)^{2vi} + (-z)^{-2vi})(z^2+1)z}{(z^2-1)^3} \right\}.$$

Finally we must consider the contribution due to the small semi-circles semi-circles centered at 1. If we make the same substitutions as in the preceeding paragraph we find the integrands become

$$[(t-1)(t+1)]^{-3} \{ [(4v + 2i)\, t^{2vi+3} + (-4v + 2i)\, t^{2vi+1}] e^{\pm 2\pi v} +$$

$$+ [(-4v + 2i)\, t^{-2vi+3} + (4v + 2i)\, t^{-2vi+1}] e^{\mp 2\pi v} \}$$

(where the top sign refers to the semi-circle above the real axis and the bottom sign refers to the lower semi-circle). Carrying out a binomial expan-

sion around $t = 1$ on each term above, we find that both integrands have the same Laurent expansion at $t = 1$ up to $0(1)$, given by

$$(e^{2\pi v} + e^{-2\pi v}) \left\{ \frac{i}{2}(t-1)^{-3} + \frac{i}{4}(t-1)^{-2} + v^2 i (t-1)^{-1} \right\} + 0(1).$$

Thus we can deal with the contribution of the two semi-circles as we would a pole (of order three, encircled in the negative direction). The contribution is

$$+ 2\pi v^2 (e^{2\pi v} + e^{-2\pi v})$$

in the limit as the semi-circles shrink to 1.

Adding the various contributions to the contour integral, we find that

$$\int_C \ldots dz = 2\pi i \cdot 2v^2 i = -4\pi v^2 =$$

$$= (-4/\sqrt{2}\,v)\, b_{1,0}(e^{2\pi v} - e^{-2\pi v}) + 2\pi v^2 (e^{2\pi v} + e^{-2\pi v})$$

or

$$b_{1,0} = +(\sqrt{2}/2)\,\pi v^3 \cdot 4\cosh^2 \pi v / 2 \sinh 2\pi v.$$

Now we are in a position to write down the zonal Plancherel measure for the remaining rank one groups. But firt it seems worthwhile to simplify the results. First we note that $\Gamma(a + iv)\,\Gamma(a - iv) = |\Gamma(a + iv)|^2$. Next note that

$$\left|\Gamma\left(\frac{1}{2} + iv\right)\right|^2 \Big/ |\Gamma(iv)|^2 = v \tanh \pi v$$

and

$$\frac{1}{|\Gamma(1+iv)|^4} \frac{v^4 |\Gamma(iv)|^2}{|\Gamma(\frac{1}{2}+iv)|^2} = \frac{1}{4\pi |\Gamma(2iv)|^2} = \frac{1}{|\Gamma(\frac{1}{2}+iv)|^4} \frac{|\Gamma(\frac{1}{2}+iv)|^2}{|\Gamma(iv)|^2}.$$

[WW, p 239, 240].

(The first formula follows from $\Gamma(z)\,\Gamma(1-z) = \pi/\sin \pi z$ and the second is just the Legendre duplication formula applied to $|\Gamma(2\sqrt{-1}\,v)|^2$.) Making these substitutions in lemma 20 and corollary 25 and putting the results of lemmas 28 and 29 into formula 26, we obtain.

Theorem 30. *The zonal Plancherel measures for the rank one groups are given by*

$$\mu(v) = \frac{\sqrt{2}}{(2\pi)^c\,\Gamma(c)} \, \frac{|\Gamma(\frac{1}{2}q + iv)|^2}{|\Gamma(iv)|^2} \qquad (if\ \ p = 0)$$

$$\mu(v) = \frac{1}{2^p} \, \frac{\sqrt{2}}{(2\pi)^c\,\Gamma(c)} \, \frac{|\Gamma(\frac{1}{2}q + \frac{1}{4}p + iv)|^2\,|\Gamma(\frac{1}{2}p + 2iv)|^2}{|\Gamma(\frac{1}{4}p + iv)|^2\,|\Gamma(2iv)|^2}$$

(for $p \neq 0$*). Here* $q = m(\lambda)$ *and* $p = m(\lambda/2)$ *and* $c = \frac{1}{2}(\dim G - \dim K)$.

(Let us recall that we have normalized the Haar measure dx on G according to the Iwasawa integration formula $dx = e^{2\rho}\,dk\,da\,dn$, where $\int dk = 1$ and da and dn are the transports of the Euclidean measures determined by the quadratic form $Q(X) = -(2p + 8q)^{-1}B(X, \theta X)$ where B is the Cartan-Killing form.)

If one expresses the Γ-factors as a monic polynomial $\times$ (1 or tanh or coth), then the constant becomes $\sqrt{2}/(2\pi)^c\,\Gamma(c)$. It is reassuring to note that Roberto Miatello has obtained recently the constant $1/\pi^c\Gamma(c)$, by entirely different methods. The seeming discrepancy is due to the fact that his Haar measure (the Riemannian measure coming from Q/vol K) is $\sqrt{2}/2^c \times$ *my* Haar measure (n is not orthogonal to k).

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