Banach f-algebras and Banach lattice algebras with unit

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Abstract. We give a necessary and sufficient condition for a Banach lattice algebra to be representable as a Banach lattice of continuous realvalued functions on a compact space, endowed with the pointwise defined multiplication. Moreover, we give a characterisation of Banach lattice algebras possessing an algebraic unit.

1. Banach f-algebras.

Let E be a (real) Banach lattice and $* : E \times E \to E$ a binary composition law, such that $(E, *)$ is a Banach algebra. The pair $(E, *)$ is called a Banach lattice algebra (see Schaefer [5]; 4, ex. 4) if every product of positive elements is positive. For the characterisation of those Banach lattice algebras, which can be represented as Banach lattices of continuous realvalued functions on a compact space, we need the concept of the center Z_E of a Banach lattice E. It is defined as the order ideal $\bigcup_{n \in \mathbb{N}} n[-I, I]$ generated by the identity operator I in $\mathscr{L}(E)$. With the structure induced from $\mathscr{L}(E)$ the center Z_E of E is a Banach lattice algebra isometrically isomorphic to $(C(K), \cdot)$ for a suitable space K (we denote with $\cdot$ the pointwise defined multiplication). It will turn out that a Banach lattice algebra is an f-algebra, in the sence of Birkhoff and Pierce ([1]), if and only if it is (contractively) embeddabe in its center. The following characterisation of Z_E was announced by Meyer ([4]) and proven by Wickstead ([7]) for Banach lattices containing a quasi-interior point. We give a simple proof for the general case, without making use of representation theorems. We use Schaefer ([5]) as a general reference for the theory of Banach lattices and Zelasko ([9]) for the theory of Banach algebras.

1.1. Lemma. *Let E be a real normed vector lattice and T a bounded positive operator on E. The following assertions are equivalent:*

(i) *$x \perp y$ implies $Tx \perp y$ for all $x, y \in E$*

(ii) *$T \leq \|T\| I$ (I is the identity operator of $\mathscr{L}(E)$*

(iii) *T maps every order ideal of E into itself.*

Proof. We show that (i) $\Rightarrow$ (ii). If T satisfies (i), also T^n satisfies (i) for every $n \in N$. Let $a \in E$; then $a^+ \perp a^-$ implies $Ta^+ \perp a^-$ and therefore $Ta^+ \perp Ta^-$.

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This means that T is a lattice morphism ([5]; 2, 2.5). Moreover, we have $(Ta+a)^+ = Ta^+ + a^+$ because $(Ta^+ + a^+) \perp (Ta^- + a^-)$ and the representation of $(Ta+a)$ as a difference of two disjoint positive elements is unique. Similarly, $(T^n a + a)^+ = T^n a^+ + a^+$ for every $n \in \mathbb{N}$. Suppose now $\|T\| < 1$: for $x \geq 0$ in E and $a = (Tx - x)$ we get $(T^2 x - x)^+ = (T(Tx-x) + (Tx - x))^+ = T(Tx-x)^+ + (Tx-x)^+$. Since $T \geq 0$ we have $T(Tx-x)^+ \geq 0$ and therefore $(T^2 x - x)^+ \geq (Tx - x)^+$. By induction we prove $(T^{2^n} x - x)^+ \geq$ $\geq (Tx-x)^+$ because $(T^{2^n} x - x)^+ = T^{2^{n-1}}(T^{2^{n-1}} x - x)^+ + (T^{2^{n-1}} x - x)^+ \geq$ $\geq (T^{2^{n-1}} x - x)^+ \geq (Tx - x)^+$. Since $(T^{2^n} x - x)^+$ is a nullsequence we have $(Tx - x)^+ = 0$, which means $Tx \leq x$. For $\|T\| = 1$ write $T = \sup_{n \in \mathbb{N}} \lambda_n T$, where $0 < \lambda_n < 1$ for every $n \in N$ and $\sup_{n \in \mathbb{N}} \lambda_n = 1$; then for $x \geq 0$ we have $Tx = (\sup_{n \in \mathbb{N}} \lambda_n T)\, x = \sup_{n \in \mathbb{N}} (\lambda_n Tx) \leq x$. We have proven $T \leq$ $\leq \|T\| I$ for every bounded positive operator T satisfying (i). The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are trivial.

Let E be a real normed vector lattice and T a bounded positive operator on E. If T satisfies the equivalent conditions of (1.1), we say that T is an f-*operator*. For *f*-operators on $C(K)$, we have the following characterisation.

1.2. Lemma. *Let K be a compact space and T an f-operator on $C(K)$. If 1_K denotes the function $1_K(t) = 1$ for every $t \in K$, we have $Tf(t) = T1_K(t) f(t)$ for all $f \in C(K)$, $t \in K$.*

Proof. Set $q = T1_K$. We know from the proof of (1.1) that T is a lattice morphism. There exists a function $\varphi: K \to K$, which on the set $U =$ $= \{t \in K \mid q(t) > 0\}$ is continuous and uniquely defined, such that $Tf(t) =$ $= q(t) f(\varphi(t))$ for all $f \in C(K)$, $t \in K$ (for a proof of this statement see Wolff [8]). Assume the existence of a $t_0 \in U$ with $\varphi(t_0) \neq t_0$. Urysohn's Lemma states that we can find functions f and g in $C(K)$ with $f \perp g$ and $f(\varphi(t_0)) = 1 = g(t_0)$. This is a contradiction, since T is an *f*-operator and so $Tf \perp g$.

We introduce now the concept of the center of a Banach lattice, following Wickstead ([7]). For an extensive treatment of the center of an archimedea vector lattice see Flösser ([2]).

1.3 Definition. Let E be a real Banach lattice. The *center* Z_E of E is the order ideal $\cup_{n \in \mathbb{N}} n[-I, I]$ generated by the identity operator I of $\mathscr{L}(E)$.

For a description of the struture of Z_E we prove a proposition, which characterises $(C(K), \cdot)$ as a Banach lattice algebra.

1.4. Proposition. *Let K be a compact space and $*: C(K) \times C(K) \to C(K)$ a binary composition law on $C(K)$, such that $(C(K), *)$ is a Banach lattice algebra. Suppose that 1_K (i.e. the function defined by $1_K(t) = 1$ for every $t \in K$) is the algebraic unit of $(C(K), *)$. Then $*$ is the pointwise defined multiplication,* i.e. $(C(K), *) = (C(K), \cdot)$.

Proof. Let f be a positive function in $C(K)$. There exists an $n \in N$ such that $f \leq n1_K$. The bipositivity of $*$ implies $f * g \leq n1_K * g = ng$, by our assumption on 1_K. The operator $T_f: C(K) \to C(K)$, defined by $T_f g = f * g$ for every $g \in C(K)$, is an *f*-operator. We deduce from (1.2) that $(f * g)(t) =$ $= (f * 1_K)(t) g(t) = f(t) g(t)$ for every $t \in K$. Since the positive cone of $C(K)$ is generating, we have proven our assertion.

Remark. This Proposition and Kakutani's Representation Theorem for AM-spaces with unit ([5]; 2, 7.4) lead to a new characterisation of $(C(K), \cdot)$: Let $(E, *)$ be a Banach lattice algebra, such that E is an AM-space with unit u and u is the algebraic unit of $(E, *)$. Then $(E, *)$ is isometrically isomorphic to $(C(K), \cdot)$ for a suitable compact space K.

1.5. Proposition. *Let E be a real Banach lattice. Every operator $T \in Z_E$ has a modulus $|T|$ in $\mathscr{L}(E)$ and $|T| \in Z_E$. If S and T are operators in Z_E, the composition $T \circ S$ is also in Z_E. Endowed with the order, the norm and the composition of operators induced from $\mathscr{L}(E)$, the pair (Z_E, o) is a Banach lattice algebra. Moreover, (Z_E, o) is isomorphic to $(C(K), \cdot)$ for a suitable compact space K.*

Proof. By (1.1) the set of positive elements in Z_E is the set of *f*-operators on E. Let $T \in Z_E$; it is readily verified that T can be written as a difference $R - S$ of two *f*-operators R and S. Take $x \in E_+$ and identify the order ideal E_x, generated by x, with $C(K_x)$ for a suitable compact space K_x (see [5], 2, §7). The operator T maps $C(K_x)$ into itself. Set $q = T1_{K_x} =$ $= R1_{K_x} - S1_{K_x}$; then (1.2) implies $Tf(t) = q(t) f(t)$ for every $f \in C(K)$, $t \in K$. From this we deduce $\sup_{|f| \leq 1_{K_x}} |Tf| = |q| = |T1_{K_x}|$. Since E_x is an order ideal of E, we have proven our first assertion. A moment's reflection shows, that if S an T are in Z_E also $T \circ S$ is in Z_E. Because the positive cone of E is normal, the norm of $\mathscr{L}(E)$ and the Minkowski functional of $[-I, I]$ coincide on Z_E. Endowed with this norm Z_E is an AM-space with unit. According to Kakutani's theorem ([5] 2, 7.4), there exists a compact space K such that Z_E and $C(K)$ are isomorphic as Banach lattices. Moreover, this isomorphysm maps I onto 1_K. The fact that I is the algebraic unit of $(Z_E, \circ)$, and (1.4) conclude our proof.

Let $(E, *)$ be a Banach lattice algebra. If $(E, *)$ is an f-algebra in the sense of Birkhoff and Pierce ([1]), we say that $(E, *)$ is a Banach f-algebra. Stated explicitely:

1.6. Definition. Let $(E, *)$ be a Banach lattice algebra. We call $(E, *)$ a Banach f-algebra if $x \perp y$ implies $a * x \perp y$ and $x * a \perp y$ for all $x, y \in E$ and $a \geq 0$ in E.

Remark. 1. $(E, *)$ is a Banach f-algebra if right and left multiplication with positive elements of E are f-operators on E.

Remark 2. Let E be a Banach lattice. It depends heavily on the Banach lattice structure of E whether it is possible to define $* : E \times E \to E$ such that $(E, *)$ is a nontrivial Bànach f-algebra (i.e. $x * y$ is not zero for all $x, y \in E$). Birkhoff and Pierce have proven that an f-algebra with trivial right and left annullators can be embedded in a direct union of totally ordered algebras, endowed with the componentwise defined multiplication. It is clear that not every Banach lattice can be compatible with the structure of a Banach f-algebra, since not every Banach lattice admits a representation as a space of realvalued functions closed under the pointwise defined multiplication. Note that such a representation implies the existence of many nontrivial multiplicative linear forms, which are necessarily lattice morphisms.

1.7. Proposition. *Let $(E, *)$ be a Banach algebra with trivial left and right annulators, i.e. for every $a \in E$ there exist $x, y \in E$ such that $a * x \neq 0$ and $y * a \neq 0$. The following assertions are equivalent.*

(a) $(E, *)$ is a Banach f-algebra

(b) The left and the right regular representations of $(E, *)$ into $\mathscr{L}^r(E)$ are (contractive) isomorphisms of $(E, *)$ into $(Z_E, \circ)$.

(c) $(E, *)$ can be identified with a sublattice algebra of $(C(K), \cdot)$ for a suitable compact space K. Under this identification the norm of E either coincides with or is finer than the norm of $C(K)$.

Proof. (a) $\Rightarrow$ (b) Let Λ denote the left regular representation of E into (E). It is clear that $\Lambda(E) \leq Z_E$. Let $a \in E$; we prove that $\Lambda|a| = |\Lambda a|$. From the proof of (1.5) we know that $|(\Lambda a)|\, x = |(\Lambda a)x| = |a * x|$ for every $x \geq 0$ in E. Since right multiplication with x is an f-operator, it is also a lattice morphism (see proof of (1.1)). Therefore $|a * x| = |a| * x$, but $|a| * x = (\Lambda|a|)x$, which proves that $\Lambda|a| = |\Lambda a|$. It follows readily from the definition of a Banach lattice algebra, that Λ is a contractive algebraic morphism of $(E, *)$ into $(Z_E, \circ)$. An analogous argument shows that also the right regular representation P is a contractive structure morphism. Since for every $a \in E$, $a \neq 0$, there exist $x, y \in E$ with $a * x \neq 0$ and $y * a \neq 0$, both Λ and P are isomorphisms.

(b) $\Rightarrow$ (c) This implication follows from the fact, that $(Z_E, \circ)$ can be identified with $(C(K), \cdot)$ for a suitable compact space K (see (1.5)).

(c) $\Rightarrow$ (a) To prove this, observe that $(C(K), \cdot)$ is a Banach f-algebra for every compact space K.

1.8. Examples.

1. Let $(E, *)$ be a Banach f-algebra with trivial right and left annullators, such that E has order continuous norm. Then E is an atomic Banach lattice. For every $1 \leq p < \infty$ the pair $(\ell^p, /)$ is a Banach f algebra (here again $\cdot$ denotes the pointwise defined multiplication). The center of ℓ^p is isomorphic to ℓ^∞.

2. Let X be a locally compact space and $C_0(X)$ the space of all continuous realvalued functions on X, which vanish at infinity. Endowed with the supremum norm and pointwise defined order $C_0(X)$ is a Banach lattice. If $\cdot$ denotes again the pointwise defined multiplication, the pair $(C_0(X), \cdot)$ is a Banach f-algebra and therefore it can be (contractively) embedded in its center. The center of $C_0(X)$ is $C(\beta X)$.

2. Banach lattice algebras with an algebraic unit.

Let $(E, *)$ be a real lattice algebra with multiplicative modulus and an algebraic unit $e \geq 0$. Birkhoff and Pierce ([1], Theorems 10 and 14) proved that $(E, *)$ is an f-algebra and, therefore, that e is a weak order unit. If E is finite dimensional, this implies that e is also an order unit. Otherwise, this is not necessarily true. Take for example $C(\mathbb{R})$, the continuous functions on $\mathbb{R}$, with the pointwise defined multiplication. It is natural to ask for additional conditions guarantying that e is also an order unit. Surprisingly, the existence of a norm on E is already sufficient. We prove the next Lemma for Banach lattice algebras: that it is true for normed lattice algebras as well, follows from the standard completion arguments.

2.1. Lemma. *Let $(E, *)$ be a Banach lattice algebra with multiplicative modulus and an algebraic unit $e \geq 0$. Then e is the norming order unit of E.*

Proof. This Lemma becomes a consequente of (1.7) if we assume that the right and left annullators are trivial and make use of Theorems 10 and 14 of Birkhoff and Pierce ([1]). In fact, they assure that, under these conditions, $(E, *)$ is a Banach f-algebra. Therefore, by (1.7) it is isomorphically embeddable in $(Z_E, *)$. Clearly, this embedding maps e onto I, and the fact that it is a lattice isomorphism completes the proof. We give an alternative simple proof for the general case.

We prove that $x \geq 0$ and $\|x\| \leq 1$ imply $x \leq e$. Suppose first that $\|x\| < 1$ and that $x \not\leq e$, which means $(x - e)^+ > 0$. Since $x \geq 0$ also $(x + e) \geq 0$ and our hypothesis implies that left and right multiplication with $(x + e)$ are lattice morphisms (see [5] 2, 2.5). We therefore have $(x^2 - e)^+ = ((x + e)(x - e))^+ = (x + e)(x - e)^+ \geq (x - e)^+ > 0$. By induction we prove

$$(x^{2^n}-e)^+ = (x^{2^{n-1}}+e)(x^{2^{n-1}}-e)^+ \geq (x^{2^{n-1}}-e)^+ \geq (x-e)^+ > 0.$$

But the continuity of all operations on $(E,*)$ and the fact that $\|x\| < 1$ imply that $\lim_{n\to\infty} (x^{2^n}-e)^+ = 0$, which is a contradiction. To prove our statement for the case that $\|x\| = 1$, take a sequence $(\lambda_n) \subseteq \mathbb{R}$ with $0 < \lambda_n < 1$ and $\sup_{n\in\mathbb{N}} \lambda_n = 1$; then $x = \sup_{n\in\mathbb{N}} (\lambda_n x) \leq e$.

In the next proposition we no longer require that $(E,*)$ has multiplicative modulus.

2.2. Proposition. *Let $(E,*)$ be a Banach lattice algebra with an algebraic unit $e \geq 0$. If e is a quasi-interior point of E (i.e. the ideal $\bigcup_{n\in\mathbb{N}} n[-e,e]$ generated by e is dense in E), then it is also the norming order unit of E.*

Proof. We prove that $(E,*)$ has multiplicative modulus and apply (2.1). Let K be the structure space of E (see |5|). With the norm defined by the gauge function of the interval $[-e,e]$, the ideal $E_e = \bigcup_{n\in\mathbb{N}} n[-e,e]$ generated by e is a Banach lattice isomorphic to $C(K)$ for a suitable compact space K (this follows from Kakutani's representation theorem for AM-spaces with unit). Since $*$ is bipositive $(E_e,*)$ is a sublattice algebra of $(E,*)$. It follows from (1.4) that $(E_e,*)$ is isomorphic to $(C(K),\cdot)$. Since $(E_e,*)$ has multiplicative modulus also $(E,*)$ has multiplicative modulus, by the joint continuity of multiplication.

Corollary. *Let $(E,*)$ be a Banach lattice algebra with an algebraic unit $e \geq 0$. Then $(E,*)$ contain a nontrivial closed sublattice algebra $(D,*)$, isomorphic to $(C(K),\cdot)$ for a suitable compact space K. This isomorphism is an isometry. If the norm of E is ordercontinuous, then D is a finite dimensional projection band.*

Proof. Let D be the closure of the ideal $E = \bigcup_{n\in\mathbb{N}} n[-e,e]$. Clearly $(D,*)$ is a Banach lattice algebra satisfying the conditions of (2.2). Let K be a compact space such that $(D,*)$ and $(C(K),\cdot)$ are isometrically isomorphic as Banach lattice algebras. Our second statement follows from the fact that the norm of $C(K)$ is order continuous if and only if K is discrete.

Remark. Let (X,Ω,μ) be a measure space, where μ is a diffuse measure and X contains more than one point. Moreover, let $*$ be a multiplication on $L^1(X,\mu)$, such that $(L^1(X,\mu),*)$ is a Banach lattice algebra. Then (2.2) implies that $(L^1(X,\mu),*)$ does not contain an algebraic unit.

References

[1] Birkhoff G. and Pierce R., *Lattice ordered rings*, An. Acad. Brasil. Ci. 28, 41-69 (1956).

[2] Flosser H. O., *Das Zentrum archimedischer Vektorverbande* (to appear).

[3] Kakutani S., *Concrete representations of abstract M-spaces*, Ann. of Math. 42, 994-1024 (1941).

[4] Meyer M., *Le stabilisateur d'un space vectorie réticulé*, C. R. Acad. Sc. Paris 283, série A, 839-841 (1976).

[5] Schaefer H. H., *Banach lattices and positive operators*, Springer, Heidelberg 1974.

[6] Stone M. H., *Applications of the theory of boolean rings to general topology*, Trans. Amer. Math. Soc., 41, 375-481 (1937).

[7] Wickstead A. W., *The structure space of a Banach lattice*, J. Math. Pures Appl.

[8] Wolff M., *Uber das Spektrum von Verbandshomomorphismen in Banachverbänden*, Math. Ann. 182, 161-169 (1969).

[9] Zelasko W., *Banach algebras*, Elszevier Publ. Comp. Amsterdam-London New York 1973.

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