BOL. SOC. BRAS. MAT., VOL. 11 Nº 1 (1980), 19-24

On Ω-stability of flows

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1. Introduction.

In this paper we consider a C^r flow ϕ on a compact manifold M, whose Birkhoff center is a disjoint union of the set F of (hyperbolic) fixed points of ϕ, and a hyperbolic set Λ. We give an extension for flows of the correspondent result for diffeomorphisms in [2]. That is, if the Birkhoff center of ϕ is as above and has no cycles, then it coincides with the non-wandering set of ϕ. In particular, ϕ is stable with respect to its non-wandering set, $\Omega(\phi)$, that is, ϕ is Ω-stable.

Smale's Axion A'requires:

(*a*) Ω is a disjoint union of the set of critical points F and the closure Λ of its periodic orbits.

(*b*) each element of F is hyperbolic and Λ is a hyperbolic set for ϕ.

We define $c(\phi)$, the Birkoff center of ϕ, as the closure of the set $\{x \in M | x \in \alpha(x) \cap \omega(x)\}$, where $\alpha(x)$ and $\omega(x)$ are the α-limit and ω-limit sets of x, respectively. We prove the following theorem.

Theorem A. *If the Birkoff center of ϕ, $c(\phi)$, is a hyperbolic set for ϕ and has the no cycle property, then ϕ satisfies Axion A' and it is Ω-stable.*

To prove theorem A we need first, as in the case of diffeomorphisms, to obtain a decomposition of $c(\phi)$ into a finite number of disjoint hyperbolic sets for the flow, each of then having local product struture. For this, we use a version for flows of Anosov's Closing lemma for $c(\phi)$. With this result the same proof of the Ω-Decomposition Theorem for flows satisfying Axiom A' [4] yields a decomposition of $c(\phi)$ into hyperbolic sets with local product structure. Next we apply some results in [2] to ϕ_1, the time one map of the flow ϕ. First we prove that $c(\phi)$ coincides with the Birkhoff center of ϕ_1, $c(\phi_1)$, i.e., with the closure of the set $\{x \in M | x \in \alpha_1(x) \cap \omega_1(x)\}$, where $\alpha_1(x)$ and $\omega_1(x)$ are the α and ω-limit sets of x for the diffeomorphism ϕ_1. From results in [1], a hyperbolic set for the flow ϕ, with local product structure is an isolated set for the diffeomorphism ϕ_1. As defined in [2], we say that a compact set $K \subset M$ is an isolated set for a homeomorphism f of M if it is invariant by f, (i.e. $f(K) = K$), and there exists a neighbourhood U of K such that K is the maximal invariant set for f in U, that is, $\bigcap_{n \in \mathbb{Z}} f^n(U) = K$.

Recebido em dezembro de 1979.

These results and the assumption of no cycle gives us a decomposition of $c(\phi_1)$ into disjoint isolated sets having no cycles. From Lemma 5 below, we conclude that $c(\phi_1)$ coincides with the non-wandering set of ϕ_1, Ω_1, and obtain a filtration for $\Omega_1 = \Omega(\phi_1)$. We finish the proof of Theorem A using Lemma 4, which says that if there exists a filtration for Ω_1, then $\Omega = \Omega(\phi)$ coincides with $\Omega_1 = \Omega(\phi_1)$.

I would like to thank J. Palis for profitable conversations.

2. Main Theorem.

Here we give the proof of Theorem A. First we establish some notation, and give some well known results which will be used later.

Throughout, ϕ is assumed to be a C^r flow, $r \geq 1$ on a compact C^∞ manifold M without boundary. For each $x \in M$, we denote by $o(x)$ the orbit of x by ϕ; i.e., $o(x) = \{\phi_t(x) \mid t \in \mathbb{R}\}$. For a subset $D \subset M$, $\bar{D}$ will denote its closure in M, and int D will denote its interior in M.

A compact invariant subset $\Lambda \subset M$ is said to be hyperbolic for ϕ if, for every $t > 0$, $T\phi_t$ leaves invariant a continuous splitting.

$$T_\Lambda M = E^u \oplus E^\phi \oplus E^s$$

expanding E^u and contracting E^s, where E^ϕ is the tangent bundle to the orbits of the flow.

Through each point $x \in \Lambda$ we have C^r injectively immersed manifolds $W^u(x)$, $W^s(x)$ tangent to E^u_x, E^s_x at x. For small $\varepsilon > 0$ we denote by $W^u_\varepsilon(x)$ the closed ε-disc in $W^u(x)$, centered at x, [1], [4].

Let $W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x)$, $W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x)$. Similarly we define $W^u_\varepsilon(\Lambda)$ and $W^s_\varepsilon(\Lambda)$.

We say that Λ has local product structure if

$$W^s_\varepsilon(\Lambda) \cap W^u_\varepsilon(\Lambda) = \Lambda$$

for some $\varepsilon > 0$.

According to a result in [1], if Λ is a hyperbolic set for ϕ, having local product structure, then Λ is an isolated set for ϕ_1, and $W^s(\Lambda) = \{x \in M \mid \omega_1(x) \subset \Lambda\} = \{x \in M \mid \omega(x) \subset \Lambda\}$. That is, the unstable manifold of Λ coincides with the unstable space associated to the isolated set Λ for ϕ_1 as in [2]. Similarly for $W^s(\Lambda)$.

Using this fact and the following lemmas we will prove Theorem A by applyng to ϕ_1 the results proved for diffeomorphisms in [2].

Lemma 1. *Let $x \in M$ and $\gamma = \{\phi_t(x) \mid 0 \leq t \leq 1\}$. If $\omega(x) \supset o(x)$ then for each $p \in \gamma$, there exists $q \in \gamma$ such that $q \in \omega_1(p)$.*

Proof. Let $t_n \to \infty$ as $n \to \infty$ such that $\phi_{t_n}(p) \to p$ as $n \to \infty$. Let $0 < \delta_n < 1$ such that $t_n + \delta_n = k_n \in N$. Put $y_n = \phi_{k_n}(p)$. We may assume that $y_n \to y$. We claim that $y \in o(p)$. For if $y \notin o(p)$, let F be a flow box such that $\phi_t(p) \in F$ for $-1 < t <$ but $y \notin F$ (if the orbit of x is a periodic orbit the result is trivial). Since $\phi_{t_n}(p) \to p$, we have $\phi_{t_n}(p) \in F$ for large n and so $y_n \in F$ for large n. Thus $y \in F$ which is a contradiction. But γ is a fundamental domain for $o(p)$, that is, there exists $n_0 \in \mathbb{Z}$ such that $\phi_{n_0}(y) \in \gamma$. So $\phi_{n_0}(y_n) = \phi_{n_0 + k_n}(p)$ $\phi_{n_0}(y) \in \gamma$.

Lemma 2. *The Birkhoff center of ϕ coincides with the Birkhoff center of ϕ_1, time one map of the flow.*

Proof. All we need to show is that, if $x \in \omega(x)$ then $x \in \omega_1(x)$.

If $p \in \omega_1(p)$ then $\phi_t(p) \in \omega_1(\phi_t(p))$ for every $t \in \mathbb{R}$. So it suffices to prove that there exists $p \in \gamma$ such that $p \in \omega_1(p)$, where $\gamma = \{\phi_t(x) \mid 0 \leq t \leq 1\}$. From Lemma 1 we can construct a sequence p_n, such that $p_1 = x$ and $p_n \in \omega_1(p_{n-1}) \cap \gamma$ for $n > 1$. Since $p_n \in \omega(p_k)$ if $n > k$, we may assume that $p_n \to p \in \gamma$. We claim that $p \in \omega_1(p)$. All we need to show is that, given $\varepsilon > 0$ and an integer $N > 0$, there exists $k > N$ such that $d(\phi_k(p), p) < \varepsilon$, where d is a metric on M. For this, given $\varepsilon > 0$, let $\delta > 0$ be such that if $|s| < \delta$ then $d(q, \phi_s(q)) < \varepsilon/2$. Since p_n, $p \in \gamma$ and $p_n \to p$ there is a number n_0 such that $d(p_{n_0}, p) < \varepsilon/2$ and $p_{n_0} = \phi_t(p)$ for $|t| < \delta$. Since $p_n \in \omega_1(p_{n_0})$ for every $n > n_0$ we conclude that $p \in \omega_1(p_{n_0})$. So there exists $k > N$ such that $d(\phi_k(p_{n_0}), p) < \varepsilon/2$. Thus we have $d(\phi_k(p), p) < d(\phi_k(p), \phi_k(p_{n_0})) + d(\phi_k(p_{n_0}), p) <$ $< d(\phi_k(p), \phi_t(\phi_k(p)) + \varepsilon/2 < \varepsilon$ since $|t| < \delta$.

Let $L_1 = L(\phi_1)$ the limit set of ϕ_1, that is, the union of the closure of the sets $\{x \in M \mid \exists\, y \in M$ such that $x \in \alpha_1(y)\}$ and $\{x \in M \mid \exists\, y \in M$ such that $x \in \omega(y)\}$.

Lemma 3. *Suppose $L_1 \subset \Lambda_1 \cup \ldots \cup \Lambda_k$, where $\{\Lambda_i\}$ $i = 1, \ldots, k$, is a disjoint family of isolated sets for ϕ_1. Let $x \in M$. If $\omega_1(x) \subset \Lambda_i$, then $\omega(x) \subset \Lambda_i$. Similarly if $\alpha_1(x) \subset \Lambda_i$, then $\alpha(x) \subset \Lambda_i$.*

Proof. First we prove the following assertion: there exists $\delta > 0$ such that if $|s| < \delta$, then $\omega_1(\phi_s(x)) \subset \Lambda_i$. To see this, let U be a neighbourhood of Λ_i such that $\bigcap_{n \in \mathbb{Z}} \phi_n(U) = \Lambda_i$ and $U \cap (\bigcup_{j \neq i} \Lambda_j) = \Phi$. Let $\varepsilon > 0$ such that the set $B(y\,;\varepsilon) = \{z \in M \mid d(y, z) < \varepsilon\}$ is contained in U for every $y \in \Lambda_i$. Take $y_1, \ldots, y_\ell \in \Lambda_i$ such that $V = \bigcup_{j=1}^{\ell} B(y_j\,; \varepsilon/2) \supset \Lambda_i$. Let $n_0 \geq 0$ be such that $\phi_n(x) \in V$ for every $n \geq n_0$, and let $\delta > 0$ be such that if $|s| < \delta$, then $d(q, \phi_s(q)) < \varepsilon/2 \ \forall \ q \in M$. Then, given $n \geq n_0$, we have $\phi_n(x) \in V$, so there is $y_j \in \Lambda_i$ such that $d(\phi_n(x), y_j) < \varepsilon/2$. Thus $d(\phi_n(\phi_s(x)), y_j) < d(\phi_{n+s}(x), \phi_n(x)) + d(\phi_n(x), y_j) < d(\phi_s(\phi_n(x)), \phi_n(x)) + \varepsilon/2 < \varepsilon$. That is, $\phi_n(\phi_s(x)) \in U$ for every $n \geq n_0$. Consequently $\omega_1(\phi_s(x)) \subset \Lambda_i$, proving the assertion.

We consider now the curve $\gamma = \{\phi_s(x) \mid 0 \leq s \leq 1\}$. For each $j = 1 \ldots k$ let $A_j = \{y \in \gamma \mid \omega_1(y) \cap \Lambda_j\}$. Since $\{\Lambda_j\}$ $j = 1, \ldots, k$ is a family of disjoint isolated sets for ϕ_1, we have that $\omega_1(y) \subset \Lambda_j$ for some j, for every $y \in M$, [2]. Thus $\gamma = \bigcup_{j=1}^{k} A_j$. From the above assertion A_j is an open set in γ, for each $j = , \ldots, k$. Thus we conclude that $\gamma = A_i$, since the sets A_j, for $j = 1, \ldots, k$ are all disjoint and $x \in A_i$. We have just proved that for each $y \in \gamma$ there exist an open segment β_y in the orbit of x ($\beta_y = \{\phi_s(y)/|s| < \delta\}$) and a number $n(y) > 0$ such that $\phi_n(\beta_y) \subset V$ for every $n \geq n(y)$. Since γ is compact there is a number $n_0 > 0$ such that $\phi_n(\gamma) \subset V$ for every $n \geq n_0$. So $\phi_t(x) \in V$ for every $t \geq n_0$ which implies that $\omega(x) \subset \Lambda_i$.

We now prove that the existence of a filtration for the non-wandering set Ω_1 of ϕ_1 implies that the non-wandering set $\Omega(\phi)$ of the flow coincides with Ω_1.

Lemma 4. *Let $\Omega_1 = \Lambda_1 \cup \ldots \cup \Lambda_k$, where $\{\Lambda_i\}$ for $1 \leq i \leq k$ is a family of disjoint isolated sets for ϕ_1. If there exists a filtration for Ω_1, that is, a family of compact sets $\Phi = K_0 \subset, \ldots, \subset K_k = M$ such that $\phi_1(K_i) \subset$ int K_i and $\bigcap_{n \in \mathbb{Z}} \phi_n(K_i - K_{i-1}) = \Lambda_i$, then $\Omega(\phi) = \Omega_1$.*

Proof. We will show that if, $x \notin \Omega_1$, then x is a wandering point for the flow ϕ. Let Λ_i such that $\omega_1(x) \subset \Lambda_i$. From Lemma 3, $\omega(x) \subset \Lambda_i$. Let $\gamma = \{\phi_s(x)/0 \leq s \leq 1\}$. Since $x \notin \Omega_1$, and so $\gamma \cap \Omega_1 = \Phi$, there are numbers $n_1, n_2 \geq 0$ such that $\phi_{-n}(\gamma) \subset M - K_i$, for every $n > n_1$, and $\phi_n(\gamma) \subset$ int K_i for every $n \geq n_2$. Let U be a neighbourhood of $\phi_{-n_1}(\gamma)$ in $M - K_i$ such that $\phi_{n_1+n_2}(U) \subset$ int K_i. Thus, if W is a neighbourhood of $\phi_{-n_1}(x)$ such that $\phi_s(W) \subset U$ for $0 \leq s \leq 1$, then $W \cap \phi_t(W) = \Phi$ for $t \geq n_1 + n_2$ proving our assertion. Indeed if y, and $\phi_t(y) \in W$, for $t \geq n_1 + n_2$, let $0 < \delta < 1$ be such that $t + \delta = m \in \mathbb{N}$. Thus $\phi_m(y) = \phi_\delta(\phi_t(y)) \in U$ which is a contradiction, since $m = t + \delta \geq t \geq n_1 + n_2$ and so $\phi_m(U) \subset$ int K_i.

With these lemmas and the facts stated below about the Birkhoff center $c(\phi)$ of ϕ, when $c(\phi)$ is hyperbolic, we can prove Theorem A.

Suppose that $c(\phi)$ is hyperbolic for the flow. If F denotes the set of fixed points of ϕ, then F is finite, and each element of F is a hyperbolic fixed point of ϕ, since $E^\phi_x = 0$ for $x \in F$. Continuity of E^ϕ implies that F is disjoint from the closure of $(c(\phi) - F)$, which we denote by Λ.

The first result that we need for $c(\phi)$ is a version for flows of the Anosov Closing Lemma, which says that if $c(\phi)$ is hyperbolic then the periodic orbits of ϕ are dense in $\Lambda = c(\phi) - F$. The proof given by Newhouse [3], for the case of a hyperbolic α-limit set of a diffeomorphyism can be adapted to obtain the above result for $c(\phi)$.

With this result, the same proof of the Ω-decomposition Theorem [4], gives us a decomposition of $c(\phi)$ into a finite number of disjoint hyperbolic sets for ϕ, $\Lambda_1, \ldots, \Lambda_k$ each of them having local product structure. Thus, from the results stated before, each Λ_j is an isolated set for ϕ_1, and $W^u(\Lambda_j) = \{x \in M \mid \alpha_1(x) \subset \Lambda_j\}$, $W^s(\Lambda_j) = \{x \in M \mid \omega_1(x) \subset \Lambda_j\}$.

Definition. A cycle for $c(\phi) = \Lambda_1 \cup \ldots \cup \Lambda_k$ is a sequence. $\Lambda_{i_1}, \ldots, \Lambda_i = \Lambda_{i_1}$ such that there are points $x_1, \ldots, x_j$, with $x_n \notin \bigcup_{\ell=1}^{j} \Lambda_{i_\ell}$, $\alpha(x_n) \subset \Lambda_{i_n}$ and $\omega(x_n) \subset \Lambda_{i_{n+1}}$ for $1 \leq n \leq j$.

Next we use a filtration lemma and a theorem for the Birkhoff center of a homeomorphism of M, which are proved in [2] (Lemma 1.8 and Theorem A), to get the following results for ϕ_1.

Lemma 5. *If $c(\phi_1)$ has a decomposition $c(\phi_1) = \Lambda_1 \cup \ldots \cup \Lambda_k$ into disjoint isolated sets for ϕ_1 having no cycles, then*

(a) *There exists a filtration for $c(\phi_1)$;*

(b) *$c(\phi_1) = \Omega(\phi_1)$.*

We now prove Theorem A.

Proof of Theorem A. From the facts above and Lemma 2 we have that $c(\phi_1)$ has a decomposition $c(\phi_1) = \Lambda_1 \cup \ldots \cup \Lambda_k$ into disjoint isolated sets for ϕ_1. By assumption, this decomposition does not have cycles. Using Lemma 2 and Lemma 5, we conclude that $c(\phi) = \Omega(\phi_1)$. Then, applying Lemma 4 we get $\Omega(\phi_1) = \Omega(\phi)$, and so $c(\phi) = \Omega(\phi)$. Thus ϕ obeys Axiom A' and has the no cyle property, and so, from [4] ϕ is Ω-stable.

Next, we give a corollary of Theorem A. We recall that a flow ϕ is called a Kupka-Smale flow if it satisfies:

(a) The periordic orbits of ϕ are hyperbolic;

(b) The transversality condition.

If in addition $\Omega(\phi)$ is the union of a finite number of periodic orbits, then ϕ is said to be a Morse-Smale flow.

As an immediate consequence of Theorem A we have the following result.

Corollary. *If ϕ is Kupka-Smale flow and $c(\phi)$ is the union of a finite number of periodic orbits, then ϕ is in fact a Morse-Smale flow.*

Finally, we observe that, as in the case of dioffeomorphism [2], we have for flows, a partial converse of Theorem A. In fact, the proof given in [2] (Theorem C) for diffeomorphisms can be adapted to obtain the following result for flows.

Theorem B. *If $c(\phi)$ is a hyperbolic set for the flow, and ϕ is Ω-stable, then $c(\phi)$ has the no cycle property. In particular $c(\phi) = \Omega(\phi)$.*

As a consequence we get:

Corollary. *Let ϕ be a C^r flow such that $c(\phi)$ is the union of a finite number of periodic orbits. Then ϕ is Ω-stable iff $\Omega(\phi) = c(\phi)$ is hyperbolic and has the no cycle property.*

References

[1] M. Hirsch, J. Palis, C. Pugh and M. Shub, *Neighbourhood of hyperbolic sets,* Invent. Math. 9, 121-134 (1970).

[2] I. P. Malta, *Hyperbolic Birkhoff Centers,* Trans. Amer. Math. Soc. (to appear), and An. Acad. Brasil de Ci., 51, 27-29 (1979).

[3] S. E. Newhouse, *Hyperbolic limits sets,* Trans. Amer. Math. Soc., Vol. 167, 125-150 (1972).

[4] C. Pugh and M. Shub, *The Ω-stability Theorem for flows,* Invent. Math. 11, 150-158 (1970).

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