On the boundedness of Ricci curvature of an indefinite metric

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1. Introduction.

Among several conditions on boundedness of Ricci curvature that play important roles in general relativity we have [1, p. 95]

null convergence condition: $R_{ab}w^a w^b \geqq 0$ *for all null vectors w:*

time-like convergence condition: $R_{ab}w^a w^b \geqq 0$ *for all time-like vectors.*

In the present paper we consider the following conditions for a Lorentzian manifold M of dimension $\geqq 3$:

(i) $R_{ab}w^a w^b = 0$ *for all null vectors w*

(ii) $|R_{ab}w^a w^b| \leqq d$ *for all time-like unit vectors w, where d is a certain positive number,*

and prove that each of these conditions implies that M is an Einstein space, that is, $R_{ab} = c\,g_{ab}$. As a matter of fact, these results are valid for any metric of signature $(-,\ldots,+,\ldots)$ and can be stated as therems in linear algebra:

Theorem 1. *Let V be an n-dimensional real vector space with non-degenerate inner preduct* $\langle\,,\rangle$ *of signature* $(-,\ldots,+,\ldots)$. *If a bilinear symmetric function f on V satisfies the condition*

(1) $f(x,x) = 0$ *for all null vectors* $x \in V$, *then there is a constant c such that*

$$f(x,y) = c\langle x,y\rangle \text{ for all } x,y \in V.$$

Theorem 2. *Let V be as in Theorem 1. If a bilinear symmetric function f on V satisfies the condition*

(2) $|f(x,y)| \leqq d$ *for all time-like unit vectors x, i.e.* $\langle x,x\rangle = -1$, *where d is a certain positive number, or*

(2') $|f(x,x)| \leqq d$ *for all space-like unit vectors x, i.e.* $\langle x,x\rangle = 1$, *where d is a certain positive number, then there is a constant c such that*

$$f(x,y) = c\langle x,y\rangle \quad \text{for all} \quad x,y \in V.$$

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In section 2 we shall prove Theorem 1 as well as an equivalent result (Theorem 1a), which we use for the proof of Theorem 2. In section 3 we give a proof of Theorem 2 and add a remark on a result of Kulkarni [2] on sectional curvature of an indefinite metric.

2. Proof of Theorem 1.

One way of proving Theorem 1 is to use the same argument as that in [1, p. 61]. For the sake of completeness we provide the argument.

Let x be a time-like vector and y a space-like vector in V, and consider

$$p(t) = \langle x + ty, x + ty\rangle = \langle x, x\rangle + 2t\langle x, y\rangle + t^2\langle y, y\rangle$$

and

$$q(t) = f(x + ty,\ x + ty) = f(x, x) + 2t\ f(x, y) + t^2\ f(y, y),$$

which are polynomials of degree 2 in t. For $t = 0$, we have $p(t) < 0$ since x is time-like. For large enough $|t|$, we have $p(t) > 0$ since x is space-like. Thus there exist $t_1 < 0 < t_2$ such that $p(t_1) = p(t_2) = 0$ and

$$t_1 t_2 = \frac{\langle x, x\rangle}{\langle y, y\rangle}.$$

By condition (1), we have $q(t_1) = q(t_2) = 0$ and hence

$$t_1 t_2 = \frac{f(x, x)}{f(y, y)}.$$

Therefore

$$\frac{\langle x, x\rangle}{\langle y, y\rangle} = \frac{f(x, x)}{f(y, y)} \quad \text{i.e.} \quad \frac{f(x, x)}{\langle x, x\rangle} = \frac{f(y, y)}{\langle y, y\rangle}, \text{ say, } c.$$

It follows that for any space-like or time-like vector z we get $f(z,z) = c\langle z, z\rangle$. This is valid for any null vector z as well. By polarization we easily get

$$f(z, w) = c\langle z, w\rangle \quad \text{for all} \quad z, w \in V.$$

In order to state an equivalent result, we consider the following conditions for a bilinear symmetric function f:

(1a) If $\langle x, x\rangle = -1$, $\langle y, y\rangle = 1$ and $\langle x, y\rangle = 0$, then $f(x, y) = 0$;

(1b) If $\langle x, x\rangle = -1$, $\langle y, y\rangle = 1$ and $\langle x, y\rangle = 0$, then $f(x, x) + f(y, y) = 0$.

Lemma 1. (1) *implies* (1a) *and* (1b).

To prove this, let x, y be two vectors as in (1a). Then $x + y$ and $x - y$ are null vectors. By (1) we get

$$f(x + y,\ x + y) = f(x - y,\ x - y) = 0,$$

i.e.

$$f(x, x) + 2f(x, y) + f(y, y) = 0$$

$$f(x, x) - 2f(x, y) + f(y, y) = 0.$$

Hence $f(x, y) = 0$ and $f(x, x) + f(y, y) = 0$.

Lemma 2. (1a) *implies* (1b) *and* (1).

Let x, y be as in (1b). Then $x_1 = \cosh tx + \sinh ty$ and $y_1 = \sinh tx + \cosh ty$ form another orthonormal pair like $\{x, y\}$. Thus by (1a) we get

$$\begin{aligned} 0 &= f(x_1, y_1) = \\ &= (\cosh t\ \sinh t)(f(x, x) + f(y, y)) + (\sinh^2 t + \cosh^2 t)\ f(x, y) = \\ &= (\cosh t \sinh t)(f(x, x) + f(y, y)), \end{aligned}$$

since $f(x, y) = 0$. Thus for $t \neq 0$ we get $f(x, x) + f(y, y) = 0$, proving (1b).

Now let u be a null vector. Then we can find x, y such that $\langle x, x\rangle = -1$, $\langle y, y\rangle = 1$, $\langle x, y\rangle = 0$ and $u = x + y$. Then

$$f(u, u) = f(x + y,\ x + y) = f(x, x) + f(y, y) + 2f(x, y) = 0$$

by virtue of (1a) and (1b).

Remark. (1b) implies (1a), as can be proved in a similar way. Thus (1), (1a) and (1b) are equivalent.

We now give an alternate proof of the following

Theorem 1a. *If a bilinear symmetric function f satisfies condition* (1a), *then there is a constant c such that*

$$f(x, y) = c\langle x, y\rangle \quad \textit{for all} \quad x, y \in V.$$

Let $\{e_1, \dots, e_r, e_{r+1}, \dots, e_n\}$ be an orthonormal basis of V such that

$$\langle e_i, e_i \rangle = -1 \quad \text{for} \quad 1 \leqq i \leqq r$$

$$\langle e_j, e_j \rangle = \ \ 1 \quad \text{for} \quad r+1 \leqq j \leqq n.$$

We shall prove that $f(e_i, e_j) = 0$ for $i \neq j$. If $1 \leqq i \leqq r$ and $r+1 \leqq j \leqq n$, then this is satisfied by virtue of condition (1a). Now assume $1 \leqq i, j \leqq r$ (the case where $r+1 \leqq i, j \leqq n$ is similar). Take any $k \geqq r+1$ and set $z = \sinh t\, e_i + \cosh t\, e_k$. Then $\langle z, z \rangle = 1$ and $\langle z, e_j \rangle = 0$. Thus condition (1a) implies

$$0 = f(z, e_j) = \sinh t\ f(e_i, e_j) + \cosh t\ f(e_k, e_j) = \sinh t\ f(e_i, e_j),$$

because $f(e_k, e_j) = 0$. For $t \neq 0$, we obtain $f(e_i, e_j) = 0$.

Now let $c_i = f(e_i, e_i)$ for $1 \leqq i \leqq n$. By condition (1b) which follows from (1a), we have $c_i + c_j = 0$ for $1 \leqq i \leqq r$ and $r+1 \leqq j \leqq n$. Thus

$$-c_1 = \dots = -c_r = c_{r+1} = \dots = c_n, \text{ say, } c.$$

It follows that $f(x, y) = c\langle x, y \rangle$ for all $x, y \in V$.

3. Proof of Theorem 2.

We shall prove Theorem 2 under assumption (2). Let x, y be two vectors such that $\langle x, x \rangle = -1$, $\langle y, y \rangle = 1$ and $\langle x, y \rangle = 0$. For $|t| > 1$, we have $\langle tx + y, tx + y \rangle = 1 - t^2 < 0$. Thus

$$u = \frac{tx + y}{(t^2 - 1)^{1/2}}$$

is a time-like unit vector. By assumption (2) we get

$$-d \leqq \frac{f(tx + y, tx + y)}{t^2 - 1} \leqq d$$

that is,

$$-d(t^2 - 1) \leqq t^2 f(x, x) + f(y, y) + 2t\, f(x, y) \leqq d(t^2 - 1).$$

Let $t \to 1$ from above. Then

$$f(x, x) + f(y, y) + 2f(x, y) = 0.$$

Let $t \to -1$ from below. Then

$$f(x, x) + f(y, y) - 2f(x, y) = 0.$$

From these two equations, we get $f(x, y) = 0$ and $f(x, x) + f(y, y) = 0$. By Theorem 1a, we get the conclusion of Theorem 2.

The proof under assumption (2′) is similar.

Remark. The above proof has been inspired by the work of Kulkarni [2]. He shows for an indefinite metric of signature $(-, \dots, +, \dots)$ that if the sectional curvature function K is bounded from below (or from above) on the set of all nondegenerate 2-planes, then K is a constant function. For boundedness of K on all time-like (or space-like) 2-planes, we may establish the following.

Proposition. *Let M be a manifold of dimension $\geqq 3$ with an indefinite metric of signature $(-, \dots, +, \dots)$. If there is some $d > 0$ such that*

$$|K(p)| \leqq d \text{ for all time-like (or space-like) 2-planes } p,$$

then K is a constant function.

We note that one-sided boundedness on all time-like (or space-like) 2-planes:

$$K(p) \geqq d \quad \text{or} \quad K(p) \leqq d$$

does not imply that K is a constant function, as can be shown by using the spaces

$$S_1^2 \times \mathbb{R} \quad \text{or} \quad H_1^2 \times \mathbb{R},$$

where S_1^2 (resp. H_1^2) is the 2-dimensional Lorentz manifold of constant sectional curvature 1 (resp. -1). The same spaces serve as examples showing that one-sided boundedness of Ricci curvature on all time-like (or space-like) unit vectors:

$$R_{ab} w^a w^b \geqq d \quad \text{or} \quad R_{ab} w^a w^b \leqq d$$

does not imply that the space is Einstein.

References

[1] Hawking, S. W., Ellis, G. F. R., *The large-scale structure of space-time*, Cambridge-London-New York-Melbourne, Cambridge University Press, 1973.

[2] Kulkarni, R. S., *The values of sectional curvature in indefinite metrics*, Comm. Math. Helv. 54, 173-176 (1979).

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