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A note on non-linearizable analitic functions

*P. R. Sad**

1. Introduction.

Let $f(z) = \lambda z + a_2 z^2 + \ldots$, where $|\lambda| = 1$, be a power series with positive radius of convergence. The local structure of orbits of f is completely determined when λ is a root of unity (see [2]). We consider here the case λ is not a root of unity. Our viewpoint consists in finding topological rather than analytical reasons that in some cases don't allow f to be linearizable. It is known that there exists a subset $A \subseteq S^1$ with measure 1 such that if $\lambda \in A$ then f is linearizable in a neighbourhood of 0 (see [7]). What does it happen if $\lambda \in S^1 - A$? We should mention first the very interesting fact (of easy proof) that if f is C^0-linearizable near 0, then f is also analytically linearizable. This suggests us to pay attention to the topological behavior of f near 0. We have the following theorem.

Theorem. *For a dense subset $\tilde{A}$ of S^1, given $\lambda \in \tilde{A}$ we can find a convergent power series $f_\lambda(z) = \lambda z + a_2 z^2 + \ldots$ such that there exists a sequence of periodic orbits of f_λ approaching 0 with arbitrarily large periods. In particular, f is not C^0-linearizable.*

This is quite different from classical ideas due to Julia ([5] and [6]). He studies generic polynomial endomorphisms $P(z) = \lambda z + \ldots + a_n z^n$ which are not linearizable and shows that a periodic orbit appears arbitrarily close to 0 but the whole orbit probably goes far from the origin. We present the proof of this beautiful result at §2.

We would like to note that, in [8], Siegel shows that for a dense subset A' of S^1, given $\lambda \in A'$ there exists $f_\lambda(z) = \lambda z + a_2 z^2 + \ldots$ non-linearizable; the proof involves the non convergence of the associated Schröder series. So we have an interesting viewpoint duality.

We obtain the power series $f(z)$ mentioned in the above theorem through a limit process. In [4], Cremer asks for a way of constructing fixed points of analytical functions that are not centers.

The techniques involved in the proof of the theorem are fairly simple. We hope they can be improved in order to prove the following.

*Departamento de Matemática, ICEx, UFMG, Belo Horizonte, MG, **BRASIL**.
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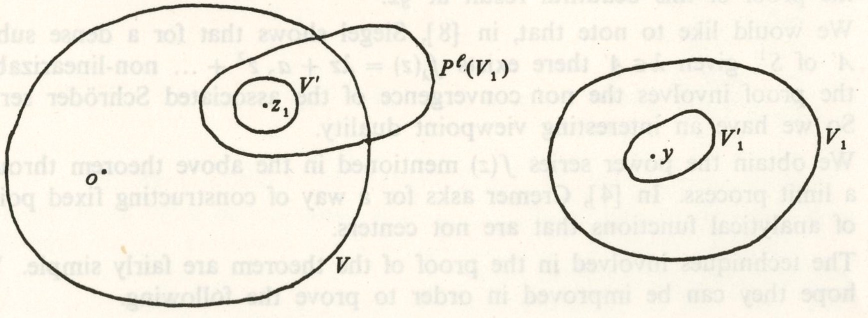
Conjecture. *Let $f(z) = \lambda z + a_2 z^2 + \dots$ be non-linearizable, $|\lambda| = 1$. If λ is not a root of unity, then 0 is accumulated by periodic orbits of f with arbitrarily large periods.*

2. Contrast with classical theory.

We consider here a polynomial endomorphism $P(z) = \lambda z + \dots + a_n z^n$ not linearizable at 0, $n \geq 2$. It has infinitely many periodic orbits; we may suppose all these orbits hyperbolic (with the exception of 0). It can be shown that a finite number of them are sinks (see [5]). We look for a periodic point close to the origin. We proceed as follows:

Step 1. let V be a neighbourhood of O; then $\mathscr{F} = \{P^n/V_1;\ n \in \mathbb{N}\}$ is not a normal family. Let's distinguish two cases: (i) λ is a root of unity. Hence, by [2] there exists z_0 em V such that $P^n(z_0) \to O$. If $\mathscr{F}$ were a normal family, it would exist $n_k \to \infty$ and an analytic function Q: $V \to \mathbb{C}$ such that $P^{n_k} \to Q$; but $P^{n_k}(z_0) \to O$ so $Q(z_0) \equiv O$, contradicting $|Q'(O)| = 1$. (ii) λ is not a root of unity. Choose $n_k \to \infty$ such that $P^{n_k} \to Q$ and $(P^{n_k})'(O)$ converges to λ_0 belonging to the Siegel set A. It follows that $PQ = QP$; but Q linearizable (and λ not a root of unity) implies P linearizable (use formal series).

Step 2. by a theorem of Montel (see [3], pg. 302), if $\# \left(\mathbb{C} - \bigcup_{n=0}^{\infty} P^n(V)\right) \geq 2$ then $\mathscr{F}$ is a normal family. So we may take $z_1 \in V$ such that some power $y = P^k(z_1)$ is a periodic source of P of period m.



Let V_1 be a small neighbourhood of y. Again, $\{P^{mn}/V_1;\ n \in \mathbb{N}\}$ can't be a normal family (because $|(P^m)'(y)| > 1$); hence some $P^l(V_1) \ni z_1$, $l \in \mathbb{N}$. Now take a disk $V' \subset P^l(V_1)$ with center z_1 such that $V_1' = P^k(V')$ is contained in V_1. By hyperbolicity of the source y some $P^{mj}(V_1')$ contains V_1 for $j \in \mathbb{N}$. Therefore some power $P^i(V') \supset V'$, $i \in \mathbb{N}$; this implies the existence of a periodic point of P inside V'. We note that these are global arguments; the periodic orbit we have found is not necessarily close to O.

3. Proof of the theorem.

Let $f: V \to \mathbb{C}$ be an analytic function defined on an open subset $V \subseteq \mathbb{C}$, $0 \in V$. If $f'(0) \neq 0$ we know that f is a diffeomorphism in some disk $D \subseteq V$. There exists $\varepsilon > 0$ such that if $g: V \to \mathbb{C}$ is analytic and $|g(z) - f(z)| < \varepsilon$ for $z \in D$ ($\|g - f\|_D < \varepsilon$) then g is a diffeomorphism from D to $g(D)$. Let $\mathscr{N} = \mathscr{N}(f, D, \varepsilon)$ be the set of such analytic functions; we consider only those which take 0 in 0.

Lemma 1. *Let $\lambda_0 = f'(0) \neq 1$ be a root of unity: ($\lambda_0^n = 1$, n being the least integer $k > 1$ such that $\lambda_0^k = 1$). There exists a disk $\tilde{D} \subseteq D$ and an analytic change of coordinates G_g defined on a neighbourhood of 0 such that $G_g \cdot g \cdot G_g^{-1}(z) = \lambda z + z^{n+1} R(z, g)$ for $z \in \tilde{D}$ and $g \in \mathscr{N}$. Furthermore, $\|G_g \cdot g \cdot G_g^{-1} - F \cdot f \cdot F^{-1}\|_{\tilde{D}} \to 0$ if $\|g - f\|_D \to 0$ ($F = G_f$, and each G_g takes 0 in 0).*

Proof.

1 — Let's first prove the lemma for f, and so extend it for g close to f. Let $f(z) = f_2(z) = \lambda_0 z + b_2'' z^2 + \dots$ we look for ψ_2 analytic such that $\psi_2 \cdot f_2 \cdot \psi_2^{-1}(z) = \lambda_0 z + b_3''' z^3 + \dots = f_3(z)$. Take $\psi_2(z) = z + a_2 z^2$; to find a_2, we note that $(z + a_2 z^2) \cdot (\lambda_0 z + b_2'' z^2 + \dots) = (\lambda_0 z + b_3''' z^3 + \dots) \cdot (z + a_2 z^2)$ impplies

$$a_2 = \frac{b_2''}{\lambda_0 - \lambda_0^2}.$$

We continue this way; then we determine a_i, $2 \leq i \leq n - 1$, such that if $\psi_i(z) = z + a_i z^i$ we have $\psi_i \cdot f_i = f_{i+1} \cdot \psi_i$ where $f_i(z) = \lambda_0 z + b_i^{(i)} z^i + \dots$ and

$$a_i = \frac{b_i^{(i)}}{\lambda_0 - \lambda_0^i}.$$

Now if $f_n(z) = \lambda_0 z + b_n^{(n)} z^n + \dots$, we choose

$$a_n = \frac{b_n^{(n)}}{\lambda_0 - \lambda_0^n}$$

and $\psi_n(z) = z + a_n z^n$ and get

$$\psi_n \cdot f_n = f_{n+1} \cdot \psi_n$$

where

$$f_{n+1}(z) = \lambda_0 z + b_{n+1}^{(n+1)} z^{n+1} \dots$$

We finally take $F = \psi_n \cdot \psi_{n-1} \cdot \ldots \cdot \psi_2$ and so

$$F \cdot f \cdot F^{-1}(z) = \lambda_0 z + z^{n+1} R(z, f).$$

It's obvious that this construction can be extended continuosly to g close to f.

Lemma 2. *Let $f(z) = \lambda_0 z + z^{n+1} R(z, f)$ $(\lambda_0^n = 1, \lambda_0 \neq 1, R(0, f) \neq 0)$ be an analytic function defined on a neighbourhood V of O. There exists a disk $D' \subseteq V$ such that if $g(z) = \lambda z + z^{n+1} R(z, g)$ is defined on V and $\|g - f\|_{D'}$ is small then $g^n(z) - z$ and $f^n(z) - z$ have the same number of roots inside D'.*

Proof.

1 — We may suppose that there exists $V' \subseteq V$ such that if $\|g - f\|_{\bar{D}}$ is small enough then g^n is defined on V; let $D' \subseteq V'$ be a disk with $0 \in D'$. We have $\|g - f\|_{D'} \to 0 \Rightarrow \|g^n - f^n\|_{D'} \to 0$, and $g^n(z) = \lambda^n z + z^{n+1} R(z, g^n)$ and $f^n(z) = \lambda_0^n z + z^{n+1} R(z, f^n)$. We know that $\|R(z, g^n) - R(z, f^n)\|_{D'} \to 0$ if $\|g - f\|_{D'} \to 0$. Now,

$$|g^n(z) - f^n(z)| \le |(\lambda^n - 1)z| + |z^{n+1}(R(z, g^n) - R(z, f^n))|.$$

We want to compare $|g^n(z) - f^n(z)|$ with

$$|f^n(z) - z| = |z^{n+1} R(z, f^n)| \text{ ou } \partial D'.$$

Let $M = \min\limits_{z \in \partial D'} |R(z, f^n)| > 0$ and $|z| = a$ if $z \in \partial D'$; then

$$\left| \frac{g^n(z) - f^n(z)}{f^n(z) - z} \right| \le \frac{|(\lambda^n - 1)\, a|}{M a^{n+1}} + \frac{a^{n+1} \|R(z, g^n) - R(z, f^n)\|_{\partial D'}}{M a^{n+1}}.$$

From that we get $\left| \dfrac{g^n(z) - f^n(z)}{f^n(z) - z} \right| < 1$ for $z \in \partial D'$ if $\|g - f\|_{D'}$ is small enough. Lemma 2 follows from a well known theorem (see [1], page 152).

Corollary. *Let f and g be as in lemma 1, and n a prime integer. If $\|g - f\|_D$ is small enough and $g'(0) \neq \lambda_0$, then g has a periodic orbit of period n close to the origin, and 0 is a fixed point of g. Furthermore, this periodic orbit tends to 0 if $\|g - f\|_D \to 0$.*

Proof.

1 — By lemma 2, we may suppose that $f(z) = \lambda_0 z + z^{n+1} R(z, f)$ and $g(z) = \lambda z + z^{n+1} R(z, f)$; as $f^n(z) = z + z^{n+1} R(z, f^n)$ we see that 0 is a root of $f^n(z) - z$ with multiplicity $n + 1$; from that it follows $g^n(z) - z$ has $n + 1$ roots close to 0. One of them is 0, and the other n roots constitute the orbit of period n. We note that 0 is an isolated fixed point of f with multiplicity 1.

Now we construct analytic local diffeomorphisms of $\mathbb{C}$ that cannot be linearized because there exists a sequence of periodic orbits of crescent period approaching the fixed point 0. We begin with $f_0(z) = e^{2\pi i \lambda_0} z + z^{n+1}$, where $e^{2\pi i \lambda_0 n} = 1$ and we choose $V(\lambda_0) \subseteq \mathbb{C}$ such that $g(z) = e^{2\pi i \lambda} z + z^{n+1}$ satisfies the corollary above for $\lambda \in V(\lambda_0)$.

Let $\lambda_1 \in V(\lambda_0)$ be such that $e^{2\pi i \lambda_1 n_1} = 1$, $n_1 > n$ being a prime integer. Choose now $V(\lambda_1) \subseteq V(\lambda_0)$ with $\lambda_0 \notin V(\lambda_1)$. We note that (i) in some disk around 0 there exists a periodic orbit of g with period n; (ii) if $V(\lambda_1)$ is small enough, there exists a periodic orbit of g with period n_1 inside a disk strictly cointained in the first one. Continuing this way we get a sequence $V(\lambda_0) \subseteq V(\lambda_1) \subseteq \ldots \subseteq V(\lambda_i)$ with $\lambda_{j-1} \notin V(\lambda_j)$ and prime integers $n < n_1 < n_2 < \ldots < n_i$ with the following property: there exist disks $D_0 \subseteq D_1 \subseteq \ldots \subseteq D_i$, with radius $(D_i) \to 0$ such that if $g(z) = e^{2\pi i \lambda} z + z^{n+1}$, $\lambda \in V(\lambda_i)$, then g has inside D_i a periodic orbit with period n_i. Let $a \in \bigcap_{i=0}^{\infty} V(\lambda_i)$ and $g(z) = e^{2\pi i a} z + z^{n+1}$. Then, the fixed point 0 of g is accumulated by a sequence of periodic orbits whose periods increase without limit, and from this it follows that g can not be linearized. We have $|e^{2\pi i a}| = 1$, and $a \notin \mathbb{Q}$ (see [2]). The construction we have done shows also that the subset of $\lambda \in S^1$ such that there exists *a* non-linearizable analytic function $f(z) = \lambda z + \ldots$ is dense in S^1.

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Departamento de Matemática
Instituto de Ciências Exatas
Universidade Federal de Minas Gerais
Cidade Universitária Pampulha
30.000 — Belo Horizonte, MG
Brasil