Perturbations of $-\Delta$ that grow at infinity in certain directions

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1. Introduction.

In this article we study the spectrum of the self-adjoint realization of the formal differential expression $-\Delta + q(x_1) + V(x_1, x_2)$ in the framework of time independent scattering theory. Here $(x_1, x_2) \in \mathbb{R}^n$, $n = n_1 + n_2$, $n_1 \geq 1$, is the laplacian in $L^2(\mathbb{R}^n)$, $V(x_1, x_2)$ falls off in the x_2 direction and is bounded in the x_1 direction, while $q(x_1)$ is bounded from below and tends to infinity as $|x_1| \to \infty$, (for the precise conditions see sections 2 and 4). The main ideas used in this paper come from the works of Kato 7, Kato and Kuroda 8, and Agmon 1. The ideas and techniques in these articles have been applied by many authors to a wide variety of differential operators with great success. For instance, applications to many-particle Schrödinger operators appear in 3, 4, 5, to uniformly propagative systems in 13, 14 to Klein-Gordon type equations in 6, and to perturbations of the laplacian in two-point homogeneous spaces in 12.

The present article is divided as follows: In section 2 we describe que free operator T_0, which is the self-adjoint closure of $-\Delta + q(x_1)$ in $L^2(\mathbb{R}^n)$. The results stated in section 2 are well-known and their proofs can be found in 2, 9 and 10. In section 3 we introduce some useful function spaces and study some properties of the free resolvent. In section 4 the conditions on the perturbation $V(x_1, x_2)$ are given and the resolvent equation is discussed. In section 5 we show that the point spectrum of the perturbed operator is countable and that its singularly continuous spectrum is empty. Finally there are two appendixes containing the more technical estimates necessary for the proofs.

Along this article the following notations and definitions will be used. If X and Y are Banach apaces, the set of all bounded (resp. compact) operators from X to Y will be denoted by $B(X, Y)$ (resp. $B_0(X, Y)$). In case $X = Y$ we write simply $B(X)$ and $B_0(X)$. If H is a Hilbert space, its norm and inner product are denote by $\|\cdot\|_H$ and $(\cdot|\cdot)_H$ respectively. In case there is no possibility of confusion the subscript H will be droped. If T is a self-adjoint operator in H, we denote its spectrum by $\Sigma(T)$. The discrete spectrum of T, i.e. the set of all isolated eigenvalues with finite multiplicity is denoted by $\Sigma_d(T)$, while its complement in $\Sigma(T)$, the essential spectrum of T is denoted by $\Sigma_e(T)$. The absolutely continuous and singularly conti-

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tinuous spectra are denoted by $\Sigma_{ac}(T)$ and $\Sigma_{sc}(T)$ respectively. The algebraic tensor product of two subspaces V_1 and V_2 of the Hilbert spaces H_1 and H_2 is denoted by $V_1 \odot V_2$ and its closure by $V_1 \otimes V_2$. Integrals without explicit limits of integration are to be taken over $\mathbb{R}^s$ where s will be clear from the context. The letter C will denote the various positive constants whose precise values are of no interest. Finally the positive integers are denoted by Z^+.

2. The Free operator.

Let Δ_i, $i = 1,2$ denote the laplacian defined in $C_0^\infty(\mathbb{R}^{n_i})$ and let $q(x_1) \in L^2{}_{\text{loc}}(\mathbb{R}^{n_1})$ be non-negative pointwise. It is wellknown that the operators $-\Delta_1 + q$ and $-\Delta_1 - \Delta_2 + q$ are essentially self adjoint in $C_0^\infty(\mathbb{R}^{n_1})$ and $C_0^\infty(\mathbb{R}^n)$ respectively, where $n = n_1 + n_2$. Let t_0, T_0 and H_0 denote the unique self adjoint realizations of $-\Delta_1 + q$, $-\Delta_1 - \Delta_2 + q$ and $-\Delta_2$ in $L^2(\mathbb{R}^{n_1})$, $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^{n_2})$ respectively. Then we have,

$$T_0 = t_0 \otimes I_2 + I_1 \otimes H_0 \tag{2.1}$$

Morever, $D(t_0) \odot D(H_0)$ is a core for T_0. We will assume the following conditions on t_0:

$(t_0 - 1)$ $\Sigma(t_0)$ consists of an infinite number of positive eigenvalves, $\lambda_1 < < \lambda_2 < \lambda_3 < \dots$ with finite multiplicity and no finite accumulation point.

$(t_0 - 2)$ There is an $\alpha \in (0, 1)$ such that

$$\sum_{i=1}^{\infty} (\lambda_i)^{-2(1-\alpha)} < \infty.$$

In particular, the series $\sum_{j=1}^{\infty} (\lambda_j)^{-2}$ converges and $(t_0{}^{-1})$ is a Hilbert-Schmidt operator. Also, $\Sigma(T_0) = \Sigma_e(T_0) = \Sigma(t_0) + \Sigma(H_0) = [\lambda_1, \infty)$.

Remarks. 1) Assumption $(t_0 - 1)$ is satisfied in case q is bounded from below and $q(x_1) \to \infty$ as $|x_1| \to \infty$ (see 10 page 249). In fact we assume $q(x_1) > 0$ for convenience and the results stated below can be easily modified to fit the case q bounded from below.

2) Assumptions $(t_0 - 1)$ and $(t_0 - 2)$ are satisfied for example by $q(x_1) = |x_1|^2 + |x_1|^{2m}$, $m = 2, 3, \dots$, This is well known for $|x_1|^2$ and follows from the mini-max principle (10, page 75) for $q(x_1)$.

Let m_i be the multiplicity of λ_i and let, $\{\phi_j^{(i)}\}$ $j = 1, \dots, m_i$ denote an orthonormal basis of the corresponding eigenspace. With this notation

the resolvent $R_{t_0}(z) = (t_0 - z)^{-1}$ and the spectral family $E_{t_0}(\lambda)$ can be written as,

$$R_{t_0}(z) = \sum_{i=1}^{\infty} \frac{P_{t_0}(\lambda_i)}{z - \lambda_i} \tag{2.2}$$

$$E_{t_0}(\lambda) = \begin{cases} 0, & \lambda < \lambda_1 \\ \sum_{\lambda_i \leq \lambda} P_{t_0}(\lambda_i), & \lambda \geq \lambda_1 \end{cases} \tag{2.3}$$

where $P_{t_0}(\lambda_i)$, the projection on the eigenspace of λ_i is given by,

$$P_{t_0}(\lambda_i) f = \sum_{j=1}^{mi} (f \mid \phi_j^{(i)})\, \phi_j^{(i)}, \quad f \in L^2(\mathbb{R}^{n_1}). \tag{2.4}$$

Let $R_{T_0}(z)$, $z \notin [\lambda_1, \infty)$ and $E_{T_0}(\lambda)$, $\lambda \in \mathbb{R}$, denote the resolvent and spectral family of T_0. Then,

$$R_{T_0}(z) = \sum_{i=1}^{\infty} P_{t_0}(\lambda_i) \otimes R_0(z - \lambda_i) \tag{2.5}$$

$$E_{T_0}(\lambda) = \sum_{i=1}^{\infty} P_{t_0}(\lambda_i) \otimes E_0(\lambda - \lambda_i) \tag{2.6}$$

Here, $R_0(z)$, $z \notin [0, \infty)$ and $E_0(\lambda)$, $\lambda \in \mathbb{R}$ denote the resolvent and spectral family of H_0. We remark that the series in (2.6) is in fact a finite sum (since $E_0(\lambda - \lambda_i) = 0$ for $\lambda - \lambda_i \leq 0$) and that $(t_0 - 2)$ implies the convergence of the series in (2.2) and (2.5) in $B(L^2(\mathbb{R}^{n_1}))$ and $B(L^2(\mathbb{R}^n))$ respectively. That (2.5) and (2.6) do represent the resolvent and spectral family of T_0 can be easily seen by direct computation or as a consequence of general results in chapter VI of 2. Note that (2.5) and (2.6) can be rewitten as

$$R_{T_0}(z) = \sum_{i=1}^{\infty} \sum_{j=1}^{mi} J_j^{(i)} R_0(z - \lambda_i)\, G_j^{(i)} \tag{2.5}$$

$$E_{T_0}(\lambda) = \sum_{i=1}^{\infty} \sum_{j=1}^{mi} J_j^{(i)} E_0(\lambda - \lambda_i)\, G_j^{(i)} \tag{2.6}$$

where $J_i^{(i)} \in B(L(\mathbb{R}^{n_2}),\ L^2(\mathbb{R}^n))$ and $G_j^{(i)} \in B(L^2(\mathbb{R}^n),\ L(\mathbb{R}^{n_2}))$ are defined by,

$$J_j^{(i)} f = \phi_j^{(i)} \otimes f \tag{2.7}$$

$$(G_j^{(i)} g)(x_2) = \int dx_1\, \bar{\phi}_j^{(i)}(x_1)\, g(x_1, x_2) \tag{2.8}$$

Note that $G_j^{(i)} J_j^{(i)} f = f$, for all $f \in L^2(\mathbb{R}^{n_2})$, $G_j^{(i)} J_j^{(i')} = 0$ for $i \neq i'$ and $\|J_i\|, \|G_i\| \le 1$.
Moreover $J_j^{(i)} G_j^{(i)} g = \phi_j^{(i)} \otimes G_j^{(i)} g$ and we have,

(2.9) $$(J_j^{(i)} f \mid g)_{L^2(\mathbb{R}^n)} = (f \mid G_j^{(i)} g)_{L^2(\mathbb{R}^n)}$$

for al $f \in L(\mathbb{R}^{n_2})$, $g \in L^2(\mathbb{R}^n)$.

Proposition 2.1. $\Sigma(T_0) = \Sigma_{ac}(T_0) = [\lambda_1, \infty)$.

Proof. It suffices to show that the function $\lambda \in [a, b] \to (E_{t_0}(\lambda)\phi \mid \phi)$ is absolutely continuous for all $\phi \in L^2(\mathbb{R}^n)$ and $[a, b] \subset \mathbb{R}$. If $b < \lambda_1$, there is nothing to prove because in this case, $E_{T_0}(\lambda) = 0$. If $b \le \lambda_1$, an easy computation shows that,

$$(E_{T_0}(\lambda)\phi \mid \phi)_{L^2(\mathbb{R}^n)} = \sum_{\lambda_i \le \lambda} \sum_{j=1}^{mi} (E_0(\lambda - \lambda_i) G_j^{(i)}\phi \mid G_j^{(i)}\phi)_{L^2(\mathbb{R}^{n_1})}.$$

Since the R. H. S. of this equality is a finite sum of absolutely continuous functions we are done.

In order to simplify the notation, we assume from now on that $m_i = 1$, $i = 1, 2, 3, \ldots$. The normalized eigenfunctions corresponding to λ_i will be written φ_i. We also let $J_i = J_1^{(i)}$, $G_i = G_1^{(i)}$. It is a trivial matter to modify the arguments and results below in order to accomodate the case $m_i > 1$.

3. Function spaces and the Free Resolvent.

Let $L^2_\delta(\mathbb{R}^{n_2})$, $\delta \in \mathbb{R}$ denote the set of all complex valued measurable functions $u(x_2)$ such that,

(3.1) $$\|u\|^2_{L^2_\delta} = \int dx_2 (1 + |x_2|^2)^\delta |u(x_2)|^2 < \infty$$

and define $\mathscr{G}_{0,\delta}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \otimes L^2_\delta(\mathbb{R}^{n_2})$, provided with the norm,

(3.2) $$\|f\|^2_{0,\delta} = \int dx_1 dx_2 (1 + |x_2|^2)^\delta |f(x_1, x_2)|^2.$$

Now let,

$$\mathscr{G}_{2,\delta}(\mathbb{R}^n) = \{ f \in \mathscr{G}_{0,\delta}(\mathbb{R}^n) \mid (1 - \Delta + q) f \in \mathscr{G}_{0,\delta}(\mathbb{R}^n) \}$$

(3.3) $$\|f\|_{2,\delta} = \|(1 - \Delta + q) f\|_{0,\delta}$$

where the laplacian is computed in the sense of distribuition theory. Note that $D(T_0) = \mathscr{G}_{2,0}(\mathbb{R}^n)$ and $\mathscr{G}_{0,0}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. In what follows M_s denotes the operator of multiplication by $M_s(x_2) = (1 + |x_2|^2)^{-s/2}$.

Theorem 3.1. *If $\delta > 0$ the inclusion $\mathscr{G}_{2,\delta}(\mathbb{R}^n) \hookrightarrow \mathscr{G}_{0,0}(\mathbb{R}^n)$ is compact.*

Proof. It suffices to prove that the operator $(1 + T_0)^{-1} M_\delta$ $B_0(L^2(\mathbb{R}^n)$ or equivalently $M_\delta(1 + T_0)^{-1} \in B_0(L^2(\mathbb{R}^n))$. The proof is similar to that of theorem (4.1) of the next section and is indicated in appendix 1.

Theorem 3.2. $R_{T_0}(z) \in B(\mathscr{G}_{0,\delta}(\mathbb{R}^n), \mathscr{G}_{2,-\delta}(\mathbb{R}^n))$, $\delta > /2$, $z \notin [\lambda_1, \infty)$. *Moreover if $\lambda \in (-\infty, \lambda_1)$ or $\lambda \in (\lambda_N, \lambda_{N+1})$ for some $N \in Z^+$, then*

$$\lim_{\varepsilon \downarrow 0} R_{T_0}(\lambda \pm i\varepsilon) = R_{T_0}(\lambda \pm io)$$

exists in the norm topology of $B(\mathscr{G}_{0,\delta}(\mathbb{R}^n), \mathscr{G}_{2,-\delta}(\mathbb{R}^n))$.

Proof. The first statement is equivalent to showing that

$$M_\delta(1 + T_0) R_{T_0}(z) M_\delta \in B(L^2(\mathbb{R}^n)),$$

$z \notin [\lambda_1, \infty)$. But since, $(1 + T_0) R_{T_0}(z) = (1 + z) R_{T_0}(z) + 1$, it follows that, $M_\delta(1 + T_0) R_{T_0}(z) M_\delta = (1 + z) M_\delta R_{t_0}(z) M_\delta + M_\delta^2$.

This proves the first statement. As for the second, if $\lambda = \operatorname{Re} z \in (-\infty, \lambda_1)$ there is nothing to prove. Suppose therefore that $\lambda \in (\lambda_N, \lambda_{N+1})$ for some $N \in Z^+$ and write:

$$M_\delta R_{T_0}(z) M_\delta = \sum_{j=i}^{N} M_\delta J_i R_0(z - \lambda_i) G_i M_\delta + M_\delta \sum_{j=N+1}^{\infty} J_i R_0(z - \lambda_i G_i M_\delta.$$

Consider first the infinite series. Using the properties of J_i, G_i and the facts $M_\delta J_i = J_i M_\delta$, $M_\delta G_i = G_i M_\delta$, $\|R_0(z - \lambda_i)\| \le (\operatorname{dist}(z - \lambda_i, \Sigma(H_0)))^{-1} \le$ $\le (\lambda_i - \lambda)^{-1}$, we see that

$$\|M \sum_{j=N+1}^{K} J_i R_0(z - \lambda_i) G_i M_\delta\|^2 \le C \sum_{j=N+1}^{K} \|R_0(z - \lambda_i)\|^2 \le$$

$$\le C \sum_{j=N+1}^{K} (\lambda_i - \lambda)^{-2}.$$

Hence the series converges by $(t_0 - 2)$. A similar estimate shows that

$$\lim_{\varepsilon \downarrow 0} M_\delta \sum_{j=N+1}^{K} J_i R_0(\lambda - \lambda_i \pm i\varepsilon) G_i M_\delta \text{ exists in } B(L^2(\mathbb{R}^n)).$$

The results for the finite sum follow from the properties of J_i, G_i the facts $M_\delta J_i = J_i M_\delta$, $M_\delta G_i = G_i M_\delta$ and theorem (4.1) of 1. In particular, this theorem implies that

$$\lim_{\varepsilon \downarrow 0} M_\delta \sum_{j=1}^{N} J_i R_0(\lambda - \lambda_i \pm i\varepsilon) G_i M_\delta$$

exists in $B(L^2(\mathbb{R}^n))$ for $\lambda \in (\lambda_N, \lambda_{N+1})$.

4. The Perturbed Operator.

Let $V(x_1, x_2), x_i \in \mathbb{R}^{n_i}$, $i = 1,2$, be a real-valued measurable function such that,

$$(4.1) \qquad |V(x_1, x_2)| \leq CU(x_2),\ U(x_2) = (U_1(x_2) + U_2(x_2))(1 + |x_2|^2)^{-\rho-\varepsilon}$$

$$U_1 \in L^\infty(\mathbb{R}^{n_2}),\ U_2 \in L^q(\mathbb{R}^{n_2}),\ q > 2,\ q > n_2/2\alpha,\ \rho > 1/2,\ \varepsilon > 0,\ C > 0$$

where α is the number in condition $(t_0 - 2)$. In appendix 1 we prove the following result,

Theorem 4.1. $V \in B_0(\mathcal{G}_{2,-\delta}(\mathbb{R}^n), \mathcal{G}_{0,-\delta+2\rho}(\mathbb{R}^n), \delta \geq 0.$

In particular (taking $\delta = 0$) it follows that V is T_0 compact so that the operator sum $T = T_0 + V$ is self-adjoint on $D(T) = D(T_0)$, $\Sigma_d(T) = \Sigma(T) \cap (-\infty, \lambda_1)$, (its only possible accumulation point being λ_1), and $\Sigma_e(T) = \Sigma_e(T_0) = [\lambda_1, \infty)$. Moreover for $z \notin [\lambda_1, \infty)$ the following identity holds,

$$(4.2) \qquad 1 + L(z) = (T - z)\, R_{T_0}(z)$$

where $L(z) = V R_{T_0}(z)$.

Theorem 4.2. $L(z) \in B_0(\mathcal{G}_{0,\rho}(\mathbb{R}^n))$, $z \notin [\lambda_1, \infty)$. *If in addition $\lambda \in (-\infty, \lambda_1)$ or $\lambda \in (\lambda_N, \lambda_{N+1})$, for some $N \in Z^+$, the limit*

$$\lim_{\varepsilon \downarrow 0} L(\lambda \pm i\varepsilon) = L(\lambda \pm io)$$

exists in the norm topology of $B(\mathcal{G}_{0,\rho}(\mathbb{R}^n))$.

Proof. The theorem follows at once from theorems 3.2 and 4.1.

5. The spectrum of T in the intervals $(\lambda_N, \lambda_{N+1})$.

First we note that $\lambda \in (-\infty, \lambda_1)$ satisties $T\psi = \lambda\psi$ if and only if $L(\lambda)\phi = -\phi$, where $\phi = V\psi$ and moreover $\psi = R_{T_0}(z)\phi$. Our aim in this section is to prove the analogues of these results for the eigenvalues of T contained in any of the intervals $(\lambda_N, \lambda_{N+1})$, $N \in Z^+$. We begin with,

Theorem 5.1. *Let $\lambda \in (\lambda_N, \lambda_{N+1})$ for some $N \in Z^+$ be such that $T\psi = \lambda\psi$. Then $L(\lambda \pm io)\phi = -\phi$ where $\phi = V\psi \in \mathcal{G}_{0,\rho}(\mathbb{R}^n)$. Moreover $\psi = -R_{T_0}(\lambda \pm io)\phi$.*

Proof. Since $T\psi = (T_0 + V)\psi = \lambda\psi$, it follows that for all $\varepsilon > 0$ we have,

$$(5.1) \qquad \psi = R_{T_0}(\lambda \pm i\varepsilon) V\psi \mp i\varepsilon R_{T_0}(u \pm i\varepsilon)\psi.$$

Therefore,

$$(5.2) \qquad \phi = V\psi = -L(\lambda \pm i\varepsilon)\phi \mp i\varepsilon L(\lambda \pm i\varepsilon)\psi.$$

The definition of $L(z)$ and theorems 3.2, 4.1, 4.2 show that the result follows from 5.1 and 5.2 if we can show that

$$\lim_{\varepsilon \downarrow 0} \varepsilon R_{T_0}(\lambda \pm io)\psi = 0$$

in $\mathcal{G}_{2,-\rho}(\mathbb{R}^n)$. This is done in appendix 2.

Theorem 5.2. *Suppose $\lambda \in (\lambda_N, \lambda_{N+1})$ for some $N \in Z^+$ and $T\psi = \lambda\psi$. Then $\psi \in \mathcal{G}_{2,\delta}(\mathbb{R}^n)$ for some $\delta > 0$ and*

$$\|\psi\|_{2,\delta} \leq C\, \|\psi\|_{0,0}$$

where C is a constant independent of ψ. In particular, if $[a, b] \subset (\lambda_N, \lambda_{N+1})$, there are only finitely many eigenvalues of T in $[a, b]$.

Proof. Applying $(1 + T_0)$ to (5.1) and taking limits as $\varepsilon \downarrow 0$ we obtain.

$$(1 + T_0)\psi = -V\psi - (1 + \lambda) R_{T_0}(\lambda \pm io) V\psi$$

By theorem (4.1), $V\psi \in \mathcal{G}_{0,2\rho}(\mathbb{R}^n)$ so that to prove the present result it suffices to show that $M_\delta R_{T_0}(\lambda \pm io) V\psi \in L^2(\mathbb{R}^n)$ for some $\delta \in (o, 2\rho)$. In order to obtain this result, note first that ψ also satisfies the equation $(T_0 - \lambda)\psi = -V\psi$. Applying G_j to both sides of this identity we set,

$$(5.3) \qquad ((\lambda_j - \lambda) + H_0) G_j = -G_j V\psi.$$

Taking the Fourier transform of this equation (with respect to x_2) we obtain,

(5.4) $$(\lambda_j - \lambda + \hat{H}_0)(G_j \psi) = -(G_j V \psi)$$

where $\hat{H}_0$ denotes the operator of multiplication by $|p_2|^2$. Because $V\psi \in \mathscr{G}_{0,2\rho}(\mathbb{R}^n)$ it follows that $G_j V\psi \in L^2_{2\rho}(\mathbb{R}^{n_2})$ and has therefore a well defined trace on the sphere $|p_2|^2 = \lambda - \lambda_j$ for $j = 1, 2, \ldots, N$, belonging to $L^2(|p_2|^2 = \lambda - \lambda_j)$. It follows from (5.4) that this trace must be zero. By theorem (I X · 4.1) page 82 of 9 we have

(5.5) $$R_0(\lambda - \lambda_j \pm io)\, G_j V\psi \in L^2_{2\rho - 1}(\mathbb{R}^n), \qquad j = 1, \ldots, N.$$

Choose $\delta = 2\rho - 1$. Writing the function as

$$\sum_{j=1}^{N} J_j R_0(\lambda - \lambda_j \pm io)\, G_j V\psi + \sum_{j=N+1}^{\infty} J_j R_0(\lambda - \lambda_j)\, G_j V$$

we see that the finite sum belongs to $\mathscr{G}_{0,\delta}(\mathbb{R}^n)$. It remains to show that the second also has this property. But this can easily be accomplished by the methods of appendix 1 by noting that $(\lambda - \lambda_j) < 0$ for $j = N+1, \ldots$ and using

(5.6) $$R_0(\lambda - \lambda_j) = -\int_0^{\infty} \exp(\lambda - \lambda_j)\, e^{-tH_0}\, dt$$

Finally, suppose that there are infinitely may eigenvalues of T in $[a, b]$. Then, by theorem 3.1 and the first part of the present theorem nay corresponding orthonormal set of eigenfunctions would have a convergent subsequence, a contradiction.

Theorem 5.3. *Suppose* $(\lambda_N, \lambda_{N+1})$ *for some* $N \in Z^+$ *is such that* $L(\lambda + io)^{-1} = -\phi \in \mathscr{G}_{0,\rho}(\mathbb{R}^n)$. *Then the function* $\psi = -R_{T_0}(\lambda + io)$ *belongs to* $D(T)$ *and satisfies* $T\psi = \lambda\psi$.

Proof. The definition of $\mathscr{G}_{2,-\rho}(\mathbb{R}^n)$ implies at once that

$$(-\Delta + q) \in B(\mathscr{G}_{2,-\rho}(\mathbb{R}^n), \mathscr{G}_{0,-\rho}(\mathbb{R}^n))$$

so that,

$$(-\Delta + q)(-R_{T_0}(\lambda + i\,o)\,\phi) = -\lim_{\varepsilon \downarrow 0} T_0 R_{T_0}(\lambda + i\,\varepsilon)\,\phi =$$

$$= \phi \quad \lambda R_{T_0}(\lambda + i\,0)\,\phi$$

where the limit is taken in $\mathscr{G}_{0,-\rho}(\mathbb{R}^n)$ and we used theorem (3.2) and the results in appendix 2. But $V\psi = -V R_{T_0}(\lambda + i\,o)\,\phi = \phi$. Thus,

(5.7) $$(-\Delta + q)\,\psi = -V\psi + \lambda\psi.$$

Therefore $\psi \in \mathscr{G}_{2,-\rho}(\mathbb{R}^n)$ is a solution of $(-\Delta + q + V)f = \lambda f$. To show that $\psi \in D(T) = D(T_0)$ it is enough to prove that $\psi \in L^2(\mathbb{R}^n)$ because in this case $(1 - \Delta + q)\,\psi = \psi + \lambda\psi - V\psi \in L^2(\mathbb{R}^n)$. In order to show $\psi \in L^2(\mathbb{R}^n)$, note that for all $\varepsilon > 0$ we have,

$$\varepsilon \|R_{T_0}(\lambda + i\,\varepsilon)\,\phi\|^2 = (2i)^{-1}[(R_{T_0}(\lambda + i\,\varepsilon)\,\phi \mid \phi) - (\phi \mid R_{T_0}(\lambda + i\varepsilon)\,\phi)].$$

Since $\phi = V\psi$ this may be rewritten as

$$\varepsilon \|R_{T_0}(\lambda + i\,\varepsilon)\,\phi\|^2 = Im\,\theta(V R_0(\lambda + i\,\varepsilon)\,\phi, \psi)$$

where $\theta(f, g) = \int dx_1\, dx_2\, f(x_1, x_2)\, g(x_1, x_2)$ for all $f \in \mathscr{G}_{0,\rho}$; $g \in \mathscr{G}_{0,\rho}(\mathbb{R}^n)$. Taking limits as $\varepsilon \downarrow 0$ we obtain,

$$\lim_{\varepsilon \downarrow 0} \varepsilon \|R_{T_0}(\lambda + i\,\varepsilon)\,\phi\|^2 = Im\,\theta(V R_{T_0}(\lambda + i\,0)\,\phi, \psi) =$$

$$= Im(-\phi, \psi) = Im(\int dx_1, dx_2\; V(x_1 x_2)\, |\psi(x_1, x_2)|^2) = 0$$

since V is real valued. Since

$$\|R_{T_0}(z)\,\phi\|^2 = \sum_{j=1}^{\infty} \|R_0(z - \lambda_j)\, G_j\, \phi\|^2 \geq \sum_{j=1}^{N} \|R_0(z - \lambda_j)\, G_j\, \phi\|^2$$

for $Im\, z \neq 0$, it follows that

(5.8) $$\lim_{\varepsilon \downarrow 0} \varepsilon \|R_0(\lambda - \lambda_j + i\,\varepsilon)\, G_j\, \phi\|^2 = 0, \qquad j = 1, 2, \ldots, N.$$

Since $\rho > 1/2$ and $(G_j\phi) \in L^2_\rho(\mathbb{R}^{n_2})$, $(G_j\phi)$, the Fourier transform with respect to x_2, has a well defined trace on the sphere $|\rho_2|^2 = \lambda - \lambda_j$, $j = 1, \ldots, N$. By (5.8) this trace is zero. Now, methods similar to those of appendix 1 show that,

(A) $R_0(\lambda - \lambda_j)\, G_j \in B(\mathscr{G}_{0,\delta}(\mathbb{R}^n), L^2_\delta(\mathbb{R}^{n_2}))$, $j = N+1, N+2, \ldots, 0 \leq \delta$.

(B) $\sum_{j=N+1}^{\infty} J_j R_0(\lambda - \lambda_j)\, G_j$ converge in the norm of $B(\mathscr{G}_{0,\delta}(\mathbb{R}^n))$.

Hence, repeated application of theorem (I X · 41) page 82 of 9, theorem 4.1, a), b) and $\phi = V\psi$ implies after a finite number of steps that $\psi \in \mathscr{G}_{0,\gamma}(\mathbb{R}^n)$ for some $\gamma > 0$ and we are done.

Theorem 5.4. *Let Q_N be the set of all eigenvalues of T contained in $(\lambda_N, \lambda_{N+1})$. Then for any $\lambda \in (\lambda_N, \lambda_{N+1}) \setminus Q_N$, the resolvent $R_T(z) = (T - z)^{-1}$, $Im\ z > 0$ has boundary values,*

$$R_T(\lambda + i o) = R_{T_0}(\lambda \pm i o)(1 + L(\lambda \pm i o))^{-1}$$

as an operator in $B(\mathscr{G}_{0,\rho}(\mathbb{R}^n), \mathscr{G}_{2,-\rho}(\mathbb{R}^n))$. Moreover,

$$\Sigma_{ac}(T) \cap (\lambda_N, \lambda_{N+1}) = (\lambda_N, \lambda_{N+1}) \setminus Q_N.$$

Proof. The first part of this result follows from theorems 3.2, 4.2, 5.1 and 5.3. As for the second, let $[a, b] \subset (\lambda_N, \lambda_{N+1}) \setminus Q_N$ and $\phi \in \mathscr{G}_{0,\rho}(\mathbb{R}^n)$. Applying Stone's formula we obtain:

$$(E([a, b])\, \phi \mid \phi) = \frac{1}{2\pi i} \int_a^b ((R_T(\lambda + i o) - R_T(\lambda - i o))\, \phi \mid \phi)\, d\lambda$$

which implies the result since $\mathscr{G}_{0,\rho}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

Theorem 5.5. $\Sigma_{sc}(T) = \Phi$.

Proof. From theorem 5.4 it follows that

$$\Sigma_{sc}(T) \subset (\cup_k \{\lambda_k\}) \cup (\bigcup_{N=1}^{\infty} Q_N) \cup \Sigma_d(T).$$

From theorem 5.2 we see that the right hand side of this inclusion is at most countable. Hence $\Sigma_{sc}(T) = \Phi$ (11, page 211).

Remark. Let $\Lambda = (\cup_k \{\lambda_k\}) \cup (\bigcup_{N=1}^{\infty} Q_N)$ and suppose that Γ is a Borel set of the real line such that $\Gamma \cap \Lambda = \Phi$. Then the local wave operators $W \pm (T, T_0 ; \Gamma)$ exist and are complete. This follows from the results in sections 2, 3, 4, 5 and theorem 4.6 of 8.

Appendix 1.

In this appendix we prove the compactness propreties stated in theorems 3.1 and 4.1. To prove theorem 4.1 note that it is enough to show that $M_{-\delta+2\rho}^{-1} V (1 + T_0)^{-1} M_\delta^{-1} \in B_0(L^2 R^n)$. We will in fact show that

(A.1.1) $$M_{-\delta-2\rho}^{-1} V (\gamma + T_0)^{-1} M_\delta^{-1} \in B_0(L^2(\mathbb{R}^n)),\ \gamma > 0.$$

Since V is a closed operator it is enough to prove,

(A) $Q_i = M_{-\delta+2\rho}^{-1} V J_i R_0(-\gamma - \lambda_i) G_i M_\delta^{-1} \in B_0(L^2(\mathbb{R}^n))$.

(B) $\sum_{i=1}^{\infty} Q_i$ converges in the norm of $B(L^2(\mathbb{R}^n))$.

Using (4.1) it is easy to see that,

$$|(Q_i f)(x_1, x_2)| < C\, |(J_i U M_{-(\delta+\varepsilon)}^{-1} R_0(-\gamma - \lambda_i) M_\delta^{-1} G_i f)(x_1, x_2)|.$$

Hence it suffices prove (A), (B) with Q_i replaced by

$$Q_i = J_i U M_{-(\delta+\varepsilon)}^{-1} e^{-tH_0} M_\delta^{-1}$$

To do this, recall that

(A.1.2) $$R_0(-\gamma - \lambda_i) = \int_0^\infty \exp(-(\gamma + \lambda_i)\, t)\, e^{-tH_0}\, dt$$

and consider

$$\tilde{S}_i(t) = J_i U M_{-(\delta+\varepsilon)}^{-1} e^{-tH_0} M_\delta^{-1} G_i.$$

We will show that $\tilde{S}_i(t) \in B_0(L^2(\mathbb{R}^n))$, that

(A.1.3) $$\int_0^\infty \exp(-(\gamma + \lambda_i)\, t)\, \tilde{S}_i(t)\, dt < \infty$$

and that this integral satisfies an appropriate estimate in terms of λ_i, so that the series $\sum_{i=1}^{\infty} Q_i$ converges in the norm of $B(L^2(\mathbb{R}^n))$. To do this note first that if $K \in B_0(L^2(\mathbb{R}^{n_2}))$, the operator $J_i K G_i \in B_0(L^2(\mathbb{R}^n))$. Hence we examine the operator $K = U M_{-(\delta+\varepsilon)}^{-1} e^{-tH_0} M_\delta^{-1}$. This operator has kernel,

$$\frac{a}{t^{n/2}}\, U(x_2)\, (1 + |x_2|^2)^{-(\delta+\varepsilon)/2} \exp \frac{(-|x_2 - y_2|^2)}{bt} (1 + |y_2|^2)^{\delta/2}$$

where a and b are positive constants. This kernel can be estimated by,

$$\frac{a}{t^{n/2}} \frac{|U_1(x_2)| + |U_2(x_2)|}{(1 + |x_2|^2)^{\varepsilon/2}} (1 + |x_2 - y_2|^2)^{\delta/2} \exp \frac{(-|x_2 - y_2|^2)}{bt}.$$

Let A_ℓ, $\ell = 1,2$ denote the integral operator with kernel

$$at^{-n/2}(1 + |x_2|^2)^{-\varepsilon/2}\, U_\ell(x_2)\, (1 + |x_2 - y_2|^2)^{\delta/2} \exp(-|x_2 - y_2|^2/\beta t).$$

Then, using Holder's inequality and Young's theorem on the convolution (9, page 28) we get:

(A.1.4) $$\|A_\ell f\|_2 \le C\,\|M_\varepsilon\|_\infty\,\|k\|_r\,\|U_\ell\|_s\,\|f\|_2\, t^{-n/2}$$

for all $f \in L(\mathbb{R}^{n_2})$ where,

(A.1.5) $$k(x_2) = (1+|x_2|^2)^{\delta/2}\exp\frac{(-|x_2|^2)}{\beta t}$$
$$\frac{1}{r} = 1 - \frac{1}{s}, \quad s = \begin{cases}\infty & \text{if } \ell = 1\\ q & \text{if } \ell = 2\end{cases}$$

and $\|\cdot\|_\theta$ denotes the norm in $L^\theta(\mathbb{R}^n)$, $1 \le \theta \le \infty$. To see that A_1 is compact, let $\{g_j\}$, $\{k_j\}$ be sequences in $C_0^\infty(\mathbb{R}^{n_2})$ converging to M_ε and k in $L^\infty(\mathbb{R}^{n_2})$ and $L^1(\mathbb{R}^{n_2})$ respectively. Then the operators $A_1^{(j)}$ with kernels $at^{-n/2}$ $U_i(x_2)\,g_j(x_2)\,k_j(x_2-y_2)$ converge to A_1 in the norm of $B(L^2(\mathbb{R}^{n_2}))$ by (A.1.3) as $j \to \infty$ and are Hilbert-Schmidt for each fixed j. A similar approximation argument holds for A_2, the only difference being that in this case we approximate U rather than M_δ.

Next let $m \ge \delta/2$ be a positive integer so that $(1+|x_2-y_2|^2)^{\delta/2} \le$ $\le (1+|x_2-y_2|^2)^m$. Using this estimate, spherical coordinates, the binominal theorem and the integration formula

$$\int_0^\infty du \exp(-u^\tau)\,u^\sigma = \tau^{-1}\,\Gamma(\tau^{-1}(\sigma+1)),\ \tau > 0,\ \sigma > -1$$

(where Γ stands for the Gamma function) we obtain,

(A.1.6) $$\|k\|_r \le C\,p_r(t)^{1/r}\,t^{n_2/2}$$

where $p_r(t)$ is a polynominal with positive coefficients. Hence (A.1.3) implies,

(A.1.7) $$\|A_\ell\| \le C\|M_\delta\|\,\|U_\ell\|\,P_r(t)^{1/r}\,t^{-n_2/2s}.$$

Therefore,

$$\|\tilde{S}_i(t)\| \le C\,\|M_\delta\|_\infty(\|U_1\|_\infty\,p_1(t) + \|U_2\|_q\,h(t)\,t^{-n_2/2q})$$

where $h(t) = p_r(t)^{1/r}$ with $r^{-1} = 1 - q^{-1}$. Thus,

(A.1.8) $$\int_0^\infty e^{-(\gamma+\lambda i)t}\,p_1(t)\,dt \le \sup_{t\ge 0}(e^{-\gamma t}p_1(t))\int_0^\infty e^{-\lambda_i t}\,dt \le C\,\lambda_i^{-1}.$$

Moreover, using Holder's inequality with respect to t, with $\frac{1}{\alpha}$ and $\beta = \frac{1}{1-\alpha}$ as conjugate exponents, we obtain,

(A.1.9) $$\int_0^\infty e^{-(\gamma+\lambda i)t}\,h(t)\,t^{-n_2/2q} \le$$
$$\le \left[\int_0^\infty e^{-\beta\lambda_i t}\,dt\right]^{1/\beta}\left[\int_0^\infty e^{-\gamma t/\alpha}h(t)^{1/\alpha}\,t^{-n_2/2q\alpha}\right]^\alpha \le$$
$$\le C(\lambda_i)^{-(1-\alpha)}$$

since the integral involving $t^{-n_2/(2q\alpha)}$ converges because $q > n_2/2\alpha$. Thus, (A.1.3) holds. Finally, an easy computation shows that,

$$\|\sum_{i=N}^{k}\tilde{Q}_i f\|^2 \le \sum_{i=N}^{k}\|U\,M^{-1}_{-(\delta+\varepsilon)}\,R_0(-\gamma-\lambda_i)\,M_\delta^{-1}\,G_i\,f\|^2 \le$$
$$\le \|f\|^2\sum_{i=N}^{k}\|U\,M^{-1}_{-(\delta+\varepsilon)}\,R_0(-\gamma-\lambda_i)\,M_\delta^{-1}\|^2.$$

Using (A.1.2), (A.1.3), estimates (A.1.8), (A.1.9) and assumption (t_0-2) we conclude that the sum on the right hand side of this inequality tends to zero as $N, K \to \infty$. This concludes the proof of theorem 4.1.

As for theorem 3.1 one has to show that $M_\delta(1+T_0)^{-1} \in B_0(L^2(\mathbb{R}^n))$. This can be handled as above by using (A.1.2) and stimating the operator $M_\delta\,e^{-tH_0}$ whose kernel is

$$\frac{a}{t^{n/2}}(1+|x_2|^2)^{-\delta/2}\exp\frac{(-|x_2-y_2|^2)}{\beta t}$$

Appendix 2.

Let $\lambda \in (\lambda_N, \lambda_{N+1})$ for some $N \in Z^+$. We will prove that $\lim_{\varepsilon\downarrow 0}\varepsilon R_{T_0}(\lambda \pm i\varepsilon)\,\psi = 0$ in $\mathscr{G}_{2,-\rho}(\mathbb{R}^n)$ for any $\psi \in L^2(\mathbb{R}^n)$. This result is equivalent to proving that $\lim_{\varepsilon\downarrow 0}\varepsilon\,M_\delta\,(1+T_0)\,R_{T_0}(\lambda \pm i\varepsilon)\,\psi = 0$ in $L^2(\mathbb{R}^n)$. Since $(1+T_0)\,R_{T_0}(\lambda \pm i\varepsilon) = (1+\lambda \pm i\varepsilon)\,R_{T_0}(\lambda \pm i\varepsilon) + 1$ it is enough to show that $\lim_{\varepsilon\downarrow 0}\varepsilon\,M_\delta\,R_{T_0}(\lambda \pm i\varepsilon)\,\psi = 0$ in $L^2(\mathbb{R}^n)$. We write $M_\delta\,R_{T_0}(\lambda \pm i\varepsilon)\,\psi$ as follows,

(A.2.1) $$\sum_{j=1}^{N} M_\delta J_j R_0(\lambda - \lambda_j \pm i\varepsilon)\,G_j\psi + \sum_{j=N+1}^{\infty} M_\delta J_j R_0(\lambda - \lambda_j \pm i\varepsilon)\,G_j\psi$$

and observe that

(A.2.2) $$\|\sum_{j=K}^{K'} M_\delta J_j R_0(\lambda - \lambda_j \pm i\varepsilon)\,G_j f\|^2 \le \|f\|^2\sum_{j=K}^{K'}\|M_\delta R_0(\lambda - \lambda_j \pm i\varepsilon)\|^2.$$

By assumption ($t_0 - 2$), the fact that $\lambda - \lambda_j < 0$ for $j \geq N + 1$ and estimate (A.2.2) we conclude that the infinite sum in (A.2.1) converges to some element of $L^2(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$ so that its product with ε tends to zero as $\varepsilon \downarrow 0$. To control the finite sum in (A.2.1) note that by (A.2.2) it is enough to prove that $\varepsilon^2 \| M_\delta R_0(\lambda - \lambda_j \pm i\varepsilon) \|^2 \to 0$ as $\varepsilon \downarrow 0$ for $j = 1, 2, \ldots, N$. Since for $z \in \mathbb{C}$, $Im\, z \neq 0$ we have $\| M_\delta R_0(z) \| = \| R_0(\bar{z}) M_\delta \|$ and

$$|Im\, z| \cdot \| R_0(z) M_\delta \phi \|^2 = Im(\phi \mid M_\delta R_0(z) M_\delta \phi) \leq \| M_\delta R_0(z) M_\delta \| \, \|\phi\|^2$$

for all $\phi \in L^2(\mathbb{R}^n)$, the result follows from proposition 2.3 of 3 (because $\lambda - \lambda_i > 0$ for $j = 1, 2, \ldots, N$ and $\| M_\delta R_0(z) M_\delta \|$ equals the norm of $R_0(z)$ as are operator from $L^2_\delta(\mathbb{R}^{n_2})$ into $L^2_{\bar{\delta}}(\mathbb{R}^{n_2})$).

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