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Going Down and Unibranchness

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1. Introduction and definitions.

The following result on Going Down rings is rather straight forward: *Let $D \subsetneq T$ be an integral extension of domains. Suppose that T is a Going Down domain and that $D \subsetneq T$ has the Going Down property. Then, D is a Going Down domain.*

Two central questions in the theory of Going Down [2, p. 287] is whether the above result stays true when:

1) the hypothesis "$D \subsetneq T$ is integral" is weakened to "$D \subsetneq T$ possesses the Going *Up* property and the Incomparability property".

2) the hypothesis "$D \subsetneq T$ is integral" is substituted by "$D \subsetneq T$ is unibranched".

The object of this paper is to show that the answer to both questions is negative. For that, it clearly suffices for us to show that the answer to the second question is negative. We shall construct an extension $D \subsetneq T$ that is unibranched (-and even geometrically unibranched-) and prossesses the Going Down property with D a non Going Down domain and T a valuation overring of D, hence in particular a Going Down domain [1, Theorem 1, p. 35].

Now, let us recall a few definitions. An extension $A \subsetneq B$ possesses the *Going Up property* (respectively the *Going Down property*) if given any prime ideals $P \subseteq Q$ of A, given any prime ideal P' of B such that $P' \cap A = P$ (resp. Q' of B such that $Q' \cap A = Q$), there exists a prime ideal Q' of B such that $P' \subseteq Q'$ and $Q' \cap A = Q$ (resp. P' of B such that $Q' \supseteq P'$ and $P' \cap A = P$). A prime ideal P of A is unibranched in B if there exists exactly one prime ideal P' of B such that $P' \cap A = P$; it is geometrically unibranched if it is unibranched and if letting $\varphi : B \to \frac{B}{P'}$ be the canonical homomorphism, the quotient field of $\varphi(A)$ is equal to the quotient field of $\frac{B}{P'}$. The extension $A \subsetneq B$ is *unibranched* (respectively *geometrically unibranched*) if every prime ideal of A is unibranched (resp. geometrically unibranched) in B. The extension $A \subsetneq B$ possesses the *Incomparability property* if given any prime ideals $P' \subseteq P''$ of B such that $P' \cap A = P'' \cap A$ we have $P' = P''$. A domain B is an *overring* of A if $A \subseteq B \subseteq$ quotient

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field of A. A domain A is a *Going Down domain* if it satisfies the following equivalent conditions [1, Theorem p. 35].

(i) $A \subsetneq B$ possesses the Going Down property for each domain B containing A.

(ii) $A \subsetneq A[u]$ possesses the Going Down property for each u in the quotient field of A.

(iii) $A \subsetneq V$ possesses the Going Down property for each valuation overring V of R.

2. Construction of the example.

The construction will make use of the following proposition which is a special case of [3, Theorem A]; for the sake of completeness, we shall give a proof.

Proposition. *Let R be a ring with exactly two maximal ideals M_1, M_2. Let K be a field and suppose that for $i = 1,2$, there exists a surjective homomorphism $\varepsilon_i : R \to K$ with kernel M_i. Let $D = \{f \in R \mid \varepsilon_1(f) = \varepsilon_2(f)\}$. Then*

a) D and R have the same quotient field and R is a finite D-module.

b) $M_1 \cap M_2$ is the unique maximal ideal of D, and it is geometrically unibranched in R_{M_i}, $i = 1,2$.

c) If P is a prime ideal of D different from $M_1 \cap M_2$, then P is geometrically unibranched in R.

Proof. a) Let $\varepsilon : R \to K^2$ be the homomorphism defined by $\varepsilon(f) = (\varepsilon_1(f), \varepsilon_2(f))$. Since ε_1 and ε_2 are comaximal, we get that ε is surjective by the Chinese Reminder Theorem. Note that by definition, D is the inverse image of the diagonal of K^2. Let $f_1 \in R$ such that $\varepsilon(f_1) = (1, 0)$; we will show that $R = D + Df_1$. Let $r \in R$; for $i = 1,2$, let $r_i \in R$ such that $\varepsilon(r_i) = (\varepsilon_i(r), \varepsilon_i(r))$; clearly we have $r_i \in D$ and $\varepsilon(r) = (\varepsilon_1(r), \varepsilon_2(r)) = (\varepsilon_1(r), \varepsilon_1(r))(1, 0) - (\varepsilon_2(r), \varepsilon_2(r))(1, 0) + (\varepsilon_2(r), \varepsilon_2(r))(1, 1) = \varepsilon(r_1)\,\varepsilon(f_1) - \varepsilon(r_2)\,\varepsilon(f_1) + \varepsilon(r_2)\,\varepsilon(1) = \varepsilon((r_1 - r_2)f_1 + r_2)$, hence $r - ((r_1 - r_2)f_1 + r_2) \in \ker \varepsilon = M_1 \cap M_2 \subseteq D$; thus $R = D + Df_1$. Now, it is clear that $M_1 \cap M_2$ is different from (0) and is a common ideal of D and R; then D and R have the same quotient field.

b) Since R is a finite D-module, the extension $D \subsetneq R$ is integral; in order to show that $M_1 \cap M_2$ is the only maximal ideal of D, it suffices to show, by Cohen-Seidenberg's theorem, that $M_1 \cap D = M_2 \cap D = M_1 \cap M_2$. The inclusion $M_1 \cap M_2 \subseteq M_1 \cap D$ is clear; conversely, if $f \in M_1 \cap D$ we have $\varepsilon_2(f) = \varepsilon_1(f) = 0$, hence $f \in M_1 \cap M_2$; thus $M_1 \cap M_2 = M_1 \cap D$. Similarly, $M_1 \cap M_2 = M_2 \cap D$. Let $i \in \{1, 2\}$ and let $\varphi_i : R_{M_i} \to \dfrac{R_{M_i}}{M_i R_{M_i}}$ be the canonical homomorphism; in order to show that $\varphi_i(D) = \varphi_i(R_{M_i})$,

it clearly suffices to show that given any $r \in R$, there exists $d \in D$ such that $\varphi_i(d) = \varphi_i(r)$; then, let $d \in R$ such that $\varepsilon(d) = (\varepsilon_i(r), \varepsilon_i(r))$; by definition itself, we have $d \in D$ and $\varepsilon_i(d) = \varepsilon_i(r)$, i.e. $(r - d) \in \ker \varepsilon_i = M_i \subseteq \ker \varphi_i$, so that indeed $\varphi_i(d) = \varphi_i(r)$.

c) Let P be a prime ideal of D different from $M_1 \cap M_2$. Since R is integral over D, there exists a non maximal prime ideal P' of R lying over P; notice that such a prime ideal P' satisfies $P = P' \cap D \supseteq P' \cap (M_1 \cap M_2) \supseteq P' \cap P = P$, so that $P = P' \cap (M_1 \cap M_2)$. If P'' is another prime ideal of R lying over P, we have $P' \cap (M_1 \cap M_2) = P'' \cap (M_1 \cap M_2)$, hence $P' = P''$. Now, let $\varphi : R \to \dfrac{R}{P'}$ be the canonical homomorphism, and let $r \in R$. Choose $d \in M_1 \cap M_2$ such that $d \notin P$; then $dr \in M_1 \cap M_2 \subseteq D$, $\varphi(d)$ is a non zero element of $\varphi(D)$ and $\varphi(d)\,\varphi(r) = \varphi(dr) \in \varphi(D)$; thus $\varphi(r)$ belongs to the quotient field of $\varphi(D)$.

Example. Let R be a Prüfer domain having exactly two maximal ideals M_1, M_2 such that: height $M_1 = 1$, height $M_2 = n \geq 2$ and $\dfrac{R}{M_1} \simeq \dfrac{R}{M_2}$ (We will check afterwards that there exists indeed such an object). Let $\varepsilon_1 : R \to \dfrac{R}{M_1}$ be the canonical homomorphism and $\varepsilon_2 : R \to \dfrac{R}{M_1}$ be a surjective homomorphism with kernel M_2. Let $D = \{f \in R \mid \varepsilon_1(f) = \varepsilon_2(f)\}$. By part a) of the preceding proposition, $D \subsetneq R$ is an integral extension, hence in particular $D \subsetneq R$ possesses the Going Up property. By parts b) and c), $M_1 \cap M_2$ is the unique maximal ideal of the domain D, M_1 and M_2 are exactly the prime ideals of R that lie over $M_1 \cap M_2$, and above any prime ideal $P \neq M_1 \cap M_2$ of D lies exactly one prime ideal of R. Since height $M_1 = 1$, we can conclude that the extension $D \subsetneq R_{M_2}$ is unibranched and possesses the Going Up property; consequently, it also possesses the Going Down property. Then, taking $T = R_{M_2}$, we have:

(1) $D \subsetneq T$ is an extension that is unibranched (and even geometrically unibranched by parts b) and c) of the proposition) and that possesses the Going Down property.

(2) T is an overring of D by a), and T is a valuation ring since it is a localization of the Prüfer domain R.

(3) D is not a Going Down domain for, clearly, the extension $D \subsetneq R$ does not possess the Going Down property.

Now we are left with showing that given an integer $n \geq 2$, there exists indeed a Prüfer domain R having exactly two maximal ideals M_1, M_2 such that height $M_1 = 1$, height $M_2 = n$ and $\dfrac{R}{M_1} \simeq \dfrac{R}{M_2}$.

Let k be a field and let $t_1, \ldots, t_m, \ldots$ be an infinite number of indeterminates over k. Let $K = k(t_1, \ldots, t_m, \ldots)$ and let $X, Y_1, \ldots, Y_n$ be indeterminates over K. Let $V_1 = K(Y_1, \ldots, Y_n)[X]_{(X)}$; it is clear that V_1 is a rank-1 valuation ring of the field $K(X, Y_1, \dots, Y_n)$ whose residue field is isomorphic to $K(Y_1, \dots, Y_n)$. Using the composition of valuations n times [4, (15.2) p. 190], it is easy to see that there exists a rank-n valuation ring V_2 of $K(X, Y_1, \dots, Y_n)$ that contains $K(X)$ and whose residue field is isomorphic to $K(X)$. Take $R = V_1 \cap V_2$; it is clear that $V_1 \not\subseteq V_2$ and $V_2 \not\subseteq V_1$; then, by [4, (18.8) p. 262], we have that R is a Prüfer domain having exactly two maximal ideals M_1 and M_2 of height 1 and n respectively; furthermore, since $V_i = R_{M_i}$ for $i = 1, 2$, we have $\frac{R}{M_1} \simeq \frac{R_{M_1}}{M_1 R_{M_1}} \simeq K(Y_1, \dots, Y_n) \simeq$ $\simeq K \simeq K(X) \simeq \frac{R_{M_2}}{M_2 R_{M_2}} \simeq \frac{R}{M_2}$.

Remarks. (1) Since the extension $D \subsetneq T$ is unibranched and possesses the Going Down property, and since T is a valuation ring, we obtain that Spec D is totally ordered by inclusion, i.e. that D is a treed quasi local domain. Then, it follows that the D we have produced is an n-dimensional treed quasi local domain that is not a Going Down domain; such an example had already been constructed by W. J. Lewis for $n = 2$ with different techniques [2, Exemple 4.4 p. 275].

(2) A treed domain of dimension $n \geq 1$ is always of grade 1. Indeed, choosing x in the maximal ideal M of D such that x belongs to no other prime ideal of D, it is clear that M is equal to the radical of (x) so that every non invertible element of $\frac{D}{(x)}$ is a zero divisor. Then, it follows that our domain D, as well as the domain constructed by Lewis, is a treed non Going Down domain of grade 1; this provides a negative answer to a question of Dobbs and Simis [5, p. 116].

(3) The extension $D \subsetneq R$ is integral; R is a Prüfer domain, hence in particular is a Going Down domain. This illustrates the fact that in general the concept of Going Down does not "descend" via an integral extension, even if the bottom domain is treed.

References

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