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Differentiable functions

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Contents

1. Introduction.

This article, based on lectures given at the Instituto de Matemática Pura e Aplicada in 1979, is an introduction to some problems in "ideals of differentiable functions" or "differential analysis". The problems are local questions in real analysis, focussing, in particular, on the relationships among differentiable functions, analytic functions and formal power series. The paper includes an exposition (with complete proofs) of some of the fundamental classical theorems, and a discussion of recent results and several important open problems.

The development of differential calculus in this century has its origin in the work of Whitney on differentiable functions. The profound theorems proved during the last three decades were motivated on the one hand by problems of Laurent Schwartz concerning division of distributions and differentiable functions, and on the other by the theory of singularities of differentiable mappings, developed at first by Thom and Whitney. Some of the most fundamental results are due to Schwartz's students Glaeser, Grothendieck and Malgrange. Schwartz's division problems were resolved by the work of Hörmander, Łojasiewicz and Malgrange concerning ideals of differentiable functions generated by analytic functions.

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The results of Mather concerning stability and generic properties of differentiable mappings will not be studied here. Outside the generic theory, we can distinguish three closely related themes: extension of functions defined in closed sets to differentiable functions, division of differentiable functions, and composition of differentiable mappings.

We will begin with an elementary theorem on differentiable even functions, which introduces some important, if simple, techniques and which provides a good illustration of the fundamental problems and the relationships among them.

Let U be an open subset of $\mathbb{R}^n$. We denote by $\mathscr{E}^m(U)$ (respectively $\mathscr{E}(U)$) the algebra of m times continuously differentiable (respectively infinitely differentiable) functions in U, with the topology of uniform convergence of functions and all their partial derivatives on compact sets. This is the topology defined by the seminorms

$$|f|_m^K = \sup_{\substack{x\in K\\ |k|\le m}} \left|\frac{\partial^{|k|} f}{\partial x^k}(x)\right|,$$

where K is a compact subset of U (and m runs through $\mathbb{N}$ in the $\mathscr{C}^\infty$ case). Here $x = (x_1, \dots, x_n)$, k denotes a multiindex $k = (k_1, \dots, k_n) \in \mathbb{N}^n$, $|k| = k_1 + \dots + k_n$, and

$$\frac{\partial^{|k|}}{\partial x^k} = \frac{\partial^{|k|}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}.$$

We will sometimes use m for either a nonnegative integer or $+\infty$, and write $\mathscr{E}^{+\infty}(U) = \mathscr{E}(U)$.

Let $\mathscr{E}^m(\mathbb{R})_{\text{even}}$ be the closed subspace of $\mathscr{E}^m(\mathbb{R})$ consisting of even functions ($m \in \mathbb{N}$ or $m = +\infty$).

Theorem 1.1. *If $f(x)$ is a $\mathscr{C}^{2m}$ even function of one variable ($m \in \mathbb{N}$ or $m = +\infty$), then there exists a $\mathscr{C}^m$ function $g(y)$ such that $f(x) = g(x^2)$. In fact there exists a continuous linear operator $L : \mathscr{E}^{2m}(\mathbb{R})_{\text{even}} \to \mathscr{E}^m(\mathbb{R})$ such that $f(x) = L(f)(x^2)$ for all $f \in \mathscr{E}^{2m}(\mathbb{R})_{\text{even}}$.*

The first assertion is due to Whitney [42]. The second follows from a theorem of Seeley [32]. It will be clear that an analogous result holds for functions of several variables that are even in some of them.

We will prove the theorem using the following elementary but important lemma.

Lemma 1.2. (Hadamard's lemma). *If $f(x) = f(x_1, \dots, x_n, x_{n+1}, \dots, x_p)$ is a $\mathscr{C}^m$ function such that*

$$f(0, \dots, 0, x_{n+1}, \dots, x_p) = 0,$$

then there exist $\mathscr{C}^{m-1}$ functions $g_i(x_1, \dots, x_p)$, $1 \le i \le n$, such that

$$f(x) = \sum_{i=1}^n x_i\, g_i(x).$$

Proof. By the fundamental theorem of calculus and the chain rule, we have

$$f(x) = \int_0^1 \frac{\partial f(tx_1, \dots, tx_n, x_{n+1}, \dots, x_p)}{\partial t}\, dt = \sum_{i=1}^n x_i\, g_i(x),$$

where

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n, x_{n+1}, \dots, x_p)\, dt.$$

It is clear that the $g_i(x)$ defined in the proof of Lemma 1.2 depend in a continuous linear way on f.

Hadamard's lemma is a very simple type of division theorem for differentiable functions. In the $\mathscr{C}^\infty$ case, the assertion of the lemma is equivalent to the statement that the ideal in $\mathscr{E}(\mathbb{R}^p)$ generated by $x_1, \dots, x_n$ is closed. Malgrange [19] proved that if U is an open subset of $\mathbb{R}^n$, then any ideal I in $\mathscr{E}(U)$ which is generated by finitely many analytic functions is closed. Malgrange's theorem has a more concrete formulation: a $\mathscr{C}^\infty$ function f on U belongs to I if and only if it "belongs formally to I". "Belongs formally to I" means that the formal Taylor series of f at each point of U is the formal Taylor series of some element of I. In fact according to Whitney's spectral theorem [43], the closure of any ideal I in $\mathscr{E}(U)$ equals the ideal of $\mathscr{C}^\infty$ functions which belong formally to I.

Proof of Theorem 1.1. Let $f(x)$ be a $\mathscr{C}^{2m}$ even function. There is a unique continuous function $g(y)$ defined in $[0, \infty)$ such that g is $\mathscr{C}^{2m}$ in $(0, \infty)$ and $f(x) = g(x^2)$. If $x \neq 0$, we have

$$\frac{dg^{(k)}(x^2)}{dx} = 2xg^{(k+1)}(x^2), \qquad 0 \le k < 2m.$$

On the other hand we can use Hadamard's lemma to define $\mathscr{C}^{2(m-k)}$ even functions h_k inductively as follows:

$$h_0 = f$$

$$h_k' = 2x\, h_{k+1}, \qquad 0 \le k < m.$$

It follows that $h_k(x) = g^{(k)}(x^2)$ outside the origin, so that each derivative $g^{(k)}$, $0 \le k \le m$, can be continued up to the origin. We will prove that g is the restriction to $[0, \infty)$ of a $\mathscr{C}^m$ function.

The problem of extending g to a differentiable function is a very special instance of Whitney's extension problem: When is a function defined in a closed subset X of $\mathbb{R}^n$ the restriction of a $\mathscr{C}^m$ function? (cf. [40], [41]). In fact we want to extend g in a continuous linear way. The existence of such an extension in the $\mathscr{C}^\infty$ case was first proved by Mityagin [27] and Seeley [32].

Let $\mathscr{E}^m([0, \infty))$ denote the space of continuous functions g in $[0, \infty)$ such that g is $\mathscr{C}^m$ in $(0, \infty)$ and all derivatives of $g \mid (0, \infty)$ extend continuously to $[0, \infty)$. Then $\mathscr{E}^m([0, \infty))$ has the structure of a Fréchet space defined by the seminorms

$$|g|_m^K = \sup_{\substack{y \in K \\ |k| \le m}} |g^k(y)|,$$

where K is a compact subset of $[0, \infty)$ (and m runs through $\mathbb{N}$ in the $\mathscr{C}^\infty$ case), and where g^k denotes the continuation of $(d^k/dy^k)(g \mid (0, \infty))$ to $[0, \infty)$. The following theorem completes the proof of Theorem 1.1.

Theorem 1.3. *There is a continuous linear operator*

$$E : \mathscr{E}^m([0, \infty)) \to \mathscr{E}^m(\mathbb{R})$$

such that $E(g) \mid [0, \infty) = g$ *for all* $g \in \mathscr{E}^m([0, \infty))$.

Proof. Our problem is to define $E(g)(y)$ when $y < 0$. If $m = 0$, we can define $E(g)(y)$ by reflection in the origin: $E(g)(y) = g(-y)$, $y < 0$. If $m = 1$, we can use a weighted sum of reflections. Consider

$$E(g)(y) = a_1 g(b_1 y) + a_2 g(b_2 y), \qquad y < 0,$$

where $b_1, b_2 < 0$. Then $E(g)$ determines a $\mathscr{C}^1$ extension of g provided that the limiting values of $E(g)(y)$ and $E(g)'(y)$ agree with those of $g(-y)$ and $g'(-y)$ as $y \to 0-$; in other words if

$$a_1 + a_2 = 1$$

$$a_1 b_1 + a_2 b_2 = 1.$$

For distinct $b_1, b_2 < 0$, these equations have a unique solution a_1, a_2. This extension is due to Lichtenstein [15].

Hestenes [10] remarked that the same technique works for any $m < \infty$: a weighted sum of m reflections leads to solving a system of linear equations determined by a Vandermonde matrix.

If $m = \infty$, we can use an infinite sum of reflections [32]:

$$E(g)(y) = \sum_{k=1}^{\infty} a_k \, \phi(b_k y) \, g(b_k y), \qquad y < 0,$$

where $\{a_k\}$, $\{b_k\}$ are sequences satisfying

(1) $b_k < 0$, $b_k \to -\infty$ as $k \to \infty$;
(2) $\Sigma_{k=1}^\infty |a_k| \, |b_k|^n < \infty$ for all $n \ge 0$;
(3) $\Sigma_{k=1}^\infty a_k b_k^n = 1$ for all $n \ge 0$;

and ϕ is a $\mathscr{C}^\infty$ function such that $\phi(y) = 1$ if $0 \le y \le 1$ and $\phi(y) = 0$ if $y \ge 2$. In fact, condition (1) guarantees that the sum is finite for each $y < 0$. Condition (2) shows that all derivatives converge as $y \to 0-$, uniformly in each bounded set, and (3) shows that the limits agree with those of the derivatives of $g(y)$ as $y \to 0+$. The continuity of the extension operator also follows from (2).

It is easy to choose sequences $\{a_k\}$, $\{b_k\}$ satisfying the above conditions. We can take $b_k = -2^k$ and choose a_k using a theorem of Mittag-Leffler: there exists an entire function $\Sigma_{k=1}^\infty a_k z^k$ taking arbitrary values (here $(-1)^n$) on a sequence of distinct points (here 2^n), provided that the sequence does not have a finite accumulation point.

It is clear that Seeley's extension operator actually provides a simultaneous extension of all classes of differentiability.

In this article we will be concerned mainly with $\mathscr{C}^\infty$ functions. Whitney's theorem on even functions in the $\mathscr{C}^\infty$ case is equivalent to the statement that the subalgebra of $\mathscr{E}(\mathbb{R})$ of functions of the form $g(x^2)$ is closed. This is a special case of Glaeser's composition theorem [9]. Let U, V be open subsets of $\mathbb{R}^n$, $\mathbb{R}^p$ (respectively), and $\phi : U \to V$ a semiproper analytic mapping. Glaeser proved that if ϕ has rank p in a dense subset of U, then $\phi^*\mathscr{E}(V)$ is closed in $\mathscr{E}(U)$. Here $\phi^* : \mathscr{E}(V) \to \mathscr{E}(U)$ is the algebra homomorphism defined by $\phi^*(g) = g \circ \phi$, where $g \in \mathscr{E}(V)$. Glaeser's theorem also has a more concrete formulation: $\phi^*\mathscr{E}(V)$ equals the subalgebra of $\mathscr{E}(U)$ of functions which are "formally compositions with ϕ". If $f \in \mathscr{E}(U)$, we denote by $T_a f$ the formal Taylor series of f at $a \in U$. We say f is "formally a composition with $\phi = (\phi_1, \ldots, \phi_p)$" if for each $b \in \phi(U)$, there is a formal power series G_b in the variables $y - b = (y_1 - b_1, \ldots, y_p - b_p)$ such that for each $a \in \phi^{-1}(b)$, $T_a f$ is obtained by substituting for each y_j in G_b, the formal Taylor series at a of the function ϕ_j.

One of the most significant open problems on differentiable functions is the following.

Conjecture 1.4. *The conclusion of Glaeser's theorem holds without the hypothesis on the rank of* ϕ.

Tougeron [35] has proved that if $\phi : U \to V$ is any analytic mapping, then the closure of $\phi^*\mathscr{E}(V)$ equals the subalgebra of $\mathscr{E}(U)$ of functions which are formally compositions with ϕ.

Theorem 1.1 in the $\mathscr{C}^\infty$ case also follows from the Malgrange-Mather division theorem [20], [23], which is an analogue for $\mathscr{C}^\infty$ functions of the classical Weierstrass division theorem. Suppose that U is an open subset of $\mathbb{R}^n$ and $u_1, \dots, u_p$ are $\mathscr{C}^\infty$ functions on U. Let

$$p(t, x) = t^p + \sum_{i=1}^{p} u_i(x)\, t^{p-i}.$$

Given a $\mathscr{C}^\infty$ function $f(t, x)$ on $\mathbb{R} \times U$, we can ask whether there exist $\mathscr{C}^\infty$ functions $q(t, x), r_1(x), \dots, r_p(x)$ such that

$$f(t, x) = p(t, x)\, q(t, x) + \sum_{j=1}^{p} r_j(x)\, t^{p-j}$$

The answer is "yes", although the solution is not unique unless all roots of the polynomial $t \to p(t, x)$ are real for each x. For example, if $p(t, x) = t^2 + 1$, then we can choose r_1, r_2 arbitrarily since $t^2 + 1$ is invertible.

The Malgrange-Mather division theorem provides a continuous linear mapping

$$\mathscr{E}(\mathbb{R} \times U) \to \mathscr{E}(\mathbb{R} \times U) \times (\mathscr{E}(U))^p$$

$$f \to (q_f, r_{1,f}, \dots, r_{p,f})$$

such that for all $f \in \mathscr{E}(\mathbb{R} \times U)$,

$$f = pq_f + r_f,$$

where

$$r_f(t, x) = \sum_{j=1}^{p} r_{j,f}(x)\, t^{p-j}.$$

In fact $\mathscr{E}(\mathbb{R} \times U)$ is a module over the ring $\mathscr{E}(U)$, and the continuous linear mapping above can be chosen $\mathscr{E}(U)$-linear.

Let us again consider a $\mathscr{C}^\infty$ even function $f(x)$. According to the division theorem we can write

$$f(x) = (x^2 - y)\, q(x, y) + x r_1(y) + r_2(y),$$

where q, r_1, r_2 are $\mathscr{C}^\infty$ functions which depend in a continuous linear way on f. Putting $y = x^2$, we have $f(x) = x r_1(x^2) + r_2(x^2)$. Since f is even, then $r_1 \equiv 0$ and we have $f(x) = r_2(x^2)$.

2. Whitney's extension theorem.

In this section we will prove the classical extension theorem of Whitney [40]. Let U be an open subset of $\mathbb{R}^n$, and X a closed subset of U. Whitney's theorem asserts that a function F^0 defined in X is the restriction of a $\mathscr{C}^m$ function in U ($m \in \mathbb{N}$ or $m = +\infty$) provided there exists a sequence $(F^k)_{|k| \le m}$ of functions defined in X which satisfies certain conditions that arise naturally from Taylor's formula.

First we consider $m \in \mathbb{N}$. By a *jet* of *order m* on X we mean a sequence of continuous functions $F = (F^k)_{|k| \le m}$ on X. Here k denotes a multiindex $k = (k_1, \dots, k_n) \in \mathbb{N}^n$. Let $J^m(X)$ be the vector space of jets of order m on X. We write

$$|F|_m^K = \sup_{\substack{x \in K \\ |k| \le m}} |F^k(x)|$$

if K is a compact subset of X, and $F(x) = F^0(x)$.

There is a linear mapping $J^m : \mathscr{E}^m(U) \to J^m(X)$ which associates to each $f \in \mathscr{E}^m(U)$ the jet

$$J^m(f) = \left(\frac{\partial^{|k|} f}{\partial x^k} \bigg|\, X \right)_{|k| \le m}$$

For each $|k| \le m$, there is a linear mapping $D^k : J^m(X) \to J^{m-|k|}(X)$ defined by $D^k F = (F^{k+\ell})_{|\ell| \le m-|k|}$. We also denote by D^k the mapping of $\mathscr{E}^m(U)$ into $\mathscr{E}^{m-|k|}(U)$ given by

$$D^k f = \frac{\partial^{|k|} f}{\partial x^k}.$$

This should cause no confusion since

$$D^k \circ J^m = J^{m-|k|} \circ D^k.$$

If $a \in X$ and $F \in J^m(X)$, then the *Taylor polynomial* (of *order m*) of F at a is the polynomial

$$T_a^m F(x) = \sum_{|k| \le m} \frac{F^k(a)}{k!} \cdot (x-a)^k$$

of degree $\le m$. Here $k! = k_1! \ldots k_n!$. We define $R_a^m F = F - J^m(T_a^m F)$, so that

$$(R_a^m F)^k(x) = F^k(x) - \sum_{|\ell| \le m - |k|} \frac{F^{k+\ell}(a)}{\ell!} \cdot (x-a)^\ell$$

if $|k| \le m$.

Definition 2.1. A jet $F \in J^m(X)$ is a *Whitney field of class* $\mathscr{C}^m$ on X if for each $|k| \le m$,

(2.1.1) $$(R_x^m F)^k(y) = o(|x-y|^{m-|k|})$$

as $|x-y| \to 0$, $x, y \in X$.

Let $\mathscr{E}^m(X) \subset J^m(X)$ be the subspace of Whitney fields of class $\mathscr{C}^m$. $\mathscr{E}^m(X)$ is a Fréchet space, with the seminorms

$$\|F\|_m^K = |F|_m^K + \sup_{\substack{x,y \in K \\ x \ne y \\ |k| \le m}} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}},$$

where $K \subset X$ is compact.

Remark 2.2. If $F \in J^m(U)$ and for all $x \in U$, $|k| \le m$ we have

$$\lim_{y \to x} \frac{(R_x^m F)^k(y)}{|x-y|^{m-|k|}} = 0,$$

then there exists $f \in \mathscr{E}^m(U)$ such that $F = J^m(f)$. This simple converse of Taylor's theorem shows that the two spaces we have denoted $\mathscr{E}^m(U)$ are equivalent. On $\mathscr{E}^m(U)$, the topologies defined by the seminorms $|\cdot|_m^k$, $\|\cdot\|_m^k$ are equivalent (by the open mapping theorem).

Theorem 2.3. (Whitney [40]). *There is a continuous linear mapping*

$$W : \mathscr{E}^m(X) \to \mathscr{E}^m(U)$$

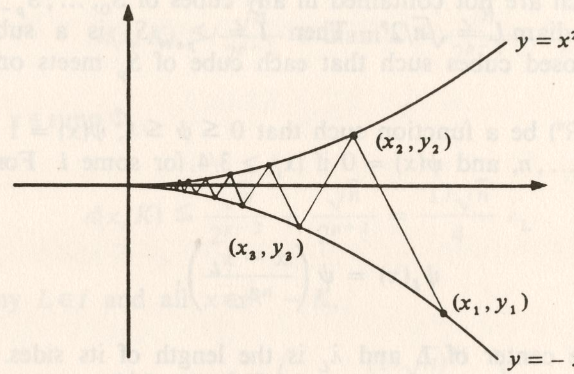
such that $D^k W(F)(x) = F^k(x)$ *if* $F \in \mathscr{E}^m(X)$, $x \in X$, $|k| \le m$, *and* $W(F) \mid U - X$ *is* $\mathscr{C}^\infty$.

Remark 2.4. The condition (2.1.1) cannot be weakened to:

(2.4.1) $$\lim_{y \to x} \frac{(R_x^m F)^k(y)}{|x-y|^{m-|k|}} = 0$$

for all $x \in X$, $|k| \le m$.

For example, let $n = m = 1$. Choose sequences of numbers $\{x_k\}$, $\{y_k\}$ as in the following figure, where the line segment joining (x_k, y_k) and (x_{k+1}, y_{k+1}) has slope $(-1)^k$.



Let $X = \{0\} \cup \{x_1, x_2, \ldots\}$. Define $F \in J^1(X)$ by $F^0(0) = 0$, $F^0(x_k) = y_k$, $F^1 \equiv 0$. Since each x_k is isolated, then (2.4.1) holds trivially for $x = x_k$. On the other hand, $(R_0^1 F)^0(x_k) = y_k$ and $(R_0^1 F)^1(x_k) = 0$, so that (2.4.1) holds for $x = 0$. But F has no $\mathscr{C}^1$ extension to $\mathbb{R}$ since

$$\frac{y_{k+1} - y_k}{x_{k+1} - x_k} = (-1)^k$$

does not approach a limit as $k \to \infty$.

The proof of Theorem 2.3 is based on the following fundamental lemma ("Whitney partition of unity").

Lemma 2.5. *Let K be a compact subset of $\mathbb{R}^n$. There exists a countable family of functions $\Phi_L \in \mathscr{E}(\mathbb{R}^n - K)$, $L \in I$, such that*

(1) $\{\operatorname{supp} \Phi_L\}_{L \in I}$ *is locally finite: in fact each x belongs to at most 3^n of the* $\operatorname{supp} \Phi_L$;

(2) $\Phi_L \ge 0$ *for all $L \in I$, and* $\Sigma_{L \in I} \Phi_L(x) = 1$, $x \in \mathbb{R}^n - K$;

(3) $2\,d(\operatorname{supp} \Phi_L, K) \ge \operatorname{diam}(\operatorname{supp} \Phi_L)$ *for all* $L \in I$;

(4) *there exist constants C_k depending only on k and n, such that if* $x \in \mathbb{R}^n - K$, *then*

$$|D^k\Phi_L(x)| \le C_k\left(1+\frac{1}{d(x,K)^{|k|}}\right).$$

Proof. The proof is based on a certain decomposition of $\mathbb{R}^n - K$ into cubes. For each nonnegative integer p, we subdivide $\mathbb{R}^n$ into closed cubes with sides of length $1/2^p$, by the hyperplanes $x_i = j_i/2^p$, where $1 \le i \le n$ and each j_i runs through the set of all integers. Let Σ_p be the set of these cubes.

Let S_0 be the subset of Σ_0 consisting of cubes L such that $d(L, K) \ge \operatorname{diam} L = \sqrt{n}$. We define S_p inductively: S_p is the subset of Σ_p consisting of cubes L which are not contained in any cubes of $S_0, \ldots, S_{p-1}$, and such that $d(L, K) \ge \operatorname{diam} L = \sqrt{n}/2^p$. Then $I = \cup_{p\in\mathbb{N}} S_p$ is a subdivision of $\mathbb{R}^n - K$ into closed cubes such that each cube of S_p meets only cubes of S_{p-1}, S_p, S_{p+1}.

Let $\psi \in \mathscr{E}(\mathbb{R}^n)$ be a function such that $0 \le \psi \le 1$, $\psi(x) = 1$ if $|x_i| \le 1/2$ for each $i = 1, \ldots, n$, and $\psi(x) = 0$ if $|x_i| \ge 3/4$ for some i. For each $L \in I$ let

$$\psi_L(x) = \psi\left(\frac{x - x_L}{\lambda_L}\right),$$

where x_L is the center of L and λ_L is the length of its sides. We define

$$\Phi_L(x) = \frac{\psi_L(x)}{\sum_{M\in I}\psi_M(x)}.$$

It is easy to see that the family $\{\Phi_L\}_{L\in I}$ satisfies (1) and (2).

If $L \in S_p$, then

$$d(\operatorname{supp}\Phi_L, K) \ge d(L, K) - \frac{\sqrt{n}}{2^{p+2}} \ge \frac{3\sqrt{n}}{2^{p+2}} \ge \frac{1}{2}\operatorname{diam}(\operatorname{supp}\Phi_L),$$

which proves (3).

To prove (4), we first estimate $|D^k\Phi_L(x)|$ in terms of λ_L. We have

$$|D^k\psi_L(x)| = \left|\frac{1}{\lambda_L^{|k|}} D^k\psi\left(\frac{x - x_L}{\lambda_L}\right)\right| \le \frac{C}{\lambda_L^{|k|}},$$

where C is a constant depending only on k and n. Also

$$1 \le \sum_{M\in I}\psi_M(x) \le 3^n$$

for all $x \in \mathbb{R}^n - K$, by (1). Using Leibniz's rule and the preceding inequalities, we get

$$|D^k\Phi_L(x)| \le \frac{C'}{\lambda_L^{|k|}}$$

for all $x \in \mathbb{R}^n - K$, where C' is a constant depending only on k and n.

If $L \in S_0$, then $\lambda_L = 1$, so that $|D^k\Phi_L(x)| \le C$. Let $L \in S_p$, $p \ge 1$. Let L' be a cube of Σ_{p-1} containing L. Then $d(L', K) < \sqrt{n}/2^{p-1}$, so that for all $x \in L$,

$$d(x, K) \le \frac{\sqrt{n}}{2^{p-1}} + \operatorname{diam} L = \frac{\sqrt{n}}{2^{p-2}},$$

and for all $x \in \operatorname{supp}\Phi_L$,

$$d(x, K) \le \frac{\sqrt{n}}{2^{p-2}} + \frac{\sqrt{n}}{2^{p+2}} = \frac{17\sqrt{n}}{4}\lambda_L.$$

Thus for any $L \in I$ and all $x \in \mathbb{R}^n - K$,

$$|D^k\Phi_L(x)| \le C'\left(1 + \frac{(17\sqrt{n})^{|k|}}{4^{|k|}d(x,K)^{|k|}}\right).$$

This proves (4).

Proof of Theorem 2.3. By a simple partition of unity argument it is enough to assume $U = \mathbb{R}^n$ and $X = K$, a compact subset of $\mathbb{R}^n$. Let $\{\Phi_L\}_{L\in I}$ be a Whitney partition of unity on $\mathbb{R}^n - K$.

For each $L \in I$, choose $a_L \in K$ such that

$$d(\operatorname{supp}\Phi_L, K) = d(\operatorname{supp}\Phi_L, a_L).$$

Let $F \in \mathscr{E}^m(K)$. Define a function $f = W(F)$ on $\mathbb{R}^n$ by

$$f(x) = F^0(x), \qquad x \in K,$$

$$f(x) = \sum_{L\in I}\Phi_L(x)T^m_{a_L}F(x), \qquad x \notin K.$$

Clearly $f = W(F)$ depends linearly on F, and is $\mathscr{C}^\infty$ on $\mathbb{R}^n - K$. We must show that f is $\mathscr{C}^m$, $D^kf \mid K = F^k$, $|k| \le m$, and W is continuous. If $|k| \le m$, we write

$$f^k(x) = F^k(x), \qquad x \in K,$$

$$f^k(x) = D^k f(x), \qquad x \notin K.$$

By a *modulus of continuity* we mean a continuous increasing function $\alpha : [0, \infty) \to [0, \infty)$ such that $\alpha(0) = 0$ and α is concave downwards. There exists a modulus of continuity α such that

$$|(R_a^m F)^k(x)| \leq \alpha(|x - a|) \cdot |x - a|^{m-|k|}$$

for all $a,\ x \in K$, $|k| \leq m$, and

$$\alpha(t) = \alpha(\operatorname{diam} K), \qquad t \geq \operatorname{diam} K,$$

$$\|F\|_m^K = |F|_m^K + \alpha(\operatorname{diam} K).$$

In fact, define $\beta : [0, \infty) \to [0, \infty)$ by $\beta(0) = 0$ and

$$\beta(t) = \sup_{\substack{x,\, y \in K \\ x \neq y \\ |x-y| \leq t \\ |k| \leq m}} \frac{|(R_x^m F)^k(y)|}{|x - y|^{m-|k|}}, \qquad t > 0.$$

Then β is increasing and continuous at 0. We get α from the convex envelope of the positive t-axis and the graph of β.

Let Λ be a cube in $\mathbb{R}^n$ such that $K \subset \operatorname{Int} \Lambda$. Let $\lambda = \sup_{x \in \Lambda} d(x, K)$. We will prove the following assertion.

(2.3.1) There exists a constant C depending only on m, n, λ such that if $|k| \leq m$, $a \in K$, $x \in \Lambda$, then

$$|f^k(x) - D^k T_a^m F(x)| \leq C\alpha(|x - a|) \cdot |x - a|^{m-|k|}.$$

Once (2.3.1) is established, the proof of the theorem can be completed as follows. Let (j) denote the multiindex whose j'th component is 1 and whose other components are 0. If $a \in K$, $x \notin K$, $|k| < m$, then

$$|f^k(x) - f^k(a) - \sum_{j=1}^{n} (x_j - a_j)\, f^{k+(j)}(a)| \leq |f^k(x) - D^k T_a^m F(x)| +$$

$$+ |D^k T_a^m F(x) - D^k T_a^m F(a) - \sum_{j=1}^{n} (x_j - a_j)\, D^{k+(j)} T_a^m F(a)|.$$

The first term in the right hand side is $o(|x - a|)$ by (2.3.1), while the second is $o(|x - a|)$ since $T_a^m F(x)$ is a polynomial. Hence f^k is continuously differentiable and $\partial f^k / \partial x_j = f^{k+(j)}$.

Applying (2.3.1) to a point $x \in \Lambda$ and a point $a \in K$ such that $d(x, K) = d(x, a)$, we have

$$|D^k f(x)| \leq |D^k T_a^m F(x)| + C\alpha(\lambda)\, \lambda^{m-|k|} \leq$$

$$\leq \sum_{|\ell| \leq m-|k|} \frac{\lambda^{|\ell|}}{\ell!} |F|_m^K + C\lambda^{m-|k|}(\|F\|_m^K - |F|_m^K).$$

Hence there is a constant C_λ (depending only on m, n, λ) such that

$$|W(F)|_m^\Lambda \leq C_\lambda \|F\|_m^K.$$

In particular, W is a continuous linear operator.

It remains to prove (2.3.1). First we claim that if $a,\ b \in K$, $|k| \leq m$, then

(2.3.2)
$$|D^k T_a^m F(x) - D^k T_b^m F(x)| \leq$$

$$\leq 2^{m-|k|}\, e^{n/2}\, \alpha(|a - b|) \cdot (|x - a|^{m-|k|} + |x - b|^{m-|k|})$$

for all $x \in \mathbb{R}^n$. To see this, we observe

$$T_a^m F(x) - T_b^m F(x) = \sum_{|k| \leq m} \frac{(x - a)^k}{k!} D^k(T_a^m F - T_b^m F)(a) =$$

$$= \sum_{|k| \leq m} \frac{(x - a)^k}{k!} (R_b^m F)^k(a),$$

so that

$$D^k T_a^m F(x) - D^k T_b^m F(x) = \sum_{|\ell| \leq m-|k|} \frac{(x - a)^\ell}{\ell!} (R_b^m F)^{k+\ell}(a).$$

Hence

$$|D^k T_a^m F(x) - D^k T_b^m F(x)| \leq$$

$$\leq \sum_{|\ell| \leq m-|k|} \frac{|x - a|^{|\ell|}}{\ell!} \cdot |a - b|^{m-|k|-|\ell|}\, \alpha(|a - b|) \leq$$

$$\leq \sum_{|\ell| \leq m-|k|} \frac{|x - b|^{m-|k|}}{\ell!} \cdot 2^{m-|k|-|\ell|}\, \alpha(|a - b|) \leq$$

$$\leq 2^{m-|k|}\, e^{n/2}\, \alpha(|a - b|) \cdot |x - b|^{m-|k|}$$

if $|x - a| \leq |x - b|$. Likewise

$$|D^k T_a^m F(x) - D^k T_b^m F(x)| \leq 2^{m-|k|} e^{n/2} \alpha(|a-b|) \cdot |x-a|^{m-|k|}$$

if $|x-b| \leq |x-a|$. Our claim follows.

Now to prove (2.3.1). We can assume $x \notin K$. Then

$$f(x) - T_a^m F(x) = \sum_{L \in I} \Phi_L(x) (T_{a_L}^m F(x) - T_a^m F(x)).$$

Hence

$$f^k(x) - D^k T_a^m F(x) = \sum_{\ell \leq k} \binom{k}{\ell} S_\ell(x),$$

where

$$S_\ell(x) = \sum_{L \in I} D^\ell \Phi_L(x) \cdot D^{k-\ell} (T_{a_L}^m F(x) - T_a^m F(x)).$$

Here $\ell \leq k$ means $\ell_j \leq k_j$, $1 \leq j \leq n$, and $\binom{k}{\ell} = \dfrac{k!}{\ell!\,(k-\ell)!}$.

To estimate $|S_0(x)|$, we note that if $x \in \operatorname{supp} \Phi_L$, then

$$|x - a_L| \leq \operatorname{diam}(\operatorname{supp} \Phi_L) + d(\operatorname{supp} \Phi_L, K) \leq$$

$$\leq 3\,d(\operatorname{supp} \Phi_L, K) \leq 3\,|x-a|,$$

by Lemma 2.5(3), so that $|a - a_L| \leq 4\,|x-a|$ and $\alpha(|a-a_L|) \leq 4\alpha(|x-a|)$ because α is concave downwards. Therefore

$$|S_0(x)| \leq C\alpha(|x-a|) \cdot |x-a|^{m-|k|},$$

where C depends only on m, n, by (2.3.2) and Lemma 2.5(1).

Now consider $|S_\ell(x)|$, $\ell \neq 0$. For all $b \in K$,

$$S_\ell(x) = \sum_{L \in I} D^\ell \Phi_L(x) \cdot D^{k-\ell} (T_{a_L}^m F(x) - T_b^m F(x))$$

since $\sum_{L \in I} D^\ell \Phi_L(x) = 0$. Choose b so that $|x-b| = d(x, K)$. As before, if $x \in \operatorname{supp} \Phi_L$, then $|x-a_L| \leq 3\,|x-b| = 3\,d(x,K)$, so that $|b-a_L| \leq 4\,d(x,K)$ and $\alpha(|b-a_L|) \leq 4\alpha(d(x,K))$. By (2.3.2) and Lemma 2.5(4),

$$|D^\ell \Phi_L(x) \cdot D^{k-\ell}(T_{a_L}^m F(x) - T_b^m F(x))| \leq C_k\, \alpha(d(x,K)) \cdot d(x,K)^{m-|k|},$$

where C_k depends only on k, m, n, λ. This completes the proof of (2.3.1), and therefore of the theorem.

We now turn to the $\mathscr{C}^\infty$ case of Whitney's extension theorem. Let U be an open subset of $\mathbb{R}^n$ and X a closed subset of U. A *jet of infinite order* on X is a sequence of continuous functions $F = (F^k)_{k \in \mathbb{N}^n}$ on X. Let $J(X)$ be the space of such jets. For each $m \in \mathbb{N}$, there is a projection $\pi_m : J(X) \to J^m(X)$ associating to each jet $(F^k)_{k \in \mathbb{N}^n}$ the jet $(F^k)_{|k| \leq m}$. Let

$$\mathscr{E}(X) = \bigcap_{m \in \mathbb{N}} \pi_m^{-1}(\mathscr{E}^m(X)).$$

An element of $\mathscr{E}(X)$ is a *Whitney field* of class $\mathscr{C}^\infty$ on X. $\mathscr{E}(X)$ is a Fréchet space, with the seminorms $\|\cdot\|_m^K$, where $m \in \mathbb{N}$ and $K \subset X$ is compact.

There is a linear mapping $J : \mathscr{E}(U) \to J(X)$ defined by

$$J(f) = \left(\frac{\partial^{|k|} f}{\partial x^k} \bigg| X \right)_{k \in \mathbb{N}^n},$$

where $f \in \mathscr{E}(U)$.

Theorem 2.6. $\mathscr{E}(X) = J(\mathscr{E}(U))$.

It is again enough to prove the theorem when $U = \mathbb{R}^n$ and $X = K$, a compact subset of $\mathbb{R}^n$. We will use the following proposition.

Proposition 2.7. *For all $m \in \mathbb{N}$, let $g_m \in \mathscr{E}^m(\mathbb{R}^n)$ such that g is $\mathscr{C}^\infty$ on $\mathbb{R}^n - K$ and $g_{m+1} - g_m$ is m-flat on K. Then there exists $g \in \mathscr{E}(\mathbb{R}^n)$ such that $g - g_m$ is m-flat on K for all $m \in \mathbb{N}$.*

(A $\mathscr{C}^m$ function is *m-flat* on K if it vanishes on K together with all its derivatives of order $\leq m$).

To obtain Theorem 2.6, let $F \in \mathscr{E}(K)$ and $F_m = \pi_m(F)$, $m \in \mathbb{N}$. By Theorem 2.3, there exists $g_m \in \mathscr{E}^m(\mathbb{R}^n)$ such that g_m is $\mathscr{C}^\infty$ on $\mathbb{R}^n - K$ and $J^m g_m = F_m$. Clearly $g_{m+1} - g_m$ is m-flat on K, so the result follows from Proposition 2.7.

We will need the following two lemmas to prove Proposition 2.7.

Lemma 2.8. *There are constants $C_k \geq 0$ (depending only on $k \in \mathbb{N}^n$) such that for any compact subset K of $\mathbb{R}^n$ and any $\varepsilon > 0$, there exists a $\mathscr{C}^\infty$ function α_ε on $\mathbb{R}^n$ satisfying:*

(1) *$0 \leq \alpha_\varepsilon \leq 1$, $\alpha_\varepsilon = 1$ in a neighborhood of K, and $\alpha_\varepsilon(x) = 0$ if $d(x, K) \geq \varepsilon$;*

(2) *for all $x \in \mathbb{R}^n$ and $k \in \mathbb{N}^n$,*

$$|D^k \alpha_\varepsilon(x)| \leq \frac{C_k}{\varepsilon^{|k|}}.$$

Proof. Let $\Phi \in \mathscr{E}(\mathbb{R}^n)$ such that $\Phi \geq 0$, $\Phi = 0$ if $|x| \geq 3/8$, and $\int \Phi = 1$. Put

$$\Phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\,\Phi\left(\frac{x}{\varepsilon}\right).$$

Let χ_ε be the characteristic function of the set $\{x \in \mathbb{R}^n : d(x, K) \le \varepsilon/2\}$. We can define α_ε by the convolution $\alpha_\varepsilon = \chi_\varepsilon * \Phi_\varepsilon$.

Lemma 2.9. *If $g \in \mathscr{E}^m(\mathbb{R}^n)$ is m-flat on K, then* $\lim_{\varepsilon\to 0} |\alpha_\varepsilon \cdot g|_m^{\mathbb{R}^n} = 0$ *(where α_ε is given by Lemma 2.8).*

Proof. Let $K_\varepsilon = \{x \in \mathbb{R}^n : d(x, K) \le \varepsilon\}$. Then

$$\sup_{x\in\mathbb{R}^n} |D^k(\alpha_\varepsilon \cdot g)(x)| = \sup_{x\in K_\varepsilon} \left| \sum_{\ell \le k} \binom{k}{\ell} D^\ell \alpha_\varepsilon(x) \cdot D^{k-\ell} g(x) \right| \le$$

$$\le \sum_{\ell \le k} \binom{k}{\ell} \frac{C_\ell}{\varepsilon^{|\ell|}} \cdot \varepsilon^{m-|k|+|\ell|}\, \beta(\varepsilon),$$

where $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$, by Lemma 2.8. Hence $|\alpha_\varepsilon \cdot g|_m^{\mathbb{R}^n} \to 0$ as $\varepsilon \to 0$.

Proof of Proposition 2.7. By Lemma 2.9 there exists a sequence of positive numbers $\varepsilon_0, \varepsilon_1, \ldots$ such that

$$|\alpha_{\varepsilon_p}(g_{p+1} - g_p)|_p^{\mathbb{R}^n} \le \frac{1}{2^p}.$$

Then the series

$$g_0 + \sum_{p\ge 0} \alpha_{\varepsilon_p}(g_{p+1} - g_p)$$

converges uniformly on $\mathbb{R}^n$ to a function g. For each $m \in \mathbb{N}$ we write

$$g = g_0 + \sum_{0\le p<m} \alpha_{\varepsilon_p}(g_{p+1} - g_p) + R_m.$$

Clearly the sum of the first two terms in the right hand side is $\mathscr{C}^m$ and coincides with g_m in a neighborhood of K. On the other hand, R_m is $\mathscr{C}^m$ and m-flat on K. Therefore g is $\mathscr{C}^\infty$ and $g - g_m$ is m-flat on K for all m.

Remark 2.10. If $X = \{a\}$, Theorem 2.6 is the generalized lemma of E. Borel: given a family of real numbers $\{\alpha_k\}_{k\in\mathbb{N}^n}$, there exists $f \in \mathscr{E}(\mathbb{R}^n)$ such that $D^k f(a) = \alpha_k$ for all $k \in \mathbb{N}^n$.

Remark 2.11. For each $m \in \mathbb{N}$, Theorem 2.3 provides an "extension operator" $W^m : \mathscr{E}^m(X) \to \mathscr{E}^m(U)$; i.e. a continuous linear mapping $W^m : \mathscr{E}^m(X) \to \mathscr{E}^m(U)$ such that $W^m(F) \mid X = F$ for all $F \in \mathscr{E}^m(X)$. The operators W^m are of increasing complexity in m, however, and therefore do not induce an extension operator on the space of $\mathscr{C}^\infty$ Whitney fields. According to Seeley's formula (Theorem 1.3), there exists an extension operator for the $\mathscr{C}^\infty$ Whitney fields on a half space (or, more generally, on a domain with $\mathscr{C}^\infty$ boundary). On the other hand, the following example of Grothendieck shows there does not in general exist a continuous linear extension operator in the $\mathscr{C}^\infty$ case.

Let $U = \mathbb{R}^n$, $X = \{0\}$. It is enough to prove there does not exist an extension operator $\mathscr{E}(0) \to \mathscr{E}(B^n)$, where B^n is the closed unit ball in $\mathbb{R}^n$. The topology of $\mathscr{E}(0)$ cannot be defined by an infinite sequence of norms (since every neighborhood of 0 contains lines). On the other hand, $\mathscr{E}(B^n)$ is topologized by an infinite sequence of norms. The result follows.

Therefore the following extension problem is interesting. Under what conditions on X is there a continuous linear extension operator $\mathscr{E}(X) \to \mathscr{E}(U)$? We will take up this question in Section 3.

We conclude this section with some remarks on the seminorms $|\cdot|_m^K$ and $\|\cdot\|_m^K$, and a problem concerning the definition of $\mathscr{C}^\infty$ functions on a domain with boundary.

Definition 2.12. Let K be a compact subset of $\mathbb{R}^n$ which is connected by rectifiable arcs, and let δ be the geodesic distance on K (if $x, y \in K$, $\delta(x, y)$ is the greatest lower bound of the lengths of the rectifiable arcs joining x and y). Let p be a positive integer. We say that K is *p-regular* if there exists a constant $C > 0$ such that

$$|x - y| \ge C\delta(x, y)^p$$

for all $x, y \in K$.

Let U be an open subset of $\mathbb{R}^n$. A closed subset X of U is *regular* if for all $x \in X$ there exists an integer p and a p-regular compact neighborhood of x in X.

Proposition 2.13. *Let K be a p-regular compact subset of $\mathbb{R}^n$. Then for each $m \in \mathbb{N}$, there exists a constant C_m such that for all $F \in \mathscr{E}^{mp}(K)$,*

$$\|F\|_m^K \le C_m |F|_{mp}^K.$$

Proof. Suppose $g \in \mathscr{E}^q(\mathbb{R}^n)$, where $q \ge 1$. If $x, y \in \mathbb{R}^n$, then

$$|g(y) - g(x)| \le \sqrt{n}\,|x - y| \sup_{\substack{\xi\in[x,y] \\ |\ell| = 1}} |D^\ell g(\xi)|,$$

according to the mean value theorem. Therefore if σ is a piecewise linear arc joining x and y, of lenght $|\sigma|$, we have

$$|g(y) - g(x)| \leq \sqrt{n}\,|\sigma| \sup_{\substack{\xi \in \sigma \\ |\ell| = 1}} |D^{\ell} g(\xi)|.$$

In fact this inequality holds for any rectifiable σ, by passage to the limit.

Suppose g is (q-1)-flat at x. Iterating the preceding inequality, we get

$$|g(y)| \leq n^{q/2}\,|\sigma|^{q} \sup_{\substack{\xi \in \sigma \\ |\ell| = q}} |D^{\ell} g(\xi)|.$$

Now let K be a compact subset of $\mathbb{R}^n$ which is connected by rectifiable arcs. Let $F \in \mathcal{E}^m(K)$. Applying the inequality above with $x, y \in K$, $q = m - |k|$, and $g = D^k(W(F) - T_x^m F)$ (where W is given by Theorem 2.3), we have

$$(2.13.1) \qquad |(R_x^m F)^k(y)| \leq n^{(m-|k|)/2}\, \delta(x,y)^{m-|k|} \sup_{\substack{\xi \in K \\ |\ell| = m}} |F^{\ell}(\xi) - F^{\ell}(x)| \leq$$

$$\leq 2n^{(m-|k|)/2}\, \delta(x,y)^{m-|k|}\, |F|_m^K.$$

Suppose K is p-regular. Let $m \in \mathbb{N}$ and $|k| \leq m$. For all $x, y \in K$ and $F \in \mathcal{E}^{mp}(K)$, there exists a constant C' such that

$$(2.13.2) \qquad |(R_x^m F)^k(y)| \leq |(R_x^{mp} F)^k(y)| + C'|x - y|^{m-|k|+1} \cdot |F|_{mp}^K.$$

But by (2.13.1) and the hypothesis, there exists a constant C'' such that

$$(2.13.3) \qquad |(R_x^{mp} F)^k(y)|\; 2n^{(mp-|k|)/2}\, \delta(x,y)^{mp-|k|}\, |F|_{mp}^K \leq C''|x - y|^{m-|k|}\, |F|_{mp}^K.$$

The proposition follows immediately from (2.13.2) and (2.13.3).

Corollary 2.14. *Let U be an open subset of $\mathbb{R}^n$. If X is a regular closed subset of U, then the topology of $\mathcal{E}(X)$ is defined by the family of seminorms $|\cdot|_m^K$, where $m \in \mathbb{N}$ and $K \subset X$ is compact.*

Remark 2.15. When $p = 1$, Proposition 2.13 has the following converse, due to Glaeser [8]. Let K be a compact subset of $\mathbb{R}^n$. If the norms $|\cdot|_1^K$ and $\|\cdot\|_1^K$ are equivalent, then K has a finite number of connected components, each of which is 1-regular.

Let U be an open subset of $\mathbb{R}^n$. If X is a closed subset of U such that $\operatorname{Int} X$ is dense in X, we can consider the following strong regularity condition.

(2.16.1) For all $a \in X$ there exists a positive integer p and a compact neighborhood K of a in X with the following property: there exists a constant C such that any two points $x, y \in K$ can be joined by a rectifiable arc σ which lies in $\operatorname{Int} X$ except perhaps for finitely many points, and satisfies

$$|x - y| \geq C|\sigma|^p.$$

In Section 6 we will prove that a closed "subanalytic" set X such that $\operatorname{Int} X$ is dense in X satisfies (2.16.1).

Proposition 2.16. *Let X be a closed subset of U such that $\operatorname{Int} X$ is dense in X. Suppose X satisfies (2.16.1). If $F \in J(X)$ and $F \mid \operatorname{Int} X \in \mathcal{E}(\operatorname{Int} X)$, then $F \in \mathcal{E}(X)$.*

This can be proved by applying estimates similar to those of the proof of Proposition 2.13, to rectifiable arcs satisfying (2.16.1).

We conjecture that the converse of Proposition 2.16 is true.

Conjecture 2.17. *Suppose that every continuous function f on X such that f is $\mathcal{C}^\infty$ in $\operatorname{Int} X$ and all partial derivatives of $f \mid \operatorname{Int} X$ extend continuously to X, is the restriction of a $\mathcal{C}^\infty$ function in U. Then X satisfies (2.16.1).*

Example 2.18. Let X be the complement of the open subset of $\mathbb{R}^2$ defined by $0 < x_2 < e^{-1/x_1^2}$, $x_1 > 0$. Let f be the continuous function on X defined by $f(x_1, x_2) = e^{-1/x_1^2}$ if $x_1 > 0$, $x_2 \geq e^{-1/x_1^2}$, and $f(x_1, x_2) = 0$ otherwise. Then f is $\mathcal{C}^\infty$ in $\operatorname{Int} X$ and all partial derivatives of $f \mid \operatorname{Int} X$ extend continuously to X (in particular, to zero at the origin). But f is not the restriction of a $\mathcal{C}^\infty$ function in $\mathbb{R}^2$ because if $x_1 > 0$, then the difference quotient

$$\frac{f(x_1, e^{-1/x_1^2}) - f(x_1, 0)}{e^{-1/x_1^2} - 0} = 1.$$

3. The linear structure of ideals of differentiable functions.

Let X be a closed subset of $\mathbb{R}^n$. In Remark 2.11 we raised the following question: Under what conditions on X is there a continuous linear extension operator $\mathcal{E}(X) \to \mathcal{E}(\mathbb{R}^n)$? In fact we can formulate a more general *lifting* problem: Let $T_X : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(X)$ be the canonical projection associating to each $\mathcal{C}^\infty$ function its jet of infinite order on X. If V is a topological vector space and $G : V \to \mathcal{E}(X)$ is a continuous linear mapping, then under what conditions is there a continuous linear mapping $G : V \to \mathcal{E}(\mathbb{R}^n)$ such that the following diagram commutes?

$$\begin{array}{ccc} & & \mathscr{E}(\mathbb{R}^n) \\ & \overset{\tilde{G}}{\nearrow} & \downarrow T_X \\ V & \xrightarrow{\;G\;} & \mathscr{E}(X) \end{array}$$

We will show that if V is a locally convex topological vector space, then a lifting $\tilde{G}$ of G exists provided there exist "pointwise" liftings, uniformly in the points of X.

Our main interest in the lifting Theorem 3.1 lies in its application to the extension problem. We will discuss the extension theorems of [1]. According to Whitney's Theorem 2.6, there is an exact sequence

$$0 \longrightarrow \mathscr{I}(X;\mathbb{R}^n) \longrightarrow \mathscr{E}(\mathbb{R}^n) \xrightarrow{\;T_X\;} \mathscr{E}(X) \longrightarrow 0,$$

where $\mathscr{I}(X;\mathbb{R}^n)$ denotes the ideal in $\mathscr{E}(\mathbb{R}^n)$ of functions which are flat on X; i.e. which vanish on X together with all their partial derivatives. The existence of an extension operator $\mathscr{E}(X) \to \mathscr{E}(\mathbb{R}^n)$ is equivalent to the splitting of this exact sequence or, in other words, to the existence of a closed linear subspace of $\mathscr{E}(\mathbb{R}^n)$ complementary to the closed ideal $\mathscr{I}(X;\mathbb{R}^n)$.

Some other theorems and problems concerning splitting properties of ideals of differentiable functions will also be surveyed in this section. The only result here which will be used in the rest of the article is E. Stein's extension theorem: a special case of Theorem 3.7 that we will prove in full.

Theorem 3.1. [3]. *Let X be a closed subset of $\mathbb{R}^n$, and V a topological vector space, topologized by a family of seminorms $\|\cdot\|_{\lambda\in\Lambda}$. Let $G: V \to \mathscr{E}(X)$ be a continuous linear mapping. Suppose that for each $a \in X$, there is a continuous linear mapping $G_a: V \to \mathscr{E}(\mathbb{R}^n)$ such that*

(1) *$G_a(\xi)^k(a) = G(\xi)^k(a)$ for all $\xi \in V$ and $k \in \mathbb{N}^n$:*

(2) *for each $m \in \mathbb{N}$ and $L \subset \mathbb{R}^n$ compact, there exist $\lambda = \lambda(m, L) \in \Lambda$ and a constant $c = c(m, L)$ such that for all $\xi \in V$,*

$$|G_a(\xi)|_m^L \leq c\,\|\xi\|_\lambda.$$

Then there exists a continuous linear mapping $\tilde{G}: V \to \mathscr{E}(\mathbb{R}^n)$ such that the diagram (3.1.1) commutes.

Idea of the proof. It is enough to assume $X = K$, a compact subset of $\mathbb{R}^n$. Let $\{\Phi_L\}_{L\in I}$ be a Whitney partition of unity on $\mathbb{R}^n - K$ (Lemma 2.5).

Let $F = G(\xi) \in \mathscr{E}(K)$. For each $L \in I$, choose $a_L \in K$ such that

$$d(\operatorname{supp}\Phi_L, K) = d(\operatorname{supp}\Phi_L, a_L).$$

Define $f = \tilde{G}(\xi) \in \mathscr{E}(\mathbb{R}^n)$ by

$$f(x) = F^0(x),\ x \in K,$$

$$f(x) = \sum_{L\in I} \Phi_L(x)\, G_{a_L}(\xi)(x),\ x \notin K.$$

We can show that $\tilde{G}$ has the required properties by an argument patterned on that of Whitney's theorem 2.3; the pointwise liftings $G_a(\xi)$ here take the place of the Taylor polynomials $T_a^m F$ in Whitney's theorem.

Let X be a closed subset of $\mathbb{R}^n$. Let $F: \mathscr{E}(\mathbb{R}^p) \to \mathscr{E}(X)$ be a continuous linear mapping. Say that F is *null* at $x \in \mathbb{R}^p$ if there exists a neighborhood U of x such that if $f \in \mathscr{E}(\mathbb{R}^p)$ and $\operatorname{supp} f \subset U$, then $F(f) = 0$. The *support* of F is the complement of the set of points where F is null. Clearly $\operatorname{supp} F$ is closed.

Corollary 3.2 [3]. *If F has compact support, then there exists a continuous linear mapping $\tilde{F}: \mathscr{E}(\mathbb{R}^p) \to \mathscr{E}(\mathbb{R}^n)$ such that the following diagram commutes:*

$$\begin{array}{ccc} & & \mathscr{E}(\mathbb{R}^n) \\ & \overset{\tilde{F}}{\nearrow} & \downarrow T_X \\ \mathscr{E}(\mathbb{R}^p) & \xrightarrow{\;F\;} & \mathscr{E}(X) \end{array}$$

In the case that X is a point, Corollary 3.2 reduces to Mather's interesting variant of Borel's lemma [24, Section 7]. The general case is a consequence of Mather's theorem and Theorem 3.1. According to [24], for each $a \in X$ there exists a continuous linear mapping $F_a: \mathscr{E}(\mathbb{R}^p) \to \mathscr{E}(\mathbb{R}^n)$ such that $F(f)^k(a) = F_a(f)^k(a)$ for all $f \in \mathscr{E}(\mathbb{R}^p)$ and $k \in \mathbb{N}^n$. Moreover, Mather's estimates show that the pointwise liftings F_a are uniform in $a \in X$, so the assertion follows from Theorem 3.1.

Let X be a closed subset of $\mathbb{R}^n$. We recall that Seeley [32] and Mityagin [27] proved that an extension operator $E: \mathscr{E}(X) \to \mathscr{E}(\mathbb{R}^n)$ exists if X is a closed half space. E. Stein [33] showed that an extension operator exists when X is a domain with boundary which is locally the graph of a function of Lipschitz class 1. Moreover, the extension operators of Seeley and Stein are universal in the sense that they simultaneously extend all classes of differentiability (in contrast with the sequence of operators W^m of increasing complexity given by Whitney's extension theorem 2.3). In fact Seeley's and Stein's formulas define extension operators from the Sobolov spaces $L_k^p(\operatorname{Int} X)$ to $L_k^p(\mathbb{R}^n)$ for all $k \in \mathbb{N}^n$ and $1 \leq p \leq \infty$. A Lipschitz condition of order 1 for the boundary of X is in the nature of the best possible for such an extension [33, p. 182].

For Whitney fields, on the other hand, extension operators exist for closed sets X such that $\operatorname{Int} X$ is dense in X and which (roughly speaking) have singularities of finite order on the boundary.

Definition 3.3. [17]. A subset A of $\mathbb{R}^n$ is *semianalytic* if for each point $x \in \mathbb{R}^n$, there exists an open neighborhood U of x in $\mathbb{R}^n$ and a finite number of real analytic functions f_{ij}, g_{ij} on U such that

$$A \cap U = \bigcup_i \{f_{ij} = 0, \quad g_{ij} > 0 \text{ for all } j\}.$$

The image of a semianalytic set by a proper analytic mapping need not be semianalytic [17]. The class of subanalytic sets is obtained by enlarging the class of semianalytic sets to include images under proper analytic mappings.

Definition 3.4 [12], [13]. A subset A of $\mathbb{R}^n$ is *subanalytic* if for each $x \in \mathbb{R}^n$, there exists an open neighborhood U of x in $\mathbb{R}^n$ and a finite system of proper real analytic mappings $f_{ij} : N_{ij} \to U$ $(j = 1, 2)$, such that

$$A \cap U = \bigcup_i (Im\, f_{i1} - Im\, f_{i2}).$$

(Here each N_{ij} is a real analytic manifold).

Theorem 3.5 [1]. *Let X be a closed subanalytic subset of $\mathbb{R}^n$. Then there exists an extension operator $E : \mathscr{E}(X) \to \mathscr{E}(\mathbb{R}^n)$ if and only if $\operatorname{Int} X$ is dense in X.*

The necessity of the hypothesis follows easily from Grothendieck's example 2.11. The theorem can be proved using Theorem 3.1 and Hironaka's resolution of singularities, by induction on the lengths of the finite sequences of local blowings-up with smooth centers needed locally to rectilinearize the singularities on the boundary of X. (The notion of "blowing-up" and the application of resolution of singularities to problems on differentiable functions will be introduced in Section 6).

Definition 3.6. Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be a function which satisfies a Lipschitz condition of order γ, $0 < \gamma \leq 1$; i.e. there is a constant $M > 0$ such that

$$|\phi(x) - \phi(x')| \leq M\,|x - x'|^\gamma$$

for all $x, x' \in \mathbb{R}^{n-1}$. We consider points in $\mathbb{R}^n$ as pairs (x, y), $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$. The open subset

$$\{(x, y) \in \mathbb{R}^n : y > \phi(x)\}$$

is called a *special Lipschitz domain* of *class* Lip γ. A rotation of such a domain will also be called a special Lipschitz domain.

Let Ω be an open subset of $\mathbb{R}^n$ and $\partial\Omega$ its boundary. We say, more generally, that Ω is a *Lipschitz domain* if for each $a \in \partial\Omega$, there exists an open neighborhood U_a of a in $\mathbb{R}^n$ and a special Lipschitz domain Ω_a, such that $\Omega \cap U_a = \Omega_a \cap U_a$. If each Ω_a is of class Lip γ (independent of a), then we say Ω is a Lipschitz domain of *class* Lip γ.

Stein's extension theorem for $\mathscr{C}^\infty$ functions is the case $k = 1$ of the following theorem.

Theorem 3.7 [1]. *If X is the closure of a Lipschitz domain Ω, then there exists an extension operator $E : \mathscr{E}(X) \to \mathscr{E}(\mathbb{R}^n)$. If Ω is of class* Lip $1/k$, *for some positive integer k, then E can be chosen so that for every compact subset L of $\mathbb{R}^n$, there exists a compact subset K of X such that E satisfies the following estimates: for each $m \in \mathbb{N}$ there is a positive constant C such that*

$$|E(F)|_m^L \leq C\,|F|_{km}^K$$

for all $F \in \mathscr{E}(X)$.

Idea of the proof. It is enough to prove the theorem in the case of a special Lipschitz domain. The general case follows using a partition of unity.

Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a function which satisfies a Lipschitz condition of order $1/k$, where k is a positive integer; i.e. there is a constant $M > 0$ such that

$$|\phi(x) - \phi(x')| \leq M\,|x - x'|^{1/k}$$

for all $x, x' \in \mathbb{R}^{n-1}$. We can assume

$$X = \{(x, y) \in \mathbb{R}^n : y \geq \phi(x)\}.$$

Let Γ be the compact subset of $\mathbb{R}^n$ defined by

$$M\,|x|^{1/k} \leq y \leq M.$$

The Lipschitz condition on ϕ implies that $a + \Gamma \subset X$ for all $a \in X$.

We claim it is enough to prove there exists an extension operator $E_0 : \mathscr{E}(\Gamma) \to \mathscr{E}(\mathbb{R}^n)$ (which satisfies estimates like those in the theorem). In fact let $E_a : \mathscr{E}(a + \Gamma) \to \mathscr{E}(\mathbb{R}^n)$ be the operator obtained by translating E_0 to a. Our theorem follows from Theorem 3.1 with $V = \mathscr{E}(X)$, G the identity mapping of $\mathscr{E}(X)$, and the pointwise lifting G_a given by composing E_a with the restriction $\mathscr{E}(X) \to \mathscr{E}(a + \Gamma)$. (For the estimates on the seminorms, it is necessary to check the estimates involved in the proof of Theorem 3.1).

When $k = 1$, Γ is defined by linear inequalities, and there exists an extension operator $S : \mathscr{E}(\Gamma) \to \mathscr{E}(\mathbb{R}^n)$ by Seeley's theorem. Hence we can use $E_0 = S$ to prove Stein's theorem.

In general, it is clear that instead of E_0, we can use an extension operator $E_0' : \mathscr{E}(\Gamma') \to \mathscr{E}(\mathbb{R}^n)$, where $\Gamma' \subset \Gamma$ is some domain with boundary containing the origin. We first find E_0' in the case $k = 2$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping given by

$$(x_1, \ldots, x_{n-1}, y) = (t_1^2, \ldots, t_{n-1}^2, y).$$

Let K be the compact subset of $\mathbb{R}^n$ defined by

$$M\,|t| \leq y \leq M,$$

and $\Gamma' = \pi(K)$. Clearly $\Gamma' \subset \Gamma$. There is an extension operator $S : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^n)$ as above. Let $\pi^* : \mathscr{E}(\Gamma') \to \mathscr{E}(K)$ be the composition $\pi^*(F) = F \circ \pi$, $F \in \mathscr{E}(\Gamma')$. Let $A : \mathscr{E}(\mathbb{R}^n) \to \mathscr{E}(\mathbb{R}^n)$ be the operator defined by taking the even part of $f(t, y) \in \mathscr{E}(\mathbb{R}^n)$ with respect to each coordinate t_i. By Theorem 1.1, there exists a continuous linear mapping $L : Im\, A \to \mathscr{E}(\mathbb{R}^n)$ such that $L(f) \circ \pi = f$ for all $f \in \operatorname{Im} A$. We can take $E_0' = L \circ A \circ S \circ \pi^*$.

Repeating this process m times, we find E_0' in the case $k = 2^m$. From these cases, our result follows for any positive integer k, but with less precise estimates on the seminorms of the extension when $k \neq 2^m$ for some m. We refer to [1] for the precise estimates.

Remark 3.8. In each of Theorems 3.5 and 3.7 we can in fact choose an extension operator which simultaneously extends all classes of differentiability, though with a certain loss of differentiability depending on the singularities of the closed set X [1]. The loss of differentiability in extending from a Lipschitz domain of class Lip $1/k$, for example, is exactly that indicated by the estimates of Theorem 3.7.

Remark 3.9. M. Tidten [34] has proved there does not exist an extension operator for the closed subset of $\mathbb{R}^2$ defined by

$$0 \leq y \leq e^{-1/x^2}, \qquad x \geq 0.$$

We have already observed that if X is a closed subset of $\mathbb{R}^n$, then there exists an extension operator $E : \mathscr{E}(X) \to \mathscr{E}(\mathbb{R}^n)$ if and only if the ideal $\mathscr{I}(X; \mathbb{R}^n)$ in $\mathscr{E}(\mathbb{R}^n)$ admits a complementary closed linear subspace. This suggests some questions concerning the linear structure of ideals of differentiable functions. Let U be an open subset of $\mathbb{R}^n$ and I a closed ideal in $\mathscr{E}(U)$.

(3.10.1) Does I admit a complementary closed linear subspace?

(3.10.2) Suppose I is generated by $\phi_1, \ldots, \phi_p$. Do there exist continuous linear operators $L_i : I \to \mathscr{E}(U)$, $1 \leq i \leq p$, such that any $g \in I$ can be written $g = \Sigma_{i=1}^p \phi_i L_i(g)$?

These questions arise also in the solution of linear equations in $\mathscr{E}(U)$:

$$\begin{pmatrix} \phi_{11} & \cdots & \phi_{1p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \phi_{q1} & \cdots & \phi_{qp} \end{pmatrix} \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ f_p \end{pmatrix} = \begin{pmatrix} g_1 \\ \cdot \\ \cdot \\ g_q \end{pmatrix}.$$

A $q \times p$ matrix $\phi = (\phi_{ij})$ of $\mathscr{C}^\infty$ functions in U determines an $\mathscr{E}(U)$-linear mapping $\phi : \mathscr{E}(U)^p \to \mathscr{E}(U)^q$. Questions (3.10.1) and (3.10.2) of course can be stated for submodules of $\mathscr{E}(U)^q$ and deal, respectively, with the existence of a projection onto $\operatorname{Im} \phi$, and with the linear structure of the solution space of the system of equations.

The solution of linear equations is obviously important in various spaces of functions. Let us recall two classical criteria for solution.

(3.11.1) If ϕ_{ij}, g_i are convergent power series in several variables (over $\mathbb{R}$ or $\mathbb{C}$ say), then there exists a convergent power series solution f_j if and only if there exists a formal power series solution. This follows from elementary properties of the completion of a local ring.

(3.11.2) Suppose the ϕ_{ij} are real analytic functions in U and $g_i \in \mathscr{E}(U)$. Malgrange's division theorem [19] (cf. [21, VI.1.1′], [36, VI.1.5]) asserts there exists a solution $f_j \in \mathscr{E}(U)$ if and only if there exists a formal solution at every point of U. Equivalently, $\operatorname{Im} \phi$ is closed, so we can ask the questions (3.10.1) and (3.10.2).

General results on the linear structure of the solution space are quite recent, even in analytic cases. The answer to both questions concerning the submodule $\operatorname{Im} \phi$ is "yes" for:

(3.12.1) Entire functions (defined over $\mathbb{R}$ or $\mathbb{C}$) when the ϕ_{ij} are polynomials. This is a beantiful elementary theorem of Djakov and Mityagin [7]. It can be proved by an explicit decomposition of the monomials in the Taylor series expansions.

(3.12.2) Convergent power series (defined over $\mathbb{R}$ or $\mathbb{C}$). This is Malgrange's privileged neighborhood theorem [22]. In this case we must be more precise about the topological structure. We can ask whether *linear* splittings in (3.10.1) and (3.10.2) induce *continuous* operators in the space of power series which converge in a given polydisk. Malgrange's theorem asserts there exist linear splittings such that the polydisks for which this is true form a fundamental system of neighborhoods of the origin.

The answer to question (3.10.2) is "yes" for submodules of $\mathscr{E}(U)^q$ generated by q-tuples of analytic functions:

Theorem 3.13 [4]. *Let $\phi = (\phi_{ij})$ be a $q \times p$ matrix of real analytic functions in U. Then the surjection*

$$\phi : \mathscr{E}(U)^p \to \operatorname{Im} \phi$$

splits.

Of course when $p = 1$ this follows from the open mapping theorem. Theorem 3.13 is a consequence of Oka's coherence theorem and results of D. Vogt and M. J. Wagner [37], [38], [39] and M. Tidten [34] concerning the splitting of exact sequences of nuclear Fréchet spaces. The latter results also provide an approach to question (3.10.1); in particular another approach to the extension problem (cf. [4], [34]).

Remark 3.14. Even for ideals generated by analytic functions, the answer to question (3.10.1) is sometimes "no". For example, the ideal I generated by $x^2 + y^2$ in $\mathscr{E}(\mathbb{R}^2)$ does not split. This follows from Grothendieck's example 2.11 since $\mathscr{I}(0; \mathbb{R}^2) \subset I$ and $\mathscr{E}(\mathbb{R}^2)/I$ is of infinite dimension.

Conjecture 3.15. *If X is a closed subanalytic subset of U, then the ideal $\mathscr{I}(X; U)$ in $\mathscr{E}(U)$ consisting of all functions which vanish on X splits.*

The space of restrictions to X of $\mathscr{C}^\infty$ functions in U has a natural Fréchet algebra structure as the quotient $\mathscr{E}(U)/\mathscr{I}(X; U)$. The exact sequence

$$0 \to \mathscr{I}(X; U) \to \mathscr{E}(U) \to \mathscr{E}(U)/\mathscr{I}(X; U) \to 0$$

shows that the extension problem for the space of restrictions to X of $\mathscr{C}^\infty$ functions is equivalent to the splitting problem for $\mathscr{I}(X; U)$. This problem is interesting when the space of Whitney fields is too large to represent a reasonable class of "smooth" functions on X; for example when X is a proper closed analytic subset of U.

In the important case that X is a coherent analytic subset of U (cf. Section 6), $\mathscr{I}(X; U)$ is (locally) generated by finitely many analytic functions. In this case Conjecture 3.15 follows from the difficult Conjecture 1.4. Special cases are treated in [2] and [4]. For example, if X is coherent and either X has isolated singularities or $\dim X \leq 2$, then $\mathscr{I}(X; U)$ splits.

Question (3.10.1) is also interesting in $\mathscr{E}^m(U)$, $m \in \mathbb{N}$. Whitney's theorem 2.3 shows that for any closed subset X of U, the ideal in $\mathscr{E}^m(U)$ of functions which are m-flat on X splits. Merrien [25] used a construction of Whitney [41] to prove that every closed ideal I in $\mathscr{E}^m(\mathbb{R})$ splits.

Conjecture 3.16. *Every closed ideal in $\mathscr{E}^m(U)$ splits.*

4. Composition of differentiable mappings.

If N is a $\mathscr{C}^\infty$ manifold, we denote by $\mathscr{E}(N)$ the Fréchet algebra of $\mathscr{C}^\infty$ functions on N, with the $\mathscr{C}^\infty$ topology.

Let N, P be $\mathscr{C}^\infty$ manifolds, and $\phi : N \to P$ a $\mathscr{C}^\infty$ mapping. There are two natural questions concerning the composition of ϕ with differentiable functions on P.

(4.1.1) If $f \in \mathscr{E}(N)$ is constant on the fibers of ϕ, does there exist $g \in \mathscr{E}(P)$ such that $f = g \circ \phi$?

(4.1.2) If $f \in \mathscr{E}(N)$ is formally a composition with ϕ (cf. Section 1), does there exist $g \in \mathscr{E}(P)$ such that $f = g \circ \phi$?

These questions, of course, have interesting analogues in various spaces of functions. For holomorphic functions, there is the following result.

Proposition 4.2. *Let N, P be complex analytic manifolds of dimensions n, p respectively, $n \geq p$. Suppose P is connected. Let $\phi : N \to P$ be a proper holomorphic mapping such that the set N' of regular points of ϕ is dense in N. Then ϕ is surjective, and every function $g : P \to \mathbb{C}$ such that $g \circ \phi$ is holomorphic on N, is holomorphic on P.*

Proof. Since ϕ is proper, then $\phi(N)$ is a closed analytic subset of P, by Remmert's proper mapping theorem [29, VII.2, Theorem 2]. Since $\phi(N)$ is the closure of the open set $\phi(N')$, then the dimension of $\phi(N)$ is p at each of its points. Therefore $\phi(N)$ is open and closed in P, so that $\phi(N) = P$.

Now g is holomorphic in $\phi(N')$, since $\phi \mid N'$ is a submersion. But g is continuous on P, because ϕ is proper. At each of its points, the dimension of the analytic set $P - \phi(N')$ is less than p. Therefore g is holomorphic on P.

The real analytic and $\mathscr{C}^\infty$ analogues of Proposition 4.2 are false. For example, let $\phi : \mathbb{R} \to \mathbb{R}$ be the proper mapping $\phi(x) = x^3$. Then $f(x) = x$ is constant on the fibers of ϕ, but is not a $\mathscr{C}^\infty$ composition with ϕ. Nevertheless, question (4.1.1) does have a positive answer for certain classes of $\mathscr{C}^\infty$ functions (cf. Corollary 4.5, Remark 4.6, and [28]), one of which will play an important part in Section 5.

The main result of this section, Glaeser's composition theorem [9], gives a positive answer to question (4.1.2) in the case that ϕ is a real analytic mapping satisfying the hypothesis of Proposition 4.2.

Let U, V be open subsets of $\mathbb{R}^n$, $\mathbb{R}^p$ respectively ($n \geq p$). A $\mathscr{C}^\infty$ mapping $\phi : U \to V$ defines a homomorphism of Fréchet algebras $\phi^* : \mathscr{E}(V) \to \mathscr{E}(U)$; $\phi^*(g) = g \circ \phi$ for all $g \in \mathscr{E}(V)$.

Theorem 4.3. *Suppose ϕ is a semiproper analytic mapping. If the set of regular points of ϕ is dense in U, then the subalgebra $\phi^*\mathscr{E}(V)$ is closed in $\mathscr{E}(U)$.*

"Semiproper" means that $\phi(U)$ is closed in V, and for every compact subset L of $\phi(U)$, there exists a compact subset K of U such that $L = \phi(K)$. For example, a projection of $\mathbb{R}^n$ onto a linear subspace is semiproper but not proper.

Remark 4.4. In the context of Theorem 4.3, there exists, moreover, a continuous linear operator $\phi^*\mathscr{E}(V) \to \mathscr{E}(V)$ which is a section for the surjection $\mathscr{E}(V) \to \phi^*\mathscr{E}(V)$. This follows from Theorem 3.5.

Corollary 4.5 [9]. *Let $\sigma_1, \ldots, \sigma_n$ be the elementary symmetric polynomials in n variables, and $\sigma = (\sigma_1, \ldots, \sigma_n)$. If $f \in \mathscr{E}(\mathbb{R}^n)$ is symmetric in its n variables, then there exists $g \in \mathscr{E}(\mathbb{R}^n)$ such that $f = g \circ \sigma$.*

Proof. The mapping $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the hypotheses of Theorem 4.3. If $f \in \mathscr{E}(\mathbb{R}^n)$, then f can be approximated in the topology of $\mathscr{E}(\mathbb{R}^n)$ by a sequence of polynomials; if f is symmetric, then by averaging over the symmetric group, we can take the polynomials symmetric too. Therefore, if f is symmetric, then $f \in \overline{\sigma^*\mathscr{E}(\mathbb{R}^n)}$; hence $f \in \sigma^*\mathscr{E}(\mathbb{R}^n)$ by Theorem 4.3.

Remark 4.6. G. W. Schwarz has extended Corollary 4.5 to functions invariant under any linear action of a compact Lie group [31]. Let G be a compact Lie group acting linearly on $\mathbb{R}^n$. Let $\mathscr{P}(\mathbb{R}^n)^G$ (respectively $\mathscr{E}(\mathbb{R}^n)^G$) be the algebra of G-invariant polynomial (respectively $\mathscr{C}^\infty$) functions on $\mathbb{R}^n$. The algebra $\mathscr{P}(\mathbb{R}^n)^G$ is finitely generated, by a classical theorem of Hilbert. Let $p_1, \ldots, p_k$ be a set of generators, and put $p = (p_1, \ldots, p_k)$. Schwarz's theorem asserts $\mathscr{E}(\mathbb{R}^n)^G = p^*\mathscr{E}(\mathbb{R}^k)$. Mather [24] has proved the analogue of Remark 4.4 for Schwarz's theorem (cf. also [4]).

In order to prove Theorem 4.3, we will first reformulate it more concretely in terms of formal composition (cf. (4.1.2)).

Let $a \in U$, $a = (a_1, \ldots, a_n)$. We denote by $\mathscr{F}_a$ the $\mathbb{R}$-algebra of formal Taylor series at a of elements of $\mathscr{E}(U)$. Then $\mathscr{F}_a$ identifies with the ring of formal power series $\mathbb{R}[[x_1 - a_1, \ldots, x_n - a_n]]$, by the lemma of E. Borel (Remark 2.10). Let $f \mapsto \hat{f}_a$ be the projection $\mathscr{E}(U) \to \mathscr{F}_a$ which associates to each function its formal Taylor series at a.

If $b = \phi(a)$, then $\phi = (\phi_1, \ldots, \phi_p)$ induces a homomorphism $\hat{\phi}_a^* : \mathscr{F}_b \to \mathscr{F}_a$ as follows: if $G = \sum_{\ell \in \mathbb{N}^p} G^\ell (y - b)^\ell / \ell!$, then $\hat{\phi}_a^*(G)$ is obtained by substituting for each y_j in G, the formal Taylor series $\hat{\phi}_{j,a}$ of ϕ_j at a; i.e. $\hat{\phi}_a^*(G) = G \circ \hat{\phi}_a$.

Let $(\phi^*\mathscr{E}(V))^\wedge$ be the subalgebra of $\mathscr{E}(U)$ of functions which are "formally" in $\phi^*\mathscr{E}(V)$; i.e. functions $f \in \mathscr{E}(U)$ such that for each $b \in \phi(U)$, there exists $G_b \in \mathscr{F}_b$ such that $\hat{f}_a = \hat{\phi}_a^*(G_b)$ for all $a \in \phi^{-1}(b)$.

In order to prove Theorem 4.3, we will show $\overline{\phi^*\mathscr{E}(V)} \subset (\phi^*\mathscr{E}(V))^\wedge$ and $\phi^*\mathscr{E}(V) = (\phi^*\mathscr{E}(V))^\wedge$. Before beginning the proof, we give three examples concerning the necessity of the hypotheses.

Example 4.7. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the mapping defined by $\phi(x) = e^{-1/x^2}$ if $x \neq 0$, $\phi(0) = 0$. Then $\overline{\phi^*\mathscr{E}(\mathbb{R})}$ is the set of even functions which are flat at 0. Let $f(x) = e^{-1/2x^2}$ if $x \neq 0$, $f(0) = 0$. Then $f \in \overline{\phi^*\mathscr{E}(\mathbb{R})}$, but $f = \phi^{1/2} \notin \phi^*\mathscr{E}(\mathbb{R})$.

Example 4.8. Let U be a non-empty proper open subset of $\mathbb{R}^n$, and let $\phi : U \hookrightarrow \mathbb{R}^n$ be the inclusion mapping (so that $\phi(U)$ is not closed in $\mathbb{R}^n$). Then $\phi^*\mathscr{E}(\mathbb{R}^n)$ is the space of restrictions to U of functions in $\mathscr{E}(\mathbb{R}^n)$. But $\overline{\phi^*\mathscr{E}(\mathbb{R}^n)} = \mathscr{E}(U) \neq \phi^*\mathscr{E}(\mathbb{R}^n)$.

Example 4.9. Let $U = (-3, -2) \cup (-1, 1) \subset \mathbb{R}$ and $V = (-1, 1) \subset \mathbb{R}$. Define $\phi : U \to V$ by $\phi \mid (-3, -2) : x \mapsto x + 2$ and $\phi \mid (-1, 1) : x \mapsto x^2$. Then $L = [-1/2, 1/2] \subset V$ is not the image of a compact subset K of U. Clearly $f \in (\phi^*\mathscr{E}(V))^\wedge$ if and only if $f \mid (-1, 1)$ is even. On the other hand, if $f \in \phi^*\mathscr{E}(V)$, then f and all its derivatives extend by continuity to $a = -2 \in U$, giving a condition of formal composition simultaneously at the points $a = -2$, $a' = 0$. Therefore $\phi^*\mathscr{E}(V) \neq (\phi^*\mathscr{E}(V))^\wedge$.

For the proof of Glaeser's theorem, which occupies the rest of the section, we will need two results which we haven't yet proved: Łojasiewicz's theorem on division by an analytic function, and the Łojasiewicz inequality [16]. We will prove these theorems in Section 6, using resolution of singularities.

Proof of Theorem 4.3. Let $F \in \overline{\phi^*\mathscr{E}(V)}$. We have to find a $\mathscr{C}^\infty$ Whitney field G on $\phi(U)$ such that $G \circ \phi = F$.

Step 1. *There exists a unique jet $G \in J(\phi(U))$ such that $G \circ \phi = F$. (In particular, $\overline{\phi^*\mathscr{E}(V)} \subset (\phi^*\mathscr{E}(V))^\wedge$). Moreover, $(D^\ell G) \circ \phi \in \mathscr{E}(U)$ for all $\ell \in \mathbb{N}^p$.*

Let $a \in U$ and $b = \phi(a)$. We denote by $\hat{m}_a$ the maximal ideal of $\mathscr{F}_a$.

Lemma 4.10. *There exists a positive integer r such that for all $q \in \mathbb{N}$,*

$$(\hat{\phi}_a^*)^{-1}(\hat{m}_a^{qr}) \subset \hat{m}_b^q.$$

In particular, the homomorphism $\hat{\phi}_a^*$ is injective.

Proof. Since ϕ is analytic and the set of regular points of ϕ is dense in U, there exists a Jacobian determinant, say

$$\delta = \frac{D(\phi_1, \dots, \phi_p)}{D(x_1, \dots, x_p)}$$

such that $\hat{\delta}_a \neq 0$. Let r be the smallest integer s such that $\hat{\delta}_a \notin \hat{m}_a^s$.

We argue by induction on q. The assertion is true when $q = 1$ since $\hat{\phi}_a^*$ is a local homomorphism. Suppose $q > 1$. Let $S \in \mathscr{F}_b$ such that $S \circ \hat{\phi}_a \in \hat{m}_a^{qr}$.

We differentiate $S \circ \hat{\phi}_a$ with respect to $x_1, \dots, x_p$:

$$\sum_{i=1}^{p} \left(\frac{\partial S}{\partial y_i} \circ \hat{\phi}_a \right) \frac{\partial \hat{\phi}_{i,a}}{\partial x_j} \in m_a^{qr-1}, \qquad 1 \leq j \leq p.$$

By Cramer's rule,

$$\hat{\delta}_a \left(\frac{\partial S}{\partial y_i} \circ \hat{\phi}_a \right) \in \hat{m}_a^{qr-1}, \qquad 1 \leq i \leq p.$$

Since $\hat{\delta}_a \notin \hat{m}_a^r$, we have

$$\frac{\partial S}{\partial y_i} \circ \hat{\phi}_a \in \hat{m}_a^{(q-1)r}, \qquad 1 \leq i \leq p.$$

By induction, $\partial S/\partial y_i \in \hat{m}_b^{q-1}$, $1 \leq i \leq p$. Clearly $S(0) = 0$, so that $S \in \hat{m}_b^q$.

We will now find G. Suppose x is another point of U such that $\phi(x) = b$. Let $m_{(a,x)}^{(s)}$ be the ideal of functions in $\mathscr{E}(U)$ which are (s-1)-flat at a, x. Clearly $m_{(a,x)}^{(s)}$ is closed and of finite real codimension. It follows that $\phi^*\mathscr{E}(V) + m_{(a,x)}^{(s)}$ is closed in $\mathscr{E}(U)$, and therefore that F belongs to this subspace. Hence there exists $S_s \in \mathscr{F}_b$ such that

$$S_s \circ \hat{\phi}_a - \hat{F}_a \in \hat{m}_a^s$$

$$S_s \circ \hat{\phi}_x - \hat{F}_x \in m_x^s.$$

If $s, s' > qr$, then $S_s - S_{s'} \in \hat{m}_b^q$, by Lemma 4.10. Therefore the sequence $S_1, S_2, \dots$ converges in $\mathscr{F}_b$ (with its $\hat{m}_b$-adic topology) to an element

$$G_b = \sum_{\ell \in \mathbb{N}^p} \frac{G^\ell(b)}{\ell!} (y - b)^\ell$$

such that $G_b \circ \hat{\phi}_a = \hat{F}_a$, $G_b \circ \hat{\phi}_x = \hat{F}_x$.

By Lemma 4.10, G_b is defined in a unique way by the single condition $G_b \circ \hat{\phi}_a = \hat{F}_a$; hence does not depend on the point $x \in \phi^{-1}(b)$. Clearly the G_b define a field of formal series G on $\phi(U)$ such that $G \circ \phi = F$.

We will now show $(D^\ell G) \circ \phi \in \mathscr{E}(U)$, for all $\ell \in \mathbb{N}^p$. (This implies each mapping $G^\ell : \phi(U) \to \mathbb{R}$ is continuous, since ϕ is semiproper.) By hypothesis, $G \circ \phi = F \in \mathscr{E}(U)$. Proceeding by induction on $|\ell|$, it is enough to prove the following lemma.

Lemma 4.11. *Let H a field of formal series on $\phi(U)$ such that $H \circ \phi \in \mathscr{E}(U)$. Then $(\partial H/\partial y_i) \circ \phi \in \mathscr{E}(U)$, $1 \leq i \leq p$.*

Proof. Let $\gamma_i = (\partial H/\partial y_i) \circ \phi$. By hypothesis, there exists $\xi \in \mathscr{E}(U)$ such that

$$H_{\phi(x)} \circ \hat{\phi}_x = \hat{\xi}_x$$

for all $x \in U$.

Let $a \in U$. There exists a Jacobian determinant, say

$$\delta = D(\phi_1, \dots, \phi_p)/D(x_1, \dots, x_p),$$

$\neq 0$ in a neighborhood of a. Differentiating the preceding equation with respect to $x_1, \dots, x_p$, and applying Cramer's rule to the resulting system of linear equations in the unknowns $\hat{\gamma}_{i,x} = (\partial H_{\phi(x)}/\partial y_i) \circ \hat{\phi}_x$, we have $\hat{\delta}_x \cdot \hat{\gamma}_{i,x} = \hat{\xi}_{i,x}$ for all $x \in U$, where $\xi_i \in \mathscr{E}(U)$. In other words, ξ_i belongs formally to the ideal generated by the analytic function δ in $\mathscr{E}(U)$. Therefore $\xi_i = \delta \cdot \gamma_i'$, where $\gamma_i' \in \mathscr{E}(U)$, by Theorem 6.14. In the neighborhood of a, we necessarily have $\gamma_i = \gamma_i'$, so that γ_i is $\mathscr{C}^\infty$ in the neighborhood of a, and hence on U.

Step 2. G is a Whitney field of class $\mathscr{C}^\infty$ on $\phi(U)$.

Our assertion is a consequence of the following lemma, which we will prove using the Łojasiewicz inequality. Glaeser (cf. [9], [36, IX.1]) proves G is a $\mathscr{C}^\infty$ Whitney field by an analytic argument based on Lemma 4.11; we will use a geometric argument and Proposition 2.16 instead.

Lemma 4.12. *Let K be a compact subset of U, and K' a compact neighborhood of K in U. Then there exists $C > 0$ and an integer $\alpha \geq 1$ such that for all a, $x \in K$, there exists a', $x' \in K'$ such that $\phi(a') = \phi(x')$ and*

$$|\phi(a) - \phi(x)|^{1/\alpha} \geq C(|a - a'| + |x - x'|).$$

Example 4.13. This example shows the reasons leading to the use of Lemma 4.12. Let $U = (-3, -1) \cup (1, 3)$ and $V = (-1, 1)$. Define $\phi : U \to V$ by $\phi \,|\, (-3, -1) : x \mapsto -(x+2)^2$ and $\phi \,|\, (1, 3) : x \mapsto (x-2)^2$. Then ϕ satisfies the hypotheses of Theorem 4.3. Positive and negative values near 0 in V have distant inverse images a, x; the points $a' = 2$, $x' = -2$ are the intermediary points involved.

Proof of Lemma 4.12. Let $\Phi : U \times U \to \mathbb{R}$ be the analytic function $\phi(a, x) = |\phi(a) - \phi(x)|^2$. We will use the distance $d((a, x), (a', x')) = |a - a'| + |x - x'|$ ou $U \times U$. By the Łojasiewicz inequality (Corollary 6.15) applied to the function Φ on the compact subset $K \times K$ of $U \times U$, there exists $C' > 0$ and an integer $\alpha \geq 1$ such that

$$|\phi(a) - \phi(x)|^{1/\alpha} \geq C' \, d((a, x), \Phi^{-1}(0))$$

for all $(a, x) \in K \times K$. We consider two cases.

Case 1. If $d((a, x), \Phi^{-1}(0)) = d((a, x), \Phi^{-1}(0) \cap (K' \times K'))$, then there exists $(a', x') \in \Phi^{-1}(0) \cap (K' \times K')$ such that

$$d((a, x), \Phi^{-1}(0)) = |a - a'| + |x - x'|.$$

Case 2. Otherwise, let $d = d(K \times K, \; U \times U - (K' \times K'))$. Then

$$d((a, x), \Phi^{-1}(0)) \geq d \geq \frac{d}{\operatorname{diam}(K' \times K')} \cdot (|a - a'| + |x - x'|),$$

where (a', x') is any point of $\Phi^{-1}(0) \cap (K' \times K')$.

Put

$$C = \inf\left(C', \; \frac{C'd}{\operatorname{diam}(K' \times K')} \right);$$

then the condition of the lemma is satisfied.

We can now complete the proof of Glaeser's theorem. By Step 1, G is $\mathscr{C}^\alpha$ in the image of the set of regular points of ϕ, and each $D^\ell G$ is continuous in $\phi(U)$. Let $X \subset U$ be the set of critical points of ϕ. As in Proposition 2.16, it will be enough to show that $\phi(U)$ satisfies the following condition: For every compact subset L of $\phi(U)$, there exists a constant c and an integer $\alpha \geq 1$ such that any two points $b, y \in L$ can be joined by a rectifiable arc of length $\geq c\,|b - y|^{1/\alpha}$, which lies in $\phi(U) - \phi(X)$ except perhaps for finitely many points.

Lemma 4.14. *If σ is a rectifiable arc in U, then*

$$|\phi(\sigma)| \leq p\sqrt{n}\,|\sigma| \cdot \sup_{x \in \sigma} \left| \frac{\partial \phi_i}{\partial x_j}(x) \right|.$$

Proof. Let $\{x^0, x^1, \ldots, x^k\}$ be a partition of σ, and let $y^i = \phi(x^i)$, $0 \leq i \leq k$. Then

$$\sum_{i=1}^{k} |y^i - y^{i-1}| = \sum_{i=1}^{k} |\phi(x^i) - \phi(x^{i-1})| \leq \sum_{i=1}^{k} \sum_{j=1}^{p} |\phi_j(x^i) - \phi_j(x^{i-1})| \leq$$

$$\leq p\sqrt{n}\,|\sigma| \cdot \sup_{x \in \sigma} \left| \frac{\partial \phi_i}{\partial x_j}(x) \right|$$

(cf. the proof of Proposition 2.13). Our assertion follows by passage to the limit.

Let L be a compact subset of $\phi(U)$. Let K be a compact subset of U such that $\phi(K) = L$, and K' a compact neighborhood of K in U.

Lemma 4.15. *There exists a constant c_1 such that any two points $a, x \in K'$ can be joined by a polygonal arc of length $\leq c_1\,|a - x|$ in U, which intersects the singular set X of ϕ in at most finitely many points.*

Proof. X is the zero set of a finite system of analytic functions ψ_i, $1 \leq i \leq k$, in U. Working locally, we can assume a, x lie in an open ball $V \subset U$. Let Λ be the perpendicular bisector of the line segment $\overline{ax}$ (Λ is an affine hyperplane in $\mathbb{R}^n$). If $\lambda \in \Lambda \cap V$, then the segments $\overline{a\lambda}$, $\overline{\lambda x}$ lie in X if and only if

$$\int_0^1 \psi_i^2((1 - t)a + t\lambda)\,dt = 0,$$

$$\int_0^1 \psi_i^2((1 - t)x + t\lambda)\,dt = 0,$$

$1 \leq i \leq k$. Since these integrals are analytic in λ, they define a proper closed analytic subset of $\Lambda \cap V$. The result follows because any line segment which does not lie in X intersects X in at most finitely many points. (This lemma also follows from a theorem of Kropman and Brown [17, Section 22]).

Given $b, y \in L$, choose $a, x \in K$ such that $\phi(a) = b$, $\phi(x) = y$. We apply Lemma 4.12 to $K \subset K'$; let a', x' be points of K' associated to a, x by the lemma. Let σ_1, σ_2 be polygonal arcs of length $\leq c_1\,|a - a'| \leq c_1\,|x - x'|$ respectively, which join a, a' and x, x' respectively, and intersect X in at most finitely many points. Then $\sigma = \phi(\sigma_1) \cup \phi(\sigma_2)$ is a rectifiable arc joining b, y in $\phi(U)$, which intersects $\phi(X)$ in at most finitely many points.

We have $|\sigma| \leq |\phi(\sigma_1)| + |\phi(\sigma_2)| \leq c_2(|\sigma_1| + |\sigma_2|)$, where c_2 is the supremum of the $p\sqrt{n}\,|(\partial\phi_i/\partial x_j)(\xi)|$ over a certain compact neighborhood of K' in U, by Lemma 4.14. Hence

$$|\sigma| \le c_1 c_2(|a - a'| + |x - x'|) \le \frac{c_1 c_2}{C}\,|\phi(a) - \phi(x)|^{1/\alpha} =$$

$$= \frac{c_1 c_2}{C}\,|b - y|^{1/\alpha}$$

by Lemma 4.12. This completes the proof.

5. The Malgrange – Mather division theorem.

There are two fundamental theorems concerning division of differentiable functions: Malgrange's theorem on ideals generated by finitely many analytic functions, and the Malgrange-Mather division theorem. In this section, we will present a recent proof of the latter due to P. Milman [26], which pinpoints the close relationship between division problems and the other main topics of this article. Stein's extension theorem (Theorem 3.7 with $k = 1$) is the only non-elementary result needed for Milman's proof; however, we will give a somewhat shorter version which also uses Glaeser's theorem 4.3 and its Corollary 4.5.

Theorem 5.1. *Suppose that U is an open subset of $\mathbb{R}^n$, and $u_1, \ldots, u_p \in \mathscr{E}(U)$. Let*

$$p(t, x) = t^p + \sum_{i=1}^{p} u_i(x)\, t^{p-i}.$$

Then there exists a continuous linear mapping

$$\mathscr{E}(\mathbb{R} \times U) \to \mathscr{E}(\mathbb{R} \times U) \times (\mathscr{E}(U)^p)$$

$$f \mapsto (q_f, r_{1,f}, \ldots, r_{p,f})$$

such that for all $f \in \mathscr{E}(\mathbb{R} \times U)$,

$$f = pq_f + r_f,$$

where

$$r_f(t, x) = \sum_{j=1}^{p} r_{j,f}(x)\, t^{p-j}.$$

In fact the above spaces are modules over the ring $\mathscr{E}(U)$, and the mapping is $\mathscr{E}(U)$-linear

The local existence of a quotient and remainder was first proved by Malgrange [20], without regard to continuous linear dependence on the function f. The stronger result was established by Mather [23]. Different proofs were subsequently given by Łojasiewicz [18] and Nirenberg [30]. G. Lassalle [14] then proved a division theorem for $\mathscr{C}^m$ functions (with a certain loss of differentiability of the quotient and remainder).

In order to prove Theorem 5.1, it will be enough to prove the following generic division theorem.

Theorem 5.2. *Let P be the generic polynomial*

$$P(t, \lambda) = t^p + \sum_{i=1}^{p} \lambda_i\, t^{p-i},$$

where $\lambda = (\lambda_1, \ldots, \lambda_p)$. If U is an open subset of $\mathbb{R}^n$, then there exists a continuous $\mathscr{E}(U)$-linear mapping

$$\mathscr{E}(\mathbb{R} \times U) \to \mathscr{E}(\mathbb{R}^{1+p} \times U) \times (\mathscr{E}(\mathbb{R}^p \times U))^p$$

$$f \mapsto (Q_f, R_{1,f}, \ldots, R_{p,f})$$

such that for all $f \in \mathscr{E}(\mathbb{R} \times U)$,

$$f(t, x) = P(t, \lambda)\, Q_f(t, \lambda, x) + R_f(t, \lambda, x),$$

where

$$R_f(t, \lambda, x) = \sum_{j=1}^{p} R_{j,f}(\lambda, x)\, t^{p-j}.$$

To see that theorem 5.1 follows from Theorem 5.2, we let $u = (u_1, \ldots, u_p)$, so that $p(t, x) = P(t, u(x))$. Then

$$f(t, x) = p(t, x)\, q_f(t, x) + \sum_{j=1}^{p} r_{j,f}(x)\, t^{p-j},$$

where $q_f(t, x) = Q_f(t, u(x), x)$ and $r_{j,f}(x) = R_{j,f}(u(x), x)$.

We will, in fact, prove a result which is more precise than Theorem 5.2, and which is formulated with a view to proving the division theorem by induction on the degree of the generic polynomial.

We consider the generic polynomial

$$P^p(t, \lambda) = t^p + \sum_{i=1}^{p} \lambda_i t^{p-i}$$

for any nonnegative integer p (we set $P^0(t, \lambda) = 1$). Then for each positive integer p, we can define a mapping

$$\lambda^p : \mathbb{R} \times \mathbb{R}^{p-1} \to \mathbb{R}^p$$

by the following polynomial identity:

$$P^p(t, \lambda^p(s, \mu)) = (t - s)\, P^{p-1}(t, \mu),$$

where $(s, \mu) \in \mathbb{R} \times \mathbb{R}^{p-1}$; i.e. the mapping λ^p is defined by

$$\lambda_1 = \mu_1 - s,$$

$$\lambda_j = \mu_j - \mu_{j-1} s, \qquad 2 \leq j \leq p - 1,$$

$$\lambda_p = - \mu_{p-1} s.$$

Theorem 5.3 [26]. *Suppose U is an open subset of $\mathbb{R}^n$. Then for each $p \in \mathbb{N}$, there exists a continuous $\mathscr{E}(U)$ – linear mapping*

$$\mathscr{E}(\mathbb{R} \times U) \to \mathscr{E}(\mathbb{R}^{1+p} \times U) \times (\mathscr{E}(\mathbb{R}^p \times U))^p$$

$$f \to (Q^p_f, R^p_{1,f}, \ldots, R^p_{p,f})$$

such that

(1) *for all $f \in \mathscr{E}(\mathbb{R} \times U)$,*

$$f(t, x) = P^p(t, \lambda)\, Q^p_f(t, \lambda, x) + R^p_f(t, \lambda, x),$$

where

$$R^p_f(t, \lambda, x) = \sum_{j=1}^{p} R^p_{j,f}(\lambda, x)\, t^{p-j};$$

(2) *for every positive integer p, and all $f \in \mathscr{E}(\mathbb{R} \times U)$,*

$$Q^p_f(t, \lambda^p(s, \mu), x) = \frac{Q^{p-1}_f(t, \mu, x) - Q^{p-1}_f(s, \mu, x)}{t - s}.$$

We will prove Theorem 5.3 using Theorem 5.4 below. The equation $P^p(t, \lambda) = 0$ defines a nonsingular closed algebraic subset $X = X^p$ of $\mathbb{R}^{1+p}$. In fact X is the graph of the function

$$\lambda_p = - t^p - \sum_{i=1}^{p-1} \lambda_i t^{p-i},$$

so that the projection $(t, \lambda) \mapsto (t, \lambda_1, \ldots, \lambda_{p-1})$ of $\mathbb{R}^{1+p}$ onto $\mathbb{R}^p$ restricts to a global coordinate system $\phi : X \to \mathbb{R}^p$ on X.

Let $\pi : \mathbb{R}^{1+p} \to \mathbb{R}^p$ be the canonical projection $\pi(t, \lambda) = \lambda$. We denote by $\mathscr{E}_\pi(X \times U)$ the closed subspace of $\mathscr{E}(X \times U)$ consisting of all functions which are constant on the fibers $\pi^{-1}(\lambda) \times x$ of $\pi \times id_U$.

Theorem 5.4. *There exists a continuous $\mathscr{E}(U)$-linear mapping*

$$J : \mathscr{E}_\pi(X \times U) \to \mathscr{E}(\mathbb{R}^p \times U)$$

such that if $h \in \mathscr{E}_\pi(X \times U)$, then

$$(Jh)(\lambda, x) = h(t, \lambda, x)$$

for all $(t, \lambda) \in X$ and $x \in U$.

We will first prove Theorem 5.3 assuming the result of Theorem 5.4, and prove Theorem 5.4 afterwards. Since the variable $x = (x_1, \ldots, x_n)$ in U will play no part in the proof of either theorem, we will simplify our notation by neglecting U and x. It will be clear that the mappings given in both theorems are $\mathscr{E}(U)$-linear.

Proof of Theorem 5.3. We will first prove the theorem in the cases $p = 0$ and $p = 1$, and then argue by induction on p.

When $p = 0$, the desired result clearly holds with $Q^0_f(t, \lambda) = f(t)$ and $R^0_f = 0$. Suppose $p = 1$, so that our generic polynomial is $P^1(t, \lambda) = t + \lambda_1$. We have

$$f(t) - f(-\lambda_1) = (t + \lambda_1) \int_0^1 \frac{\partial f}{\partial t}(st - (1 - s)\lambda_1)\, ds$$

(this is just Hadamard's formula). Hence we can define

$$Q^1_f(t, \lambda) = \int_0^1 \frac{\partial f}{\partial t}(st - (1 - s)\lambda_1)\, ds$$

$$R^1_f(t, \lambda) = R^1_{1,f}(\lambda) = f(-\lambda_1).$$

Then since $\lambda^1(s, \mu) = -s$, we have

$$Q^1_f(t, \lambda^1(s, \mu)) = \frac{f(t) - f(s)}{t - s} = \frac{Q^0_f(t, \mu) - Q^0_f(s, \mu)}{t - s}.$$

Now assume the theorem has been proved for $0, 1, \ldots, p-1$ $(p \geq 2)$. We denote points in $\mathbb{R}^p$, $\mathbb{R}^{p-1}$ and $\mathbb{R}^{p-2}$ respectively by

$$\lambda = (\lambda_1, \ldots, \lambda_p),$$

$$\mu = (\mu_1, \ldots, \mu_{p-1}),$$

$$\nu = (\nu_1, \ldots, \nu_{p-2})$$

and write

$$\lambda(s, \mu) = \lambda^p(s, \mu),\ (s, \mu) \in \mathbb{R} \times \mathbb{R}^{p-1},$$

$$\mu(r, \nu) = \lambda^{p-1}(r, \nu),\ \ (r, \nu) \in \mathbb{R} \times \mathbb{R}^{p-2}.$$

Then

$$P^p(t, \lambda(s, \mu(r, \nu))) = (t-s)\, P^{p-1}(t, \mu(r, \nu)) = (t-s)(t-r)\, P^{p-2}(t, \nu),$$

so that $\lambda(s, \mu(r, \nu))$ is symmetric in (s, r).

By the induction hypothesis,

(5.3.1) $$f(t) = P^{p-1}(t, \mu)\, Q_f^{p-1}(t, \mu) + R_f^{p-1}(t, \mu) =$$

$$= P^p(t, \lambda(s, \mu))\ \frac{Q_f^{p-1}(t, \mu) - Q_f^{p-1}(s, \mu)}{t-s} +$$

$$+ \sum_{j=1}^{p} R_{jf}(s, \mu)\, t^{p-j},$$

where $R_{1f}, \ldots, R_{pf}$ are given by

$$\sum_{j=1}^{p} R_{jf}(s, \mu)\, t^{p-j} = P^{p-1}(t, \mu)\, Q_f^{p-1}(s, \mu) + R_f^{p-1}(t, \mu).$$

We will show each $R_{jf}(s, \mu(r, \nu))$ is symmetric in (s, r). By (5.3.1) and the symmetry of $\lambda(s, \mu(r, \nu))$, it is enough to show

$$\frac{Q_f^{p-1}(t, \mu(r, \nu)) - Q_f^{p-1}(s, \mu(r, \nu))}{t-s}$$

is symmetric in (s, r). But the latter equals

$$\frac{1}{t-s}\left(\frac{Q_f^{p-2}(t, \nu) - Q_f^{p-2}(r, \nu)}{t-r} - \frac{Q_f^{p-2}(s, \nu) - Q_f^{p-2}(r, \nu)}{s-r}\right) =$$

$$= \frac{(s-r)\, Q_f^{p-2}(t, \nu) - (t-r)\, Q_f^{p-2}(s, \nu) + (t-s)\, Q_f^{p-2}(r, \nu)}{(t-r)(t-s)(s-r)},$$

which is clearly symmetric in (s, r).

The mapping

$$(s, \mu_1, \ldots, \mu_{p-1}) \mapsto (s, \lambda_1^p(s, \mu), \ldots, \lambda_{p-1}^p(s, \mu)) =$$

$$= (s,\ \mu_1 - s,\ \mu_2 - \mu_1 s, \ldots, \mu_{p-1} - \mu_{p-2} s)$$

of $\mathbb{R} \times \mathbb{R}^{p-1}$ to $\mathbb{R}^p$ is an invertible polynomial mapping. Let η be its inverse. For each $j = 1, \ldots, p$, let $h_{jf} \circ \eta \circ \phi \in \mathscr{E}(X)$.

We will show that each $h_{jf} \in \mathscr{E}_\pi(X)$. Consider two points $(s, \lambda), (r, \lambda) \in X$, $s \neq r$. Then there exists $\nu \in \mathbb{R}^{p-2}$ such that

$$P^p(t, \lambda) = (t-s)(t-r)\, P^{p-2}(t, \nu),$$

so that

$$\lambda = \lambda(s, \mu(r, \nu)) = \lambda(r, \mu(s, \nu)).$$

Hence

$$\eta \circ \phi(s, \lambda) = (s, \mu(r, \nu)),$$

$$\eta \circ \phi(r, \lambda) = (r, \mu(s, \nu)),$$

and we have

$$h_{jf}(s, \lambda) = R_{jf}(s, \mu(r, \nu)) = R_{jf}(r, \mu(s, \nu)) = h_{jf}(r, \lambda).$$

This shows that $h_{jf} \in \mathscr{E}_\pi(X)$.

Let $R_{jf}^p = J(h_{jf})$, $1 \leq j \leq p$, where $J : \mathscr{E}_\pi(X) \to \mathscr{E}(\mathbb{R}^p)$ is the mapping given by Theorem 5.4. For each j, the mapping $f \mapsto R_{jf}^p$ of $\mathscr{E}(\mathbb{R})$ into $\mathscr{E}(\mathbb{R}^p)$ is continuous and linear.

We will finally show that $f(t) - \Sigma_{j=1}^p R_{jf}^p(\lambda)\, t^{p-j}$ is divisible by $P^p(t, \lambda)$. By Hadamard's lemma, it is enough to show this function vanishes on the zero set of $P(t, \lambda)$. If $P(t, \lambda) = 0$, then $\lambda = \lambda(t, \mu)$ for some $\mu \in \mathbb{R}^{p-1}$, so that

$$R_{jf}^p(\lambda) = (Jh_{jf})(\lambda(t, \mu)) = h_{jf}(t, \lambda(t, \mu))$$

since $(t, \lambda(t,\mu)) \in X$; i.e. $R^p_{jf}(\lambda) = R_{jf}(t,\mu)$. Hence

$$f(t) - \sum_{j=1}^{p} R^p_{jf}(\lambda)\, t^{p-j} = f(t) - \sum_{j=1}^{p} R_{jf}(t,\mu)\, t^{p-j} = 0,$$

by (5.3.1). In other words, $f(t) - \Sigma_{j=1}^p R^p_{jf}(\lambda)\, t^{p-j}$ vanishes on the zero set of $P^p(t,\lambda)$.

We now have

$$f(t) = P^p(t,\lambda)\, Q^p_f(t,\lambda) + \sum_{j=1}^{p} R^p_{j,f}(\lambda)\, t^{p-j},$$

where

$$Q^p_f(t,\lambda) = \frac{f(t) - \sum_{j=1}^{p} R^p_{jf}(\lambda)\, t^{p-j}}{P^p(t,\lambda)}.$$

The mapping $f \mapsto Q^p_f$ of $\mathscr{E}(\mathbb{R})$ to $\mathscr{E}(\mathbb{R}^{1+p})$ is clearly continuous and linear. Moreover if $\lambda = \lambda(s,\mu)$, we obtain

$$Q^p_f(t, \lambda(s,\mu)) = \frac{Q^{p-1}_f(t,\mu) - Q^{p-1}_f(s,\mu)}{t-s}$$

from (5.3.1). This completes the proof of Theorem 5.3, assuming Theorem 5.4.

Proof of Theorem 5.4. The mapping $\pi \mid X$ is proper, and is a diffeomorphism in some neighborhood of any point $(t,\lambda) \in X$ such that $(\partial P/\partial t)(t,\lambda) \neq 0$. If p is odd, then $\pi(X) = \mathbb{R}^p$ since for each $\lambda \in \mathbb{R}^p$, the polynomial $P(t,\lambda) = P^p(t,\lambda)$ has at least one real root. On the other hand, if p is even, then $\pi(X) \subsetneq \mathbb{R}^p$. In this case, $\mathbb{R}^p - \pi(X)$ is convex since P is linear in λ, and $\lambda \in \mathbb{R}^p - \pi(X)$ if and only if $P(t,\lambda) > 0$ for all $t \in \mathbb{R}$. Therefore by Stein's extension theorem (Theorem 3.7 with $k = 1$), it will be enough to show that for all $f \in \mathscr{E}_\pi(X)$, there exists $g \in \mathscr{E}(\pi(X))$ such that $f = (\pi \mid X)^*(g)$.

Our proof is by induction on p. By Glaeser's theorem 4.3, it is enough to show that if $f \in \mathscr{E}_\pi(X)$, then f is formally a composition with $\pi \mid X$ over every point $\lambda \in \pi(X)$.

Let $\sigma_i(w)$, $1 \leq i \leq p$, be the elementary symmetric polynomials in $w = (w_1, \ldots, w_p)$. Put

$$\sigma = (-\sigma_1, \sigma_2, \ldots, (-1)^p \sigma_p).$$

Then $\sigma(\mathbb{R}^p) \subset \pi(X)$ is the set of λ such that $P(t,\lambda)$ has p real roots. Since

$$\prod_{i=1}^{p} (t - w_i) = t^p - \sigma_1(w)\, t^{p-1} + \sigma_2(w)\, t^{p-2} + \ldots + (-1)^p \sigma_p(w),$$

there are mappings $\phi_i : \mathbb{R}^p \to X$ defined by $\phi_i(w) = (w_i, \sigma(w))$, $1 \leq i \leq p$.

We can work with any one of the mappings ϕ_i. We have $\sigma = \pi \circ \phi_i$. If $f \in \mathscr{E}_\pi(X)$, then $\phi_i^*(f)$ is symmetric in $(w_1, \ldots, w_p)$. Hence by Corollary 4.5, there exists $h \in \mathscr{E}(\sigma(\mathbb{R}^n))$ such that

$$\phi_i^*(f) = \sigma^*(h) = \phi_i^*((\pi \mid X)^*(h)).$$

In particular, f is formally a composition with $\pi \mid X$ at the fiber over every point λ such that $P(t,\lambda)$ has p real roots.

On the other hand, we can use the induction assumption to show f is formally a composition with $\pi \mid X$ over every other point of $\pi(X)$. Consider $(t^0, \lambda^0) \in \mathbb{R} \times \mathbb{R}^p$ such that t^0 is a real root of $P(t,\lambda^0)$ of multiplicity $k < p$. Then

$$P(t,\lambda^0) = (t - t^0)^k P^{p-k}(t, \eta^0)$$

for some $\eta^0 \in \mathbb{R}^{p-k}$. Therefore $P^{p-k}(t^0, \eta^0) \neq 0$.

We define a mapping $\lambda : \mathbb{R}^k \times \mathbb{R}^{p-k} \to \mathbb{R}^p$ by the following polynomial identity:

$$P^p(t, \lambda(\xi,\eta)) = P^k(t,\xi)\, P^{p-k}(t,\eta),$$

where $(\xi,\eta) \in \mathbb{R}^k \times \mathbb{R}^{p-k}$. Then λ is a local diffeomorphism at all points (ξ,η) where the resultant of $P^k(t,\xi)$ and $P^{p-k}(t,\eta)$ (as polynomials in t) is nonzero, because this resultant is the Jacobian determinant $D\lambda(\xi,\eta)/D(\xi,\eta)$. (We recall that the resultant of two polynomials is nonzero if and only if the polynomials have no common factors).

Define $\xi^0 \in \mathbb{R}^k$ by $P^k(t,\xi^0) = (t - t_0)^k$. Then $\lambda^0 = \lambda(\xi^0, \eta^0)$, and the mappings $\lambda(\xi,\eta)$, $(t, \lambda(\xi,\eta))$ are diffeomorphisms in some neighborhoods of the points (ξ^0,η^0), (t^0,ξ^0,η^0) respectively. Since $P^{p-k}(t^0,\eta^0) \neq 0$, then $P^{p-k}(t,\eta) \neq 0$ in a neighborhood of (t^0,η^0). If

$$P^p(t, \lambda(\xi,\eta)) = P^k(t,\xi)\, P^{p-k}(t,\eta) = 0,$$

it follows that $P^k(t,\xi) = 0$. Hence the mapping $(t,\xi,\eta) \mapsto (t, \lambda(\xi,\eta))$ induces a commutative diagram

$$\begin{array}{ccc} X^k \times \mathbb{R}^{p-k} & \longrightarrow & X^p \\ {\scriptstyle \pi^k \times id} \downarrow & & \downarrow {\scriptstyle \pi^p} \\ \mathbb{R}^k \times \mathbb{R}^{p-k} & \xrightarrow{\ \lambda\ } & \mathbb{R}^p, \end{array}$$

where the upper and lower arrows are diffeomorphisms in some neighborhoods of the points (t^0, ξ^0, η^0) and (ξ^0, η^0) respectively. By the induction hypothesis on p, it follows that f is formally a composition with $\eta | X$ over λ^0.

Remark 5.5. Theorem 5.4 suggests two interesting problems. Let $f(x) = f(x_1, \dots, x_n)$ and $F(x, u) = F(x_1, \dots, x_n, u_1, \dots, u_p)$ be polynomials such that F is a "miniversal unfolding" of f; i.e. $F(x, 0) = f(x)$, and

$$1, \frac{\partial F}{\partial u_1}(x, 0), \dots, \frac{\partial F}{\partial u_p}(x, 0)$$

form a basis for the real vector space $\mathbb{R}[[x]]/(\partial f/\partial x_i)$. (Here $\mathbb{R}[[x]] = \mathbb{R}[[x_1, \dots, x_n]]$ denotes the ring of formal power series in the variables $x_1, \dots, x_n$, and $(\partial f/\partial x_i)$ is the ideal generated by $\partial f/\partial x_i$, $1 \le i \le n$).

Let $\Sigma(F)$ be the closed algebraic subset of $\mathbb{R}^{n+p}$ defined by

$$\frac{\partial F}{\partial x_i} = 0; \qquad 1 \le i \le n.$$

Then $\Sigma(F)$ is nonsingular and of dimension p. Let $\pi_F : \Sigma(F) \to \mathbb{R}^p$ be the restriction of the projection $\pi(x, u) = u$. We ask the following questions.

(5.5.1) If $g \in \mathscr{E}(\Sigma(F))$ is constant on the fibers of π_F, does there exist $h \in \mathscr{E}(\mathbb{R}^p)$ such that $g = h \circ \pi_F$?

(5.5.2) Does every $\mathscr{C}^\infty$ vector field in $\mathbb{R}^p$ which is tangent to the set of critical values of π_F lift to a $\mathscr{C}^\infty$ vector field in $\Sigma(F)$?

Theorem 5.4 shows the answer to (5.5.1) is "yes" for the "simple singularities" of type A_k. P. Milman has shown it is also "yes" for D_k.

We can show that the answer to (5.5.2) is "yes" for all the simple singularities.

6. Resolution of singularities.

Some ideas from analytic geometry which play an important part in local differential analysis are introduced in this final section. We will show how Hironaka's powerful desingularization theorems can be used to prove the division theorem and inequality of Łojasiewicz, as well as the strong regularity property (2.16.1) for subanalytic sets.

Let N be a real analytic manifold.

Definition 6.1. A subset X of N is *analytic* if every point of X has an open neighborhood U such that $X \cap U$ is the set of common zeros of a finite family of analytic functions in U.

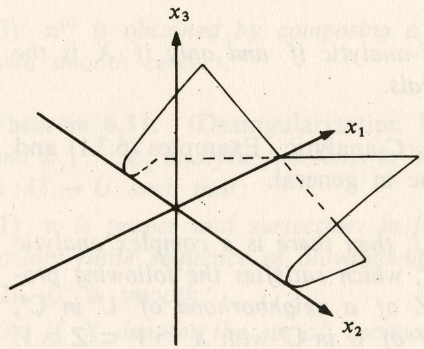
Let $\mathcal{O}$ (respectively $\mathscr{E}$) be the sheaf of germs of analytic (respectively $\mathscr{C}^\infty$) functions in N. If X is a closed analytic subset of N, we denote by $\mathscr{I}_X$ the sheaf of germs of analytic functions vanishing on X. Then $\mathscr{I}_X$ is a sheaf of ideals in $\mathcal{O}$.

Definition 6.2. Let X be a closed analytic subset of N, and $a \in X$. We say that X is *coherent* at a if there exists an open neighborhood U of a, and a finite number of analytic functions $f_1, \dots, f_k$ in U, which vanish on X and have the following property: for any $b \in U$, the germs of $f_1, \dots, f_k$ at b generate $\mathscr{I}_{X,b}$ (the stalk of $\mathscr{I}_X$ at b).

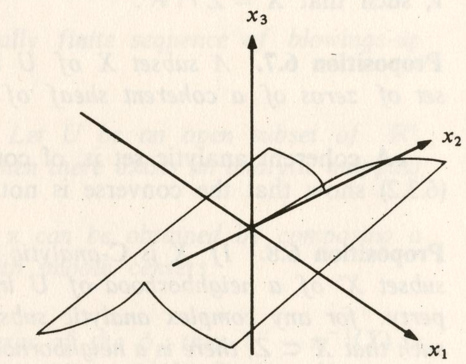
Contrary to the complex analytic case, this property is not satisfied by all real analytic sets.

Examples (6.3.1). "Whitney's umbrella" $X = \{x_3^2 - x_1 x_2^2 = 0\}$ in $\mathbb{R}^3$ is not coherent at 0, since X intersects the halfspace $\{x_1 < 0\}$ in the line $\{x_2 = x_3 = 0\}$.

(6.3.2) The closed analytic subset X of $\mathbb{R}^3$ defined by $x_3^3 - x_1^2 x_2^3 = 0$ is not coherent at 0, since $x_3^3 - x_1^2 x_2^3$ does not generate $\mathscr{I}_{X,b}$ at nonzero points b of the x_1-axis.



$x_3^2 - x_1 x_2^2 = 0$



$x_3^3 - x_1^2 x_2^3 = 0$

We say X is *coherent* if it is coherent at each of its points. Then X is coherent if and only if $\mathscr{I}_X$ is a coherent sheaf of ideals; i.e. for each point of N, there is an open neighborhood U and an exact sequence

$$(\mathcal{O} \,|\, U)^q \to \mathscr{I}_X \,|\, U \to 0.$$

The following theorem indicates the importance of coherence from the point of view of ideals of differentiable functions.

Theorem 6.4 [21, VI.3.10], [36, VI.4.2]. *Let $\mathcal{J}_X$ be the sheaf of germs of $\mathscr{C}^\infty$ functions vanishing on X. Then X is coherent at a if and only if*

$$\mathcal{J}_{X,a} = \mathcal{J}_{X,a} \cdot \mathscr{E}_a.$$

Definition 6.5. Let N be a real analytic manifold, and X a closed analytic subset of N. We say that X is *smooth* at $a \in X$ if there is an open neighborhood U of a such that $X \cap U$ is an analytic submanifold of U.

Real analytic sets may exhibit very irregular behavior (cf. [6], [5]). For example, there are real analytic sets X such that any analytic set containing the set of nonsmooth points of X contains the whole of X. To avoid such irregularities, we restrict our attention to real analytic sets which can be realized as the zero sets of coherent sheaves of ideals. By Definition 6.1, any real analytic set has this property locally.

For simplicity, we will restrict our attention to subsets of $\mathbb{R}^n$.

Definition 6.6. [29, Chapter V]. Let U be an open subset of $\mathbb{R}^n$ (which we regard as a subset of $\mathbb{C}^n$). A closed subset X of U is called *$\mathscr{C}$-analytic* if there exists an open subset V of $\mathbb{C}^n$, and a complex analytic subset Z of V, such that $X = Z \cap \mathbb{R}^n$.

Proposition 6.7. *A subset X of U is $\mathscr{C}$-analytic if and only if X is the set of zeros of a coherent sheaf of ideals.*

A coherent analytic set is, of course, C-analytic. Examples (6.3.1) and (6.3.2) show that the converse is not true in general.

Proposition 6.8. *If X is C-analytic in U, then there is a complex analytic subset X′ of a neighborhood of U in $\mathbb{C}^n$, which satisfies the following property: for any complex analytic subset Z of a neighborhood of U in $\mathbb{C}^n$, such that $X \subset Z$, there is a neighborhood V of U in $\mathbb{C}^n$ with $X' \cap V \subset Z \cap V$.*

Definition 6.9. Let X be C-analytic in U. We define the *singular set* Sing X of X as the intersection with X of the set of nonsmooth points of X' (where X' is given by Proposition 6.8).

Sing X is a C-analytic subset of X. Note that X may be smooth at some of its singular points: in Example (6.3.1), Sing X is the x_1-axis; in Example (6.3.2), Sing X is the union of the x_1- and x_2-axes, although X is smooth at all nonzero points of the x_1-axis. If X is coherent, however, then Sing X coincides with the set of nonsmooth points of X.

Let U be an open subset of $\mathbb{R}^n$, and X a C-analytic subset of U. We can define a sequence of subsets of X by $X^{(0)} = X$ and $X^{(i+1)} = \operatorname{Sing} X^{(i)}$, $i = 0, 1, 2, \ldots$. Then the sequence $\{X^{(i)}\}$ is a *smooth analytic filtration* of X in the sense that:

(1) $X^{(0)} = X$ and $X^{(i+1)}$ is a C-analytic subset of $X^{(i)}$;

(2) $\{X^{(i)}\}$ is finite;

(3) $X^{(i)} - X^{(i+1)}$ is smooth everywhere.

The following two theorems are the main results of Hironaka's great paper [11], for real analytic sets (see also [12], [13, Section 5]). The notion of blowing-up involved in these theorems will be discussed below.

Theorem 6.10 (Desingularization I). *Let U be an open subset of $\mathbb{R}^n$, and X a C-analytic subset of U. Then there exists an analytic mapping $\pi : X' \to X$ such that π is proper and surjective, and X′ is smooth everywhere.*

In more details, given any smooth analytic filtration $\{X^{(i)}\}$ of X, we can choose π in such a way that X′ is a disjoint union of analytic subsets $X'^{(i)}$, each open and closed in X′, and π induces mappings $\pi^{(i)} : X'^{(i)} \to X^{(i)}$ having the following properties:

(1) $(\pi^{(i)})^{-1}(\operatorname{Sing} X^{(i)})$ *is nowhere dense in* $X'^{(i)}$;

(2) $\pi^{(i)}$ *induces an isomorphism*

$$X'^{(i)} - (\pi^{(i)})^{-1}(\operatorname{Sing} X^{(i)}) \xrightarrow{\ \simeq\ } X^{(i)} - \operatorname{Sing} X^{(i)};$$

(3) $\pi^{(i)}$ *is obtained by composing a locally finite sequence of blowings-up with smooth centers.*

Theorem 6.11. (Desingularization II). *Let U be an open subset of $\mathbb{R}^n$, and $\phi_1, \ldots, \phi_k$ analytic functions in U. Then there exists an analytic mapping $\pi : U' \to U$ such that:*

(1) *π is proper and surjective; in fact, π can be obtained by composing a locally finite sequence of blowings-up with smooth centers;*

(2) *U′ is smooth;*

(3) *if X denotes the set of common zeros of the ϕ_i, then $U' - \pi^{-1}(X)$ is dense in U′, and π induces an isomorphism $U' - \pi^{-1}(X) \xrightarrow{\simeq} U - X$;*

(4) *for all $x' \in U'$, there exists a local coordinate system $(z_1, \ldots, z_n)$ of U′ centered at x′, such that the germs at x′ of the $\phi_i \circ \pi$ generate a principal ideal, which is generated by a monomial $z_1^{\ell_1} \ldots z_n^{\ell_n}$ with nonnegative integers ℓ_j (we say $\pi^{-1}(X)$ is locally everywhere normal crossings).*

6.12. Blowing up. Let $\mathbb{P}^r$ denote real projective space of dimension r. There is a natural mapping $p_0 : \mathbb{R}^n - \{0\} \to \mathbb{P}^{n-1}$ such that for all $\xi \in \mathbb{P}^{n-1}$, $p_0^{-1}(\xi) \cup \{0\}$ is a line through the origin in $\mathbb{R}^n$. By assigning to each $\xi \in \mathbb{P}^{n-1}$, the line obtained in this way, we get a real line bundle $p : L \to \mathbb{P}^{n-1}$, and a natural mapping $\pi_0 : L \to \mathbb{R}^n$ which is an isomorphism outside the zero section of p, and such that the zero section is mapped to the origin of $\mathbb{R}^n$.

L has the structure of a real analytic manifold. With respect to the coordinates $(x_1, \ldots, x_n)$ of $\mathbb{R}^n$, this structure can be defined by a covering $L = \cup_{i=1}^n L_i$, where $L_i \cong \mathbb{R}^n$ and L_i has a coordinate system $(z_{i1}, \ldots, z_{in})$ in which $\pi_0 | L_i$ is given by $x_j \circ \pi_0 = z_{ii}$ if $j = i$, $x_j \circ \pi_0 = z_{ii} z_{ij}$ if $j \neq i$. The mapping $\pi_0 : L \to \mathbb{R}^n$ is the *blowing-up* of $\mathbb{R}^n$ with *center* 0.

Let $Z = \mathbb{R}^n \times \mathbb{R}^p$ and $Z' = L \times \mathbb{R}^p$. Then $\pi = \pi_0 \times id_{\mathbb{R}^p} : Z' \to Z$ is the *blowing-up* of Z with *center* $0 \times \mathbb{R}^p$. More generally, if Z is a real analytic manifold and Y a smooth analytic subset of Z, then we can define the *blowing-up* $\pi : Z' \to Z$ with *center* Y: π is defined as before in a neighborhood of each point of Y, and is defined to be an isomorphism outside Y.

Now let Z be a real analytic manifold, and let $Y \subset X$ be analytic subsets of Z such that Y is smooth, but X perhaps singular. Let $\pi : Z' \to Z$ be the blowing-up of Z with center Y. By the *strict transform* X' of X by π, we mean the smallest analytic subset of $\pi^{-1}(X)$ such that π induces an isomorphism $X' - \pi^{-1}(Y) \xrightarrow{\sim} X - Y$. The mapping $p : X' \to X$ induced by π is the *blowing-up* of X with *center* Y.

For example, suppose that at $a \in Y$, X is a hypersurface, defined by an analytic equation $f = 0$. Pick a local coordinate system $(y_1, \ldots, y_r, x_1, \ldots, x_s)$ for Z centered at a, such that Y is given by $x_1 = \ldots = x_s = 0$. Then over some neighborhood of a, Z' is covered by s coordinate charts Z_i' in which can choose coordinates

$$(y_1, \ldots, y_r, z_1, \ldots, z_s) = \left(y_1, \ldots, y_r, \frac{x_1}{x_i}, \ldots, x_i, \ldots, \frac{x_s}{x_i}\right).$$

The *order* p *of* f *along* Y *at* a is the greatest integer q such that $f \in J_a^q$, where J_a is the ideal of germs at a generated by $x_1, \ldots, x_s$. Over a neighborhood of a, the strict transform X' of X is covered by $\cup_{i=1}^s X_i'$, where X'_i is defined in Z_i' by the equation

$$\frac{1}{z_i^p} f(y, z_i z_1, \ldots, z_i, \ldots, z_i z_s) = 0.$$

Examples (6.13.1). In Example (6.3.1), the strict transform of $X^{(0)} = X$ by the blowing-up of $\mathbb{R}^3$ with center the x_1-axis, is the smooth hypersurface $X'^{(0)} = \{z_1 = z_3^2\}$ in $\mathbb{R}^3$, where the induced mapping $\pi^{(0)} : X'^{(0)} \to X^{(0)}$ is determined by $(x_1, x_2, x_3) = (z_1, z_2, z_2 z_3)$. Let $X^{(1)}$ be the x_1-axis and $\pi^{(1)} : X'^{(1)} \to X^{(1)}$ the identity. If X' is the disjoint union of $X'^{(0)}$ and $X'^{(1)}$, and $\pi : X' \to X$ is the mapping defined by $\pi^{(0)}$ and $\pi^{(1)}$, then π is a resolution of the singularities of X, in the sense of Desingularization I.

(6.13.2). Example (6.3.2) can be desingularized by two blowings-up. The blowing-up of X with center the x_1-axis is the hypersurface $\{z_3^3 - z_1^2 = 0\}$ in $\mathbb{R}^3$, together with the mapping induced by $(x_1, x_2, x_3) = (z_1, z_2, z_2 z_3)$. The blowing-up of the latter hypersurface with center the z_2-axis is the smooth hypersurface $\{u_3 - u_1^2 = 0\}$, together with the mapping induced by $(z_1, z_2, z_3) = (u_1 u_3, u_2, u_3)$. The composition of these two blowings-up is a desingularization of X.

We will use Desingularization II to prove Łojasiewicz's division theorem and, as a corollary, Łojasiewicz's inequality. Let U be an open subset of $\mathbb{R}^n$, and $\phi_1, \ldots, \phi_k$ analytic functions in U. Let I be the ideal in $\mathscr{E}(U)$ generated by $\phi_1, \ldots, \phi_k$. We denote by $\hat{I}$ the ideal in $\mathscr{E}(U)$ of functions which formally belong to I; i.e. functions f such that for all $a \in U$, $\hat{f}_a$ belongs to the ideal generated by the $\hat{\phi}_{i,a}$ in $\mathscr{F}_a$. Clearly $\bar{I} \subset \hat{I}$ (in fact, $\bar{I} = \hat{I}$ according to Whitney's spectral theorem [21, II.1.7], [36, V.1.6]). Malgrange's theorem: $I = \hat{I}$ [19], [21, VI.1.1] was first proved by Łojasiewicz [16] in the particular case $k = 1$.

Theorem 6.14. *Let ϕ be analytic in U, and let I be the ideal generated by ϕ in $\mathscr{E}(U)$. Then $I = \hat{I}$.*

Proof. We apply Theorem 6.11 with $k = 1$ and $\phi_1 = \phi$. According to the theorem, there is an open convering $U' = \cup_\alpha U'_\alpha$ of U', with isomorphisms $U'_a \cong \mathbb{R}^n$, such that if $z = (z_1, \ldots, z_n)$ denotes the coordinates in U'_α, then $(\phi \circ \pi)(z) = z_1^{\ell_{\alpha 1}} \ldots z_n^{\ell_{\alpha n}} u(z)$, where the $\ell_{\alpha i}$ are nonnegative integers, and u is a unit.

Suppose $f \in \hat{I}$; i.e. for all $a \in U$, there exists $G_a \in \mathscr{F}_a$ such that

$$\hat{f}_a = \hat{\phi}_a \cdot G_a. \tag{6.14.1}$$

Then $\pi^*(f) | U'_\alpha$ belongs formally to ideal generated by $\pi^*(\phi) | U'_\alpha = z_1^{\ell_{\alpha 1}} \ldots z_n^{\ell_{\alpha n}} u$ in $\mathscr{E}(U'_\alpha)$. By Hadamard's lemma, $\pi^*(f) = \pi^*(\phi) \cdot h$, where $h \in \mathscr{E}(U')$.

It follows from (6.14.1) that h is formally a composition with π. We would like to use Glaeser's theorem 4.3 to conclude that there exists $g \in \mathscr{E}(U)$ such that $\hat{g}_a = G_a$ for all $a \in U$. But we must avoid a circular argument: Łojasiewicz's division theorem and inequality were used in essential steps (Lemmas 4.11 and 4.12 respectively) of the proof of Glaeser's theorem. However, π is the composition of locally finite sequence of blowings-up with smooth centers, so we need Theorem 4.3 only in the special case of such a blowing-up. In this case, Lemma 4.12 is clearly not needed, and for Lemma 4.11, it is enough to prove Theorem 6.14 in the special case that ϕ is the Jacobian determinant δ of the mapping

$$(z_1, \ldots, z_p, z_{p-1}, \ldots, z_n) \to (z_1, \ldots, z_p, z_{p+1}, z_{p+1} z_{p+2}, \ldots, z_{p+1} z_n)$$

Then δ is a power of z_{p+1}, so the result follows from Hadamard's lemma.

Corollary 6.15. *Let U be an open subset of $\mathbb{R}^n$, and ϕ a real analytic function in U. Let $X = \{x \in U : \phi(x) = 0\}$. Then for any compact subset K of U, there exists $C > 0$ and an integer $\alpha \geq 1$, such that for all $x \in K$,*

$$|\phi(x)| \geq C\, d(x, X)^\alpha.$$

Proof. By Theorem 6.15, $(\phi) \cdot \mathscr{E}(U)$ is closed. Therefore, by the open mapping theorem, for every $K \subset U$ compact and $m \geq 0$, there exists $K' \subset U$ compact and $m' \geq 0$ such that if $f \in (\phi) \cdot \mathscr{E}(U)$, there exists $g \in \mathscr{E}(U)$ such that $f = \phi \cdot g$ and

$$(6.15.1) \qquad |g|_m^K \leq c\,|f|_{m'}^{K'},$$

where c is independent of f.

If $x_0 \in K - X$, we can find $f \in \mathscr{E}(U)$ such that $f(x_0) = 1$, $f = 0$ in a neighborhood of X, and $|f|_{m'}^{K'} \leq c'\, d(x_0, X)^{-\alpha}$, where $c' > 0$ and $\alpha \geq 1$ are independent of x_0, but depend only on K, K'. Then (6.15.1) implies

$$\sup_K \left|\frac{f}{\phi}\right| \leq cc'\, d(x_0, X)^{-\alpha};$$

in particular,

$$|\phi(x_0)| \geq (cc')^{-1}\, d(x_0, X)^\alpha.$$

Hironaka [13] has given proofs of Corollary 6.15 and several related inequalities of Łojasiewicz, using his "rectilinearization theorem" [13, Theorem 7.1]. The rectilinearization theorem asserts that every subanalytic set can be transformed locally into unions of quadrants in Euclidean spaces, by means of a locally finite family of finite sequences of "local blowings-up" applied to the ambient space. (A subset B of $\mathbb{R}^n$ is called a *quadrant* if there exists a disjoint partition $\{1, \ldots, n\} = I_0 \cup I_+ \cup I_-$ such that B is the set of points $x = (x_1, \ldots, x_n)$ satisfying $x_i = 0$, $i \in I_0$, $x_i > 0$, $i \in I_+$, and $x_i < 0$, $i \in I_-$). Hironaka's proof of the rectilinearization theorem uses the desingularization theorems, as well as his "local flattening theorem".

We conclude by stating the rectilinearization theorem and applying it to prove the strong regularity property (2.16.1) for a closed subanalytic set X such that Int X is dense in X. R. Hardt has shown me another proof of this regularity condition, using geometric measure theory.

Theorem 6.16. *Let N be a real analytic manifold, and A a subanalytic subset of N. Let L be a compact subset of N. Then there exists a finite number of real analytic mappings $\pi_j : U_j \to N$ such that:*

(1) *U_j is isomorphic to $\mathbb{R}^{n_j}$, for some n_j;*

(2) *there exists a compact subset K_j of U_j, such that $\cup_j \pi_j(K_j)$ is a neighborhood of L in N;*

(3) *$\pi_j^{-1}(A)$ is union of quadrants in $\mathbb{R}^{n_j}$.*

Theorem 6.17. *Let V be an open subset of $\mathbb{R}^n$, and A a closed subanalytic subset of V such that Int A is dense in A. For every compact subset L of A, there exists $c > 0$ and an integer $\alpha \geq 1$ such that any two points b, $y \in L$ can be joined by a semianalytic arc σ in A such that:*

(1) $|\sigma| \leq c\,|b - y|^{1/\alpha}$.

(2) *σ intersects ∂A in at most finitely many points.*

Proof. It follows from Theorem 6.16 that there exists a finite number of analytic mappings $\pi_j : U_j \to V$ such that:

(1) $U_j = \mathbb{R}^n$ and rank $\pi_j = n$;

(2) $\pi_j(U_j) \subset A$;

(3) there is a closed ball K_j centered at the origin in U_j, such that $\cup_j \pi_j(K_j)$ is a neighborhood of L in A.

Since closed subanalytic subsets of V are "regularly situated" [13, Section 9], it is enough to prove that for each j, there exists $c > 0$ and an integer $\alpha \geq 1$ such that any two points b, $y \in \pi_j(K_j)$ can be joined by a semianalytic arc σ in A, satisfying (1), (2) of the theorem (cf. [17, Section 18]).

The argument is similar to Step 2 of our proof of Theorem 4.3. Write $\phi = \pi_j$. Let $X = Z \cup \phi^{-1}(\partial A)$, where Z is the set of critical points of ϕ. Clearly $\dim X < n$. Given b, $y \in \phi(K_j)$, choose a, $x \in K_j$ such that $\phi(a) = b$, $\phi(x) = y$. Let a', x' be points of $2K_j$ associated to a, x by Lemma 4.12. For every $c_1 > 1$, there are broken line segments σ_1, σ_2 of length $\leq c_1|a - a'|$, $\leq c_1|x - x'|$ respectively, which join a to a', x to x' respectively, and intersect X in at most finitely many points (cf. Lemma 4.15). Then $\sigma = \phi(\sigma_1) \cup \phi(\sigma_2)$ is a semianalytic arc joining b, y in A, which meets ∂A in at most finitely many points. The required estimate on $|\sigma|$ follows as in the proof of Theorem 4.3.

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