Some generic properties of Riemannian immersions

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1. Introduction.

We will use "smooth", "differentiable" and "C^∞" interchangeably in this paper. Let $f : M^m \to N^n$ be a smooth immersion, where M and N are differentiable manifolds. Whenever N has a given Riemannian structure, we will consider in M the Riemannian structure induced by f. Denote by $C(M, N)$ the set of C^∞ maps from M to N with the "fine" C^∞ topology. Alternatively, we could endow $C(M, N)$ with the uniform convergence on compact sets topology. We know that these two topologies are equivalent if and only if M is compact, the advantage of the former being that the immersions $Im(M, N)$ and imbeddings become open subsets of $C(M, N)$, even if M is not compact. For details on these topologies we refer to [3, pp. 220-223] and to [7, ch. II].

The object of this paper is to give the proof of some results concerning geometric singularity theory which we mention in [1], and then make further applications of these results to the generic study of specific geometric singularities which were firstly estudied by Feldman [5] and Little [9]. By *generic* we mean properties which hold for a dense (and eventually open) subset of $Im(M, N)$. We prove

(I) — The mean curvature vector of an immersion $f : M^m \to N^{2m}$ has, generically, isolated singularities.

(II) — The locus $\mathscr{P}(f)$ of parabolic points of an immersion $f : M^2 \to N^3$ is, generically, a closed one-dimensional submanifold of M^2.

(III) — If M^2 is compact, then the cardinallity $\# \mathscr{U}(f)$ of the locus of the umbilic points of an immersion $f : M^2 \to N^3$ satisfies, generically, the inequalities $2|\mathscr{X}(M)| \leq \# \mathscr{U}(f) < +\infty$.

We should say some words about this paper's organization. The next section is devoted to the revision of some results on jet transversality theory and to the proof of Theorem 2.11, which is a general result on geometric transversality theory. The reader who wants to avoid jet transversality language may go directly to Section 3, where we give a complete geometric description of Theorem 2.11, and where we prove (I). In Section 4 we give complete statements and proofs of (II) and (III).

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2. Review of Feldman's and Little's results.

Theorem 2.11 was firstly proved by Little [9, Theo. 2.20] in the case $N = R^n$. Our contribution in proving the general case is Proposition 2.8 below, which generalizes Proposition 2.16 of [9], and which will enable us to substitute R^n by any Riemannian manifold N^n in Little's result. In order to proof Proposition 2.8 we need to look back some definitions and techniques of [3].

2-a. *Tangent bundles, osculating maps.* Let $(x_1, \ldots, x_m)$ be coordinate functions defined on a coordinate neighbourhood U of M, and denote by $F(U)$ the sheaf of germs of smooth functions on U (see [8, p. 81]). Let $\left.\frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{ik}}\right|_x$ represent the linear functional which sends each germ $[f]$ of $F(U)$ into $\frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}$, for each $x \in U$ and $\leq i_1 \leq \ldots \leq i_k \leq m$. Let $(T_pM)_x$ be the real vector space spanned by the functionals

$$\left\{ \left.\frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}}\right|_x ; \leq k \leq p \right\}.$$

We give a natural structure of smooth vector bundle over M to the set $T_pM = \bigcup_{x \in M} (T_pM)_x$: if $(y_1, \ldots, y_m)$ are coordinates on another neighbourhood V of x, we relate the functionals $\left.\frac{\partial^k}{\partial y_{i_1} \ldots \partial y_{i_k}}\right|_x$ with the functionals $\left.\frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{ik}}\right|_x$ through the well known formulas given by the chain rule for partial derivatives. With this differentiable structure T_pM is called the *pth order tangent bundle* of M: the fibre $(T_pM)_x$ is called the *pth order tangent space* of M at x with fibre dimension

$$v(m, p) = \sum_{k=1}^{p} \frac{(m + k - 1)!}{(m - 1)!\, k!}$$

Usually T_1M and $(T_1M)_x$ are denoted by TM and $T_x(M)$, respectively. The definitions above, as well as further definitions concerning the pth order tangent bundle, may be found in [3] or [11].

A differentiable map $f : M \to N$ induces, for each $p = 1, 2, \ldots$ a differentiable homomorphism from T_pM to T_pN which covers f. Explicitly, if $\pi : T_pM \to M$ and $\pi' : T_pN \to N$ denote the respective canonical projections, then there exists a differentiable homomorphism $T_p(f) : T_pM \to T_pN$ called the *pth order differential* of f, such that $\pi' \circ T_p(f) = f \circ \pi$. Denote by

$$T = \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_s \leq s \\ 1 \leq s \leq p}} \mathcal{A}_{i_1 \ldots i_s} \frac{\partial^s}{\partial x_{i_1} \ldots \partial x_{is}}$$

an element of T_pM relative to coordinates defined in $U \subset M$. Then

$$[T_p(f) \cdot T]([g])\Big|_x = \Sigma\, \mathcal{A}_{i_1 \ldots i_s}(x) \left.\frac{\partial^s (gof)}{\partial x_{i_1} \ldots \partial x_{is}}\right|_x,$$

for each $x \in U$ and $[g] \in F(V)$, where $V \subset N$ is a coordinate neighbourhood around $y = f(x)$.

Let O^pTM denotes the p-fold symmetric tensor product of TM. Remember that O^pTM is a vector bundle over M whose fibre is generated by the functionals $\left.\frac{\partial}{\partial x_i}\right|_x \circ \ldots \circ \left.\frac{\partial}{\partial x_i}\right|_x$. If $p \geq 2$ then $T_{p-1}M$ may be viewd as a subbundle of T_pM via the inclusion map $I_{p-1} : T_{p-1}M \to T_pM$. Now we define, with the help of local coordinates, an epimorphism $P_p : T_pM \to O^pTM$: assuming that $T \in T_pM$ is given by the above local representation, then

$$P_p \cdot T = \sum_{1 \leq i_1 \leq \ldots \leq i_p \leq p} \mathcal{A}_{i_1 \ldots i_p} \frac{\partial}{\partial x_{i_1}} \circ \ldots \circ \frac{\partial}{\partial x_{i_p}}$$

In [10, p. 174], is proved that $T_pM / T_{p-1}M$ is isomorphic to O^pTM and that the sequence

$$O \to T_{p-1}M \xrightarrow{I_{p-1}} T_pM \xrightarrow{P_p} O^pTM \to O$$

is exact.

From now on we will restrict our study to the case when $p = 2$, although most of the definitions and results hold for any p.

There is a correspondence which assigns each couple $u, v \in TM$ to an element $uv \in T_2M$ which we describe in terms of coordinates on a neighbourhood $U \subset M$: if

$$u = \Sigma\, u_i \frac{\partial}{\partial x_i} \quad \text{and} \quad v = \Sigma\, v_i \frac{\partial}{\partial x_i},$$

then

$$uv(x) = \sum_{i,j=1}^{m} u_i(x) \frac{\partial v_j}{\partial x_i}(x) \left.\frac{\partial}{\partial x_j}\right|_x + \sum_{i,j=1}^{m} u_i(x)\, v_j(x) \left.\frac{\partial^2}{\partial x_i \partial x_j}\right|_x,$$

for each $x \in U$.

Define now a map $\Phi : O^2TN \to T_2N$ by $\Phi(u \circ v) = uv - D_u v$, where we suppose that N is a Riemannian manifold with Riemannian connection D. The next proposition is proved in [3], p. 191.

Proposition 2.1. *The mapping Φ is a splitting of the short exact sequence of vector bundles*

$$O \longrightarrow TN \xrightarrow{I_1} T_2N \xrightarrow{P_2} O^2TN \longrightarrow O,$$

that is, $P_2 \circ \Phi$ is the identity map of O^2TN.

Then there exists a map $D_2 : T_2N \to TN$ such that $D_2 \cdot I_1$ is the identity map of TN.

Since $T - \Phi \circ P_2(T)$ is in the kernel of P_2 for each $T \in T_2N$, it is not hard to see that $D_2(T) = I_1^{-1}(T - \Phi \circ P_2(T))$. If $f : M \to N$ is a differentiable map, the *second order osculating map of f with respect to D* is the composition $D_2 \cdot T_2(f)$, which is a vector bundle homomorphism from T_2M to TN covering f. The *second order osculating space of f at x with respect to D* is the vector subspace $D_2T_2(f)_x(T_x(M))$ of $T_{f(x)}(N)$.

Let $x \in M$, $y = f(x) \in N$, and let $(x_1, \dots, x_m)$, $(y_1, \dots, y_n)$ be coordinates on neighbourhoods U of x and V of y, respectively. We want to know the expression of $D_2T_2(f)_x(T)$ for a vector T of the type

$$T(x) = \frac{\partial^2}{\partial x_i \, \partial x_j}\bigg|_x.$$

If $[g] \in F(V)$, then

$$T_2(f)_x\left(\frac{\partial^2}{\partial x_i \, \partial x_j}\bigg|_x\right)([g]) = \frac{\partial^2 (g \circ f)}{\partial x_i \, \partial x_j}(x) =$$

$$= \frac{\partial^2}{\partial x_i \, \partial x_j}(g(y_i \circ f(x), \dots, y_n \circ f(x))) =$$

$$= \frac{\partial}{\partial x_j}\left[\sum_{k=1}^{n} \frac{\partial (y_k \circ f)}{\partial x_i}(x)\, \frac{\partial g}{\partial y_k}(y_1 \circ f(x), \dots, y_n \circ f(x))\right] =$$

$$= \sum_{k=1}^{n} \frac{\partial^2 (y_k \circ f)}{\partial x_i \, \partial x_j}(x)\, \frac{\partial g}{\partial y_k}(y_1 \circ f(x), \dots, y_n \circ f(x)) +$$

$$+ \sum_{k,l=1}^{n} \frac{\partial (y_k \circ f)}{\partial x_i}(x)\, \frac{\partial (y_l \circ f)}{\partial x_j}(x)\, \frac{\partial^2 g}{\partial y_k \, \partial y_l}(y_1 \circ f(x), \dots, y_n \circ f(x)),$$

or, deleting the g's,

$$T_2(f)_x\left(\frac{\partial^2}{\partial x_i \, \partial x_j}\bigg|_x\right) = \sum_{k=1}^{n} \frac{\partial^2 f_k}{\partial x_i \, \partial y_j}(x)\, \frac{\partial}{\partial y_k}\bigg|_y +$$

$$+ \sum_{k,l=1}^{n} \frac{\partial f_k}{\partial x_i}(x)\, \frac{\partial f_l}{\partial x_j}(x)\, \frac{\partial^2}{\partial y_k \, \partial y_l}\bigg|_y,$$

where $f_k = y_k \circ f$. It follows that

$$D_2 \cdot T_2(f)_x\left(\frac{\partial^2}{\partial x_i \, \partial x_j}\bigg|_x\right) =$$

$$= T_2(f)\left(\frac{\partial^2}{\partial x_i \, \partial x_j}\bigg|_x\right) - \Phi \circ P_2\left[T_2(f)_x\left(\frac{\partial^2}{\partial x_i \, \partial x_j}\bigg|_x\right)\right] =$$

$$= T_2(f)_x\left(\frac{\partial^2 j}{\partial x_i \, \partial x_j}\bigg|_x\right)$$

$$- \Phi\left[\left(\sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i}(x)\, \frac{\partial}{\partial y_k}\bigg|_y\right) \circ \left(\sum_{l=1}^{n} \frac{\partial f_l}{\partial x_j}(x)\, \frac{\partial}{\partial y_l}\bigg|_y\right)\right],$$

by the definition of P_2. By using the definition of Φ and after bringing about the necessary calculations, we come to the following formula

$$D_2T_2(f)_x\left(\frac{\partial^2 j}{\partial x_i \, \partial x_j}\bigg|_x\right) = \sum_{k=1}^{n} \frac{\partial^2 f_k}{\partial x_i \, \partial x_j}(x)\, \frac{\partial}{\partial y_k}\bigg|_y + \tag{1}$$

$$+ \sum_{k,l=1}^{n} \frac{\partial f_k}{\partial x_i}(x)\, \frac{\partial f_l}{\partial x_j}(x)\, D_{\frac{\partial}{\partial y_k}|_y}\left(\frac{\partial}{\partial y_l}\bigg|_y\right).$$

2-b. *$J^2(m)$ and $J^2(m) \times O(m)$ actions.* Let us consider now the bundle $\mathrm{Hom}(T_2M, TN)$ which is a vector bundle with base space $M \times N$ whose fibre F over $(x, y) \in M \times N$ is the set of linear transformations from $(T_2M)_x$ to $T_y(N)$. Given a map $f : M \to N$,

$$\{(x, L), \text{ such that } x \in M \text{ and } L : (T_2M)_x \to T_{f(x)}(N) \text{ is linear}\}$$

is a vector bundle with base M, which is usually denoted by

$$f^{-1}(\mathrm{Hom}(T_2M, TN)).$$

Since $D_2T_2(f) : T_2M \to TN$ is a homomorphism of the bundles T_2M and TN, there exists a cross section $M \to f^{-1}(\mathrm{Hom}\,(T_2M, TN))$, namely $x \to (x, D_2T_2(f)_x)$. Moreover the inclusion map

$$\bar{f} : f^{-1}(\mathrm{Hom}(T_2M, TN)) \to \mathrm{Hom}(T_2M, TN)$$

is such that the diagram

$$\begin{array}{ccc} f^{-1}(\mathrm{Hom}(T_2M, TN)) & \xrightarrow{\ \bar{f}\ } & \mathrm{Hom}(T_2M, TN) \\ \downarrow \pi' & & \downarrow \pi \\ M & \xrightarrow{\ \mathrm{id} \times f\ } & M \times N \end{array}$$

is commutative, where π, π' are the canonical projections. We call $\tilde{f}$ the cross section $M \to f^{-1}(\mathrm{Hom}(T_2M, TN))$ composed with $\bar{f}$; explicitly, $\tilde{f}$ is given by $\tilde{f}(x) = (x, f(x), D_2T_2(f)_x)$.

Observe that the structural group of T_2M is the group of linear transformations in the fibre induced by all possible coordinate changes on the base, i.e., the Lie group $J^2(m)$ of invertible 2-jets from R^m to R^m with source and target at the origin (see [12], §1). Since TN may be taken as a bundle with structural group $O(n)$, the orthogonal transformations of R^n, we see that $\mathrm{Hom}(T_2M, TN)$ may be taken as a bundle with group $J^2(m) \times O(n)$.

Given a point $x_0 \in M$ and coordinates $(x_1, \dots, x_m)$ on a neighborhood U of x_0, then for each $x \in U$ we have the two sets of linear functionals

(2)
$$X_1|_x = \left\{ \frac{\partial}{\partial x_1}\bigg|_x, \dots, \frac{\partial}{\partial x_m}\bigg|_x \right\}$$
$$X_2|_x = \left\{ \frac{\partial^2}{\partial x_1^2}\bigg|_x, \frac{\partial^2}{\partial x_1 \partial x_2}\bigg|_x, \dots, \frac{\partial^2}{\partial x_m^2}\bigg|_x \right\}.$$

Given another neighbourhood V of x_0 with coordinates $(y_1, \dots, y_m)$, we will have the analogous sets $Y_1|_y$, $Y_2|_y$, for each $y \in V$. Then we would have two bases

$$X|_x = X_1|_x \cup X_2|_x, \qquad Y|_x = Y_1|_x \cup Y_2|_x$$

for the fibre $(T_2M)_x$, $x \in U \cap V$. It will be of interest to describe the element $L \in J^2(m)$ which carries X into Y. By the chain rule,

(3)
$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{m} \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j},$$
$$\frac{\partial^2}{\partial y_i \partial y_j} = \sum_{k,l=1}^{m} \frac{\partial x_k}{\partial y_i} \frac{\partial x_l}{\partial y_j} \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{k=1}^{m} \frac{\partial^2 x_k}{\partial y_i \partial y_j} \frac{\partial}{\partial x_k},$$

for $1 \le i \le j \le m$. Since $L(X) = Y$, we have

$$L(X_1) = L_{11}X_1 = Y_1,$$

$$L(X) = L_{21}X_1 + L_{22}X_2 = Y_2,$$

where the entries of the matrices L_{ij} are easily obtained by Eq. (3). Then we may write the matrix of L relative to the basis X on the block form

$$L = \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix}.$$

Suppose now that $n \ge m$. Let A be an element of the fibre

$$F \cong \mathrm{Hom}(T_2M, TN)_{(x,y)}$$

such that $\dim[A(T_x(M))] = m$, i.e., A is 1-1 when restricted to the tangent space $T_x(M)$. Let $E = \{e_1, \dots, e_n\}$ be an orthonormal basis of $T_y(N)$ chosen in a way that $E_1 = \{e_1, \dots, e_m\}$ spans $A(T_x(M))$. Such a basis is said to be *adapted* to A. We may write

$$A(X_i) = \sum_{j=1}^{2} A_{ij}E_j,$$

where $E_2 = \{e_{m+1}, \dots, e_n\}$. Then the matrix of A in terms of the bases X and E has the following block form

$$A = \begin{pmatrix} A_{11} & O \\ A_{21} & A_{22} \end{pmatrix}$$

Let A and L be linear transformations as above. Denote by the same letters the matrix of L relative to X, and the matrix of A relative to X and E. If $Y = L(X)$, then the matrix of A relative to Y and E is LA. If we change also E by $E' = O(E)$, where $O \in O(n)$, then the matrix of A relative to Y and E' will be LAO^t. We believe that these remarks will clear up the meaning of the action of $J^2(m) \times O(n)$ over the fibres of $\mathrm{Hom}(T_2M, TN)$.

The proof of the following theorem is found in [9], pp. 297-299.

Theorem 2.2. *Let A be an elemente of F such that A is 1-1 when restricted to $T_x(M)$, and let $E = E_1 \cup E_2$ be an orthonormal basis of $T_y(N)$ adapted to A. It is possible to choose a basis X for $(T_2M)_x$, which is induced from a coordinate system $(x_1, \dots, x_m)$ around x, with the property: the matrix of A relative to the bases X and E takes the block form*

$$A = \begin{pmatrix} I & O \\ O & A_{22} \end{pmatrix}$$

where I is the $m \times m$ identity matrix. Furthermore X is the unique basis of $T_x(M)$ satisfying the above conditions.

That is, given A and E as in Theorem 2.2, it is possible to choose one and only one basis $X = X_1 \cup X_2$ of $(T_2M)_x$, induced from coordinates around x, in such a way that

$$A\left(\frac{\partial}{\partial x_i}\bigg|_x\right) = e_i, \quad i = 1, \dots, m;$$

$$A\left(\frac{\partial^2}{\partial x_i \partial x_j}\bigg|_x\right) = \sum_{\alpha=m+1}^{n} \mathscr{A}^{\alpha}_{ij}(x)\, e_\alpha, \quad 1 \leq i \leq j \leq m.$$

Suppose now that are given a fibre $F = \mathrm{Hom}(T_2M, TN)|_{(x,y)}$ and bases $X = X_1 \cup X_2$ of $(T_2M)_x$, $E = E_1 \cup E_2$ of $T_y(N)$. It is convenient to denote by $\tilde{F}$ the (open) subset of F whose elements are the linear transformations $A : (T_2M)_x \to T_y(N)$ such that $\dim [A(T_x(M))] = m$. Denote by Z the set of linear transformations $A \in F$ such that, relative to X and E, A has the block form prescribed by Theorem 2.2. Note that $Z \subset \tilde{F}$.

Let us denote by $\mathcal{O}$ the subgroup of $J^2(m) \times O(n)$ whose elements are the pairs (L, O) such that, L relative to the basis X, and O relative to the basis E, have the respective block forms

$$L = \begin{pmatrix} L_{11} & O \\ O & L_{22} \end{pmatrix}, \qquad O = \begin{pmatrix} O_{11} & O \\ O & O_{22} \end{pmatrix},$$

with $L_{11}O^t_{11} = I$. By using the fact that we may express the entries of L_{22} as a first degree polynomial function with integer coefficients of the entries of L_{11}, it is not hard to prove that $\mathcal{O}$ is isomorphic to the subgroup $O(m) \times O(n-m)$ of $O(n)$. The following propositions may be found in [9, pp. 302-304]. The definition of subvariety is reminded in the next paragraph.

Proposition 2.3. *The subgroup of $J(m) \times O(n)$ which leaves Z setwise fixed is $\mathcal{O}$.*

Proposition 2.4. *The subvarieties of $\tilde{F}$ invariant under $J^2(m) \times O(n)$ are in 1-1 correspondence with the subvarieties of Z invariant under $\mathcal{O}$, where the correspondence ρ is given by $\rho(K) = K \cap Z$. Also, if K is an orbit of $\tilde{F}$ under $J^2(m) \times O(n)$, then $\rho(K)$ is an orbit of Z under $\mathcal{O}$.*

2-c. *Jet transversality.* We recall the definition of *submanifold collection* (see [3], p. 194), which we will name here briefly by *subvariety*. Let N be a smooth manifold. A subvariety of N is a finite union $M = M_1 \cup \dots \cup M_s$ of differentiable submanifolds $M_1, \dots, M_s$ of N, satisfying

1. $M_i \cap M_j = \phi$, if $i \neq j$;
2. $\dim M_{i-1} > \dim M_i$, for $i = 2, \dots, s$;
3. $M_j \cup \dots \cup M_s$ is closed in N, for each $j \geq 1$.

The points of M_1 are the *regular points* of M and we convention that $\dim M = \dim M_1$. Let P be another differentiable manifold and let $g : P \to N$ be a differentiable map. Whenever one of the following conditions

1. $g(x) \notin M$; or
2. if $g(x) \in M_i \subset M$, then $dg_x[T_x(P)] + T_{g(x)}(M_i) = T_{g(x)}(N)$;

holds, we will say that g is *transversal* to M at $x \in P$. Here $dg_x = T_1(g)_x$.

Definition 2.5. (i) A subvariety K of F, invariant under the action of $J^2(m) \times O(n)$, is a (jet) *model singularity*. Note that, in this case, K induces a subbundle $K(M, N)$ of $\mathrm{Hom}(T_2M, TN)$ with fibre K and structural group $J^2(m) \times O(n)$. (ii) If K is a model singularity, we will say that a point $x \in M$ is a *K-jet singular point* of a map $f : M \to N$ if $\tilde{f}(x) \in K(M, N)$. (iii) If $\tilde{f}$ is transversal to $K(M, N)$ at $x \in M$, then x is said to be a *K-jet transversal singular point* of f. (iv) In the case that every $x \in M$ is a K-jet transversal point of f, we will say that f is *K-jet transversal* or *K-generic.*

Since being a jet singular point is a local property, in order to study a singular point x we may choose neighbourhoods U of x in M and V of $f(x)$ in N such that $\pi^{-1}(U \times V) \cong U \times V \times F$, where $\pi : \mathrm{Hom}(T_2M, TN) \to M \times N$ is the usual projection. It is not hard to see that $\tilde{f}$ is transversal to $K(M, N)$ at x if and only if $\tau \circ \tilde{f}$ is transversal to K at x, where $\tau : U \times V \times F \to F$ is the projection on the third factor. Then we may change $\tilde{f}$ by $\tau \circ \tilde{f}$ in order to study jet transversality at x. We have [4, Prop. 3.2].

Theorem 2.6. *Let $K \subset F$ be a model singularity. Then the set of K-jet transversal functions from M to N is dense in $C(M, N)$.*

Suppose now that the set $\mathrm{Im}(M, N)$ is nonempty. Since it is an open set of $C(M, N)$, it follows from Theorem 2.6 that the K-jet transversal immersions are dense in $\mathrm{Im}(M, N)$. If $n \geq 2m$, this set is dense also in $C(M, N)$ for, in this case, $\mathrm{Im}(M, N)$ is open and dense in $C(M, N)$.

2-d *Improvement of a result of Little.* From now on $f : M \to N$ will be always an immersion. Then the map $D_2T_2(f)_x : (T_2M)_x \to T_y(N)$ belongs to $\tilde{F}$, for each x in M, $y = f(x)$. Given an orthonormal basis $E = E_1 \cup E_2$ of $T_y(N)$, adapted to $D_2T_2(f)_x$, there exists an unique basis $X = X_1 \cup X_2$ of $(T_2M)_x$

such that the matrix of $D_2T_2(f)_x$ relative to X and E has the block form prescribed by Theorem 2.2. Let us call now by Z the family of $\nu(m,2)\times n$ matrices of the form $\begin{pmatrix} I & O \\ O & A \end{pmatrix}$. Under this new approach we may think in F as being the space of all $\nu(m,2)\times n$ matrices; $\tilde{F}$ may now be described as follows. Let $A\in F$ and let $A_1,\dots,A_m,\dots,A_n$ be the column vectors of A. Then $A\in\tilde{F}$ if and only if $A_1,\dots,A_m$ are linearly independent. Observe that the previous Z (resp. F and $\tilde{F}$) was a family of linear transformations. The results obtained with the linear transformation approach are valid for the matrix approach.

Let U, V be neighbourhoods of x, y such that the bundle Hom (T_2M, TN) is trivial over $U\times V$. Remind that for the study of jet transversality at x it is sufficient to study the transversality of the map $\tau\circ\tilde{f}$. Since f is an immersion, $\tau\circ\tilde{f}(x)\in\tilde{F}$ for all $x\in U$. Here and below we are using $\tilde{f}(x)$ and $\tau\tilde{f}(x)$ to denote the same element $D_2T_2(f)_x$ of F.

Let $B(M)\xrightarrow{\pi} M$ be the osculating frame field relative to f. The elements of $B(M)$ are the triples (x,E_1,E_2) where $x\in M$, $E_1=\{e_1,\dots,e_m\}$ is an orthonormal subset of $T_{f(x)}(N)$ tangent to $f(M)$, and $E_2=\{e_{m+1},\dots,e_n\}$ is an orthonormal subset of $T_{f(x)}(N)$ normal to $f(M)$. Given $\bar{x} = (x,E_1,E_2)\in B(M)$, we set $\mu(\bar{x}) =$ matrix of $\tilde{f}(x)$ relative to the bases X and $E=E_1\cup E_2$, where X is the basis picked up by Theorem 2.2. This defines a map $\mu: B(M)\to Z$. Let ρ be the map defined by sending a point of $\tilde{F}$ into the intersection of Z with the orbit of that point under $J^2(m)\times O(n)$. This map ρ induces a map of orbits of $\tilde{F}$ which is just the equally named 1-1 correspondence given in Proposition 2.4. Let U, V be as above and consider the bundle $B(U)\xrightarrow{\pi} U$. Since the next proposition is only stated in [9], we will prove it here.

Proposition 2.7. *For each* $x\in U$, $\mu(F_x)=\rho(\tau\tilde{f}(x))$ *where* F_x *denotes the fibre of* $B(U)$ *over* x.

Proof. If $\bar{x}=(x,E_1,E_2)$ is a given element of F_x, then $\rho(\tilde{f}(x))=\mathcal{O}(\mu(\bar{x}))$, the orbit of $\mu(\bar{x})$ under $\mathcal{O}$. In fact, given $A=L\mu(\bar{x})P^t\in\rho(\tilde{f}(x))$, then $A\in Z$ and, since $\mu(\bar{x})\in Z$, it follows from Proposition 2.3 that $(L,P)\in\mathcal{O}$, from where we get $\rho(\tilde{f}(x))\subset\mathcal{O}(\mu(\bar{x}))$. Also, by the same proposition we have $\mathcal{O}(\mu(\bar{x}))\subset\rho(\tilde{f}(x))$ which proves our affirmation. Thus, to prove this proposition it is sufficient to show that $\mu(F_x)=\mathcal{O}(\mu(\bar{x}))$. In order to do that, we will introduce the following notation: $A]^{\mathscr{B}_2}_{\mathscr{B}_1}$ denotes the matrix of a linear transformation $A\colon W_1\to W_2$ relative to the bases $\mathscr{B}_1$, $\mathscr{B}_2$ of the vector spaces W_1, W_2. If $\tilde{x}=(x,E'_1,E'_2)$ is another element of F_x, we may write

$$\mu(\bar{x})=\tilde{f}(x)]^X_E,\qquad \mu(\tilde{x})=\tilde{f}(x)]^Y_{E'},$$

where X and Y are bases of $(T_2M)_x$ given by Theorem 2.2 applied to $E=E_1\cup E_2$ and $E'=E'_1\cup E'_2$, respectively. Choose an element (L,P) of the group $J^2(m)\times O(n)$ such that $L(X_i)=Y_i$, $P(E_i)=E'_i$. Then

$$\mu(\tilde{x})=\tilde{f}(x)]^Y_{E'}=L(\tilde{f}(x)]^X_E)\,P^t=L\cdot\mu(\bar{x})\,P^t,$$

that is, $(L,P)\in\mathcal{O}$, which proves that $\mu(F_x)\subset\mathcal{O}(\mu(\bar{x}))$. On the other hand, if $L\cdot\mu(\bar{x})\cdot P^t\in\mathcal{O}(\mu(\bar{x}))$ with (L,P) in $\mathcal{O}$, we consider the bases $Y=L(X)$ and $E'=P(E)$ of $(T_2M)_x$ and $T_{f(x)}(N)$, respectively. Clearly the adapted frame $\tilde{x}=(x,E')$ is in F_x. To conclude the proof, we only need to show that $L\cdot\mu(\bar{x})\cdot P^t=\mu(\tilde{x})$: this will imply that $\mathcal{O}(\mu(\bar{x}))\subset\mu(F_x)$. Given the adapted orthonormal basis E', there exists a unique basis Y' induced from coordinates around x such that $\mu(\tilde{x})=\tilde{f}(x)]^{Y'}_{E'}\in Z$. But

$$\tilde{f}(x)]^{Y'}_{E'}=L(\tilde{f}(x)^X_E)\,P^t=L\cdot\mu(\bar{x})\cdot P^t,$$

where $L\cdot\mu(\bar{x})\cdot P^t\in Z$, since $(L,P)\in\mathcal{O}$. By the unicity in Theorem 2.2, $Y=Y'$ from where we have $\mu(\tilde{x})=L\cdot\mu(x)\cdot P^t$.

The map μ and the above proposition indicate that it is possible to express, by the method of Elie Cartan, the entries of a matrix A of Z in terms of the dual and the connection forms w_A, w_{AB}, associated to a frame $\bar{x}=(x,E)$ of $B(M)$. In fact, we will soon see that the entries of the matrix $\mu(\bar{x})$ are given by the coefficients of the second fundamental form of f at $\bar{x}$.

Let $f: M\to N$ be an immersion, where N is a Riemannian manifold with connection D. As usual, we identify $f(M)$ with M and $T_x(M)$ with the subspace $df_x(T_x(M))$ of $T_{f(x)}(N)$. The *normal space* $\nu_x(M)$ of f at x, is defined by setting $T_x(N)=T_x(M)\oplus\nu_x(M)$. The union $\nu M=\bigcup_{x\in M}\nu_x(M)$ is the *normal bundle* of the immersion f. The *second fundamental form* $B: TM\times TM\to\nu M$ may be pointwise defined by

$$B(u,v)|_x=(D_v u|_x)^{\perp},$$

where $x\in M$, u, $v\in TM$ and $\perp$ denotes the projection of $T_x(N)$ onto $\nu_x(M)$. We will give another definition of B in terms of an orthonormal frame $\bar{x}=(x,e_1,\dots,e_n)$ adapted to f. Let $\langle\,,\rangle$ denotes the Riemannian inner product of N, and let h^α_{ij}, $1\le i,j\le m$, $m+1\le\alpha\le n$, be the real functions defined on $B(M)$ by

$$h^\alpha_{ij}(\bar{x})=\langle B(e_i,e_j),e_\alpha\rangle|_x.$$

The $h^\alpha_{ij}(\bar{x})$ are the coefficients of B at $\bar{x}$. Therefore

$$B(u,v)|_x=\sum_{\alpha=m+1}^{n}\left(\sum_{i,j=1}^{m}h^\alpha_{ij}(\bar{x})\,w_i(u)\,w_j(v)\right)e_\alpha,$$

where $\{w_1, \ldots, w_n\}$ is the dual basis of $\{e_1, \ldots, e_n\}$. It can be shown that this definition is independent of the choice of the particular $\bar{x}$ in the fibre F_x.

Proposition 2.8. *Let $\bar{x} = (x, e_1, \ldots, e_n)$ be an arbitrarily choosen element of the fibre F_x of $B(M)$ over x. Denote by*

$$f(x) = \begin{pmatrix} I & O \\ O & A \end{pmatrix}$$

the image of $\bar{x}$ under μ, where I is the $m \times m$ identity matrix and A is a $m(m-1)/2 \times (n-m)$ matrix. Then the entries of A are the coefficients of the second fundamental form of f at $\bar{x}$.

Proof. It is a well known fact that it is possible to choose coordinates $(y_1, \ldots, y_n)$ around $y = f(x)$ in such a way that

$$\left.\frac{\partial}{\partial y_A}\right|_y = e_A, \qquad D_{\frac{\partial}{\partial y_B}}\left(\left.\frac{\partial}{\partial y_A}\right|_y\right) = 0, \qquad \text{for} \qquad 1 \le A, B \le n.$$

For any coordinate system $(x_1, \ldots, x_n)$ around x, we have

$$\frac{\partial^2 f_\alpha}{\partial x_i \partial x_j}(x) = \left\langle B\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), e_\alpha \right\rangle(x), \qquad m+1 \le \alpha \le n,$$

where $f_A = y_A \circ f$. In fact

$$\frac{\partial}{\partial x_i}(q) = df_q\left(\frac{\partial}{\partial x_i}(q)\right) = \sum_{A=1}^{n} \frac{\partial f_A}{\partial x_i} \frac{\partial}{\partial y_A}(f(q)),$$

for any q near x. Then, deletting the point q,

$$D_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial x_i}\right) = D_{\frac{\partial}{\partial x_j}}\left(\sum_A \frac{\partial f_A}{\partial x_i} \frac{\partial}{\partial y_A}\right) =$$

$$= \sum_A \left[\frac{\partial}{\partial x_j}\left(\frac{\partial f_A}{\partial x_i}\right)\frac{\partial}{\partial y_A} + \frac{\partial f_A}{\partial x_j} D_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial y_A}\right)\right] =$$

$$= \sum_A \frac{\partial^2 f_A}{\partial x_i \partial x_j}\frac{\partial}{\partial y_A} + \sum_A \frac{\partial f_k}{\partial x_i} D_{\sum_B \frac{\partial f_B}{\partial x_j}\frac{\partial}{\partial y_B}}\left(\frac{\partial}{\partial y_A}\right).$$

By the choice of $(y_1, \ldots, y_n)$ we have

$$D_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial x_i}\right)\bigg|_x = \sum_A \frac{\partial^2 f_A}{\partial x_i \partial x_j}(x) \cdot e_A,$$

hence

$$\left\langle B\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), e_\alpha \right\rangle(x) = \left\langle \left(D_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial x_i}\right)\right)^{\perp}, e_\alpha \right\rangle(x) =$$

$$= \left\langle \left(\sum_A \frac{\partial^2 f_A}{\partial x_i \partial x_j} e_A\right)^{\perp}, e_\alpha \right\rangle(x) = \frac{\partial^2 f_A}{\partial x_i \partial x_j}(x),$$

which proves our claim. Now, again by the choice of $(y_1, \ldots, y_n)$ and by using formula (1), we deduce that

$$\tilde{f}(x)\left(\frac{\partial^2}{\partial x_i \partial x_j}(x)\right) = D_2\, T_2(f)_x\left(\frac{\partial^2}{\partial x_i \partial x_j}(x)\right) =$$

$$= \sum_A \frac{\partial^2 f_A}{\partial x_i \partial x_j}(x)\, e_A.$$

When we choose $(x_1, \ldots, x_m)$ as the coordinate system picked out by Theorem 2.2, we get

$$f(x)\left(\frac{\partial^2}{\partial x_i \partial x_j}(x)\right) = \sum_{\alpha=m+1}^{n} \frac{\partial^2 f_\alpha}{\partial x_i \partial x_j}(x)\, e_\alpha,$$

$$e_i = f(x)\left(\frac{\partial}{\partial x_i}(x)\right) = df_x\left(\frac{\partial}{\partial x_i}(x)\right) = \frac{\partial}{\partial x_i}(x),$$

for $1 \le i, j \le m$. It follows that

$$A = \begin{bmatrix} \frac{\partial^2 f_{m+1}}{\partial x_1^2}(x) & \cdots & \frac{\partial^2 f_n}{\partial x_1^2}(x) \\ \frac{\partial^2 f_{m+1}}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f_n}{\partial x_1 \partial x_2}(x) \\ \vdots & & \vdots \\ \frac{\partial^2 f_{m+1}}{\partial x_m^2}(x) & \cdots & \frac{\partial^2 f_n}{\partial x_m^2}(x) \end{bmatrix}$$

and therefore

$$h_{ij}^{\alpha}(\bar{x}) = \langle B(e_i, e_j), e_\alpha \rangle(x) = \left\langle B\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), e_\alpha \right\rangle(x) = \frac{\partial^2 f_\alpha}{\partial x_i \partial x_j}(x),$$

which finishes the proof.

We remark that, in view of the above proposition, the set Z may be taken as the space of the second fundamental forms at a point, that is,

$$Z = \left\{ \left(\sum_{i,j=1}^{m} h_{ij}^{m+1} x_i x_j, \dots, \sum_{i,j=1}^{m} h_{ij}^{n} x_i x_j \right) \middle| (x_1, \dots, x_m) \in R^m \right\}.$$

2-e. *Geometric transversality.* Suppose that $f : M^m \to N^n$ is a given immersion of manifolds, where N is Riemannian. Let K be a subvariety of $\tilde{F}$ invariant under $J^2(m) \times O(n)$, and let $\rho(K) = K \cap Z$ be the corresponding subvariety of Z invariant under $\mathcal{O}$. A point x is said to be a *$\rho(K)$-geometrically singular point* of f if $\mu(F_x) \cap \rho(K) \neq \phi$; x is called a *$\rho(K)$-geometrically transversal singular point* if $\mu(B(M))$ meets $\rho(K)$ transversally along the entire fibre F_x; f is *$\rho(K)$-geometrically transversal* if every $\rho(K)$-singular point of f is also a $\rho(K)$-transversal point.

In the proof of the next lemma, we use $\mathcal{G}(A)$ to denote the orbit of $A \in F$ under $J^2(m) \times O(n)$.

Lemma 2.9. *The locus of K-jet singular points is equal to the locus of $\rho(K)$-geometrically singular points, for any immersion $f : M^m \to N^n$.*

Proof. If x is a K-jet singular point, them $\tilde{f}(x) \in K$ and, since K is a model singularity, $\mathcal{G}(\tilde{f}(x)) \subset K$. Using the fact that $\tilde{f}(x) \in \tilde{F}$ and Proposition 2.3, we get $\mathcal{G}(\tilde{f}(x)) \cap Z \neq \phi$. Therefore, from Proposition 2.4 it follows that

$$\mu(F_x) \cap \rho(K) = (\mathcal{G}(\tilde{f}(x)) \cap Z) \cap (K \cap Z) = \mathcal{G}(\tilde{f}(x)) \cap K \cap Z =$$
$$= \mathcal{G}(\tilde{f}(x)) \cap Z \neq \phi,$$

that is, x is $\rho(K)$-geometrically singular. On the other hand, if

$$\mu(F_x) \cap \rho(K) \neq \phi,$$

then

$$(\mathcal{G}(\tilde{f}(x) \cap Z) \cap (K \cap Z) \neq \phi,$$

which shows that $\mathcal{G}(\tilde{f}(x)) \cap K \neq \phi$. This proves the lemma.

The following theorem says that jet transversality is transformed in geometric transversality by projection of $\tilde{F}$ on Z along the orbits of $J^2(m) \times \times O(n)$. The proof is found in [9, pp. 314-315].

Theorem 2.10. *Every K-jet transversal point of an immersion f is also a $\rho(K)$-geometrically transversal point of f.*

We finally have

Theorem 2.11. *Let K be a subvariety of Z invariant under $\mathcal{O}$, where Z is the space of second fundamental forms. Then the set $\mathrm{Im}_K(M, N)$, of K-geometrically transversal immersions from M to N, is dense in $\mathrm{Im}(M, N)$.*

Proof. By using Lemma 2.9 and Theorem 2.10, it is not hard to conclude that every $\rho^{-1}(K)$-jet transversal immersion is also a K-geometrically transversal immersion. By the remarks made after Theorem 2.2, the former immersions are dense in $\mathrm{Im}(M, N)$. This proves the theorem.

3. General facts on geometric transversality.

Let $f : M^m \to N^n$ be an immersion and let $B(M)$ be the bundle of orthonormal frames adapted to f. Recall that the elements of $B(M)$ are those elements $\bar{p} = (p, e_1, \dots, e_n)$ where $p \in M$, $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_p(N)$ and $\{e_1, \dots, e_m\}$ is an orthonormal basis of $T_p(M)$. The fibre F_p of $B(M)$ over p may be identified with the subgroup $\mathcal{O} = O(m) \times O(n-m)$ of the orthogonal group $O(n)$.

We shall make use of the following convention on the range of indices

$$1 \leq A, B, C, \dots, \leq n; \qquad 1 \leq i, j, k, \dots, \leq m; \qquad m+1 \leq \alpha, \beta, \gamma, \dots, \leq n.$$

Given $\bar{p} = (p, e_1, \dots, e_n)$ in $B(M)$, we may extend the frame $(e_1, \dots, e_n)$ to a frame field on a neighbourhood V of p in N in such a way that, if $(e_A(q))$ denotes the field at $q \in M \cap V$, then $\bar{q} = (q, (e_A(q))$ remains in $B(M)$. Let w_A be the associate frame field of dual forms. The structure equations of N are given by (see [2])

$$dw_A = \sum_B w_B \wedge w_{BA}, \; w_{AB} + w_{BA} = 0,$$

$$dw_{AB} = \sum_C w_{AC} \wedge w_{CB} + \Omega_{AB}, \; \Omega_{AB} = -\tfrac{1}{2} \sum_{C,D} R_{ABCD} \, w_C \wedge w_D,$$

$$R_{ABCD} + R_{ABDC} = 0.$$

Suppose that these forms are restricted to M. Then $w_\alpha \equiv 0$ and hance

$$0 = dw_\alpha = \sum_i w_i \wedge w_{i\alpha}.$$

By Cartan's Lemma we conclude that there are real functions h_{ij}^α defined in $B(M)$ such that

$$w_{i\alpha} = \sum_j h_{ij}^\alpha w_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha.$$

The second fundamental form of f may be regarded as the vector valued quadratic form

$$B(\bar{p}) = \sum_{\alpha} \left(\sum_{i,j} h_{ij}^{\alpha}(\bar{p})\, w_i w_j \right) \cdot e_{\alpha},$$

for each $\bar{p}$ in $B(M)$. This motivates us to pick out the $n - m$ quadratic forms $u_{m-1}, \ldots, u_n$ from $\mathbb{R}^m$ to $\mathbb{R}$ given by

$$u_{\alpha}(x_1, \ldots, x_m) = \sum_{i,j} h_{ij}^{\alpha}(p)\, x_i x_j, \qquad \alpha = m+1, \ldots, n.$$

Then we have a natural map $\mu : B(M) \to Z$, $\mu(\bar{p}) = (u_{m+1}, \ldots, u_n)$, where Z is the family of all $(n-m)$-uples of quadratic forms from $\mathbb{R}^m$ to $\mathbb{R}$ of the type

$$(x_1, \ldots, x_m) \mapsto \sum_{i,j} a_{ij}\, x_i x_j, \qquad a_{ij} = a_{ji}.$$

Observe that we may identify Z with $\mathbb{R}^d$

$$d = \frac{m(m+1)(n-m)}{2}.$$

We will describe two actions of the group $\mathcal{O}$. The first, which we will call η, is over Z and the second, which we will call ζ, is over the fibres F_p. Whenever $R = (R_{ij}) \in O(m)$ and $T = (T_{\alpha\beta}) \in O(n-m)$, then (R, T) will denote the element $\begin{pmatrix} R & O \\ O & T \end{pmatrix}$ of $\mathcal{O}$. Given $(u_{\alpha}) = (u_{m+1}, \ldots, u_n) \in Z$, we wish to define $\eta((R, T), (u_{\alpha})) \in Z$. For this, let $(x_i) = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and let T^t be the transpose of T. Denote by (u'_{α}) and (x'_i) the images of (u_{α}) under T^t and (x_i) under R, respectively:

$$(u'_{m+1}, \ldots, u'_n) = T^t(u_{m+1}, \ldots, u_n),$$

$$(x'_1, \ldots, x'_n) = R(x_1, \ldots, x_n).$$

Let $\bar{u}_{\alpha}$, $\alpha = m+1, \ldots, n$, be the quadratic form defined by sending $(x_1, \ldots, x_n)$ into $u'(x'_1, \ldots, x'_m)$. Then $\eta((R, T), (u_{\alpha})) = (\bar{u}_{\alpha})$. It can be easily seen that η is a reformulation of the action of $\mathcal{O}$ defined in the previous section. The action ζ is described as follows. Given $\bar{p} = (p, e_1, \ldots, e_n) \in F_p$ and $(R, T) \in \mathcal{O}$, we take the element $\tilde{p} = (p, \varepsilon_1, \ldots, \varepsilon_n)$ of F_p such that

$$\varepsilon_i = \sum_j R_{ji} e_j, \qquad \varepsilon_{\alpha} = \sum_{\beta} T_{\beta\alpha} e_{\beta}.$$

Then $\zeta((R, T), \bar{p}) = \tilde{p}$.

Lemma 3.1. *For each point p of M, the diagram*

$$\begin{array}{ccc} \mathcal{O} \times F_p & \xrightarrow{\ \zeta\ } & F_p \\ \big\downarrow{\scriptstyle id \times \mu} & & \big\downarrow{\scriptstyle \mu} \\ \mathcal{O} \times Z & \xrightarrow{\ \eta\ } & Z \end{array}$$

is commutative, that is $u \circ \zeta = \eta \circ (id \times \mu)$.

Proof. Given $\bar{p} = (p, e_1, \ldots, e_n) \in F_p$ and $(R, T) \in \mathcal{O}$, let $\tilde{p} = (p, \varepsilon_1, \ldots, \varepsilon_n)$ be the image of $((R, T), \bar{p})$ under ζ. Write $\mu(\bar{p}) = (\bar{h}_{ij}^{\alpha})$, $\mu(\tilde{p}) = (\tilde{h}_{ij}^{\alpha})$ where, for instance, $(\bar{h}_{ij}^{\alpha})$ denotes the coeficients of the corresponding $(n-m)$-uple of quadratic forms in Z. Denote by $\bar{W}$ and $\tilde{W}$ the matrices $(\bar{w}_{AB})$ and $(\tilde{w}_{AB})$, respectively, where $\bar{w}_{AB}$ are the connection forms of N relative to the frame given by $\bar{p}$. We have the very known formula (see [2], p. 34)

$$\tilde{W} = dU \cdot U^t + U \bar{W} U^t,$$

where U is the transpose of (R, T). A consequence of this formula is that we may write $\tilde{w}_{i\alpha}$ as a function of the variables R_{ij}, $T_{\alpha\beta}$ and $\bar{w}_{i\alpha}$. Using the relations

$$\bar{h}_{ij}^{\alpha} = \bar{w}_{i\alpha}(e_j), \qquad \tilde{h}_{ij}^{\alpha} = \tilde{w}_{ij}(\varepsilon_j),$$

we may express $\tilde{h}_{ij}^{\alpha}$ as a function of the variables $\bar{h}_{ij}^{\alpha}$, R_{ij} and $T_{\alpha\beta}$. Finally, and using now the definition of η, a long but straightforward computation will show that this expression of $(\tilde{h}_{ij}^{\alpha})$ is equal to $\eta((R, T), (\bar{h}_{ij}^{\alpha}))$. This will complete the proof.

We are now in a position to make several definitions.

(i) An algebraic subvariety K of Z, invariant under η, is called a (geometric) *model singularity*. Observe that this is independent of the immersion f. Assume from now on that K is a model singularity.

(ii) A point p such that $\mu(F_p) \cap K \neq \phi$ is a *K-geometrically singular point* of f. In this case we have $\mu(F_p) \subset K$, by Lemma 3.1.

(iii) If $\mu(B(M))$ meets K transversally along the entire fibre F_p, then we say that p is a *K-geometrically transversal singular point* of f.

(iv) The immersion $f : M \to N$ is said to be *K-geometrically transversal* (or *K-generic*) if each K-singular point of f is a geometrically singular point. It can be checked that (i)-(iv) is a reformulation of the definitions given in paragraph 2-e.

We shall now derive some results concerning geometrically transversal immersions which will be useful in the proof of I-III of the Introduction. Suppose that the model singularity K is given by the zeros of the polynomials $\Phi_i(z)$, $i = 1, \dots, r$ and $z \in Z$. By this we mean that $\{\Phi_1, \dots, \Phi_r\}$ is a basis for the ideal $I(K)$ of all polynomials in Z vanishing in K. Let Φ be the map which carries $z \in Z$ into $(\Phi_1(z), \dots, \Phi_r(z)) \in \mathbb{R}^n$, and denote by ρ the maximal rank of the differentials $d\Phi_z$, when $z \in K$. Denote by $\Sigma(K)$ the locus of *singular* points of K, i.e., the points z of K where rank $(d\Phi_z)$ fails to be maximal. It is known [10] that $\Sigma(K)$ is a proper algebraic subvariety of K and that the points of K-$\Sigma(K)$ form a differentiable submanifold of Z called the *regular* points of K. We convention that $\dim(K) = \dim(K\text{-}\Sigma(K))$.

Lemma 3.2. *If p_0 is a K-geometrically transversal singular point of f, then $\mu' = u \circ \tau$ is transversal to K at p_0, for any local cross section τ of $B(M)$. Moreover, if $\{\Phi_1, \dots, \Phi_r\}$ denotes a basis for the ideal $I(K)$, then rank $(d\Phi \cdot d\mu')_{p_0}$ = rank $(d\Phi\mu'(p_0))$, where $\Phi = (\Phi_1, \dots, \Phi_r)$.*

The proof is given in detail in [1, Lemma 4].

Let us denote by k the locus of K-singular points of a K-generic immersion. Then we have

Theorem 3.3. *The set $Im_K(M, N)$ is dense in $Im(M, N)$. Moreover, if $\mu(B(M)) \cap \Sigma(K) = \phi$ for some K-generic immersion, then or $k = \phi$ or k is a closed differentiable submanifold of M, with codim k = codim K.*

Proof. The first half of the statement is Theorem 2.11 of paragraph 2-e. For the second part, let $U \subset M$ be a coordinate neighbourhood of a point p of k, and let τ be a local cross section of $B(M)$ defined on U. Since $\mu(B(M)) \cap \Sigma(K) = \phi$, the map $\mu' = \mu \circ \tau : U \to Z$ is transversal to the smooth submanifold K-$\Sigma(K)$ of Z. Then $k \cap U = (\mu \circ \tau)^{-1}(K\text{-}\Sigma(K))$ is a smooth submanifold of U (actually of M) whose codimension is the same as that of K in Z. Since this holds for any point p of k, we conclude that k is a smooth submanifold of M with codim k = codim K. Now let (p_n) be a sequence of points of k such that (p_n) converges to a point p in M. Let U be a neighbourhood of p small enough to be possible to define a cross section $\tau : U \to B(M)$. The subsequence of (p_n) which lies in U we also call (p_n). Then $(\mu'(p_n))$ lies in K and $\mu'(p_n) \to \mu'(p)$, whence $\mu(\tau(p))$ belongs to K. Therefore p belongs to k and this proves the theorem.

Remarks. 1. If $\mu(B(M)) \cap \Sigma(K) = \phi$ holds for any K-generic immersion f, or else, if K is a differentiable submanifold of Z, then $Im_K(M, N)$ is open (and dense, of course) in $Im(M, N)$.

2. Without the condition $\mu(B(M)) \cap \Sigma(K) = \phi$, we must change "differentiable manifold" by "submanifold collection" (see paragraph 2-c) in the second part of Theorem 3.3.

3. In the case where M and N are orientable, we may consider the bundle $B^+(M)$ instead of $B(M)$; the elements of $B^+(M)$ are those elements $\bar{p} = (p, e_1, \dots, e_n)$ of $B(M)$ such that $\{e_1, \dots, e_m\}$ is an oriented basis of $T_p(M)$, and $\{e_1, \dots, e_n\}$ is an oriented basis of $T_p(N)$. The actions η and ζ, as well as the definitions (i)-(iv) above, are easily given with this new bundle. It is not hard to see that $k^+ = k$ and that $Im_K(M, N) \subset Im_K(M, N)^+$, where K is a model singularity invariant under $\mathcal{O}$ (and hence a model singularity invariant under $\mathcal{O}^+ = O^+(m) \times O^+(n-m)$, as can be easily seen). It follows that $Im_K(M, N)^+$ is also dense in $Im(M, N)$. Since the algebraic varieties considered in this paper are invariant under $\mathcal{O}$, we will not make any distinction between $Im_K(M, N)^+$ and $Im_K(M, N)$, in the hope that the context will make clear which bundle and which set of immersions we have in mind.

We return now to statement I of the Introduction. The *mean curvature vector* $\mathscr{H}(p)$ of an immersion $f : M \to N$ at p is $1/m$ times the trace of the second fundamental B form of f at p. Relative to a frame $\bar{p} = (p, e_1, \dots, e_n)$, one has

$$\mathscr{H}(p) = \frac{1}{m} \sum_i B(e_i, e_i)(p) = \frac{1}{m} \sum_\alpha \left(\sum_i h_{ii}^\alpha(\bar{p}) \right) e_\alpha.$$

Then in order to study the points where $\mathscr{H}$ vanishes, we must look for the points p of M such that

$$\sum_i h_{ii}^\alpha(\bar{p}) = 0, \qquad m+1 \le \alpha \le n,$$

for all $\bar{p} \in F_p$. This indicates that we must consider the subspace H of Z given by

$$H = \left\{ (a_{11}^{m+1}, \dots, a_{1m}^{m+1}, \dots, a_{mm}^n) \in Z \text{ such that } \sum_i a_{ii}^\alpha = 0 \text{ for } m+1 \le \alpha \le n \right\}.$$

It is not hard to see that H is a vector subspace of Z whose dimension is $d - n + m$. A long but easy calculation shows that H is invariant under $\mathcal{O}$, that is, H is a model singularity. By Theorem 3.3 and Remark 1 above, $Im_H(M, N)$ is open and dense in $Im(M, N)$. Moreover $\mathscr{H}(p) = 0$ if and only if $\mu(F_p) \subset H$, so the locus h of H-geometrically singular points coincides with the set of points of M where $\mathscr{H}$ vanishes. By the second part of Theorem 3.3, codim h = codim $H = n - m$, for H-generic immersions. In summary, we have

Theorem 3.4. *There exists an open and dense subset of $Im(M, N)$, namely $Im_H(M, N)$, with the following property: for each f in $Im_H(M, N)$, the locus h, of singular points of the mean curvature vector $\mathscr{H}$ of f, is either* *empty or is a closed submanifold of M with* $\operatorname{codim} h = n - m$. *Then* $h = \phi$ *if* $n > 2m$, *and* h *is an isolated set of points of* M *if* $n = 2m$.

4. Parabolic and umbilic points of immersions $M^m \to N^{m-1}$.

Throughout this section, $\dim N = \dim M + 1$. The results and techniques presented in sections 2 and 3 are used here to simplify the proofs of some results of Feldman [5] concerning parabolic and umbilic points of hypersurfaces of R^{m+1}. The proofs we present here hold when we replace the $(m+1)$-Euclidean space by any Riemannian manifold N^{m+1}, so ours statements are done in this more general situation. The fundamental assumption throughout this section is that the set $Im(M^m, N^{m+1})$ is nonempty: this is unnecessary when $m = 2$.

4-a. *Definitions and notations.* Denote by D, $\langle\,,\rangle$, respectively, the Riemannian connection and metric of N. Given $p \in M$ and $v \in \nu_p(M)$ with $|v| = 1$, we may construct a vector field V in N in such a way that $V(p) = v$ and $V(q)$ belongs to $\nu_q(M)$ for all point q of M near p. The map $A_p^v : T_p(M) \to T_p(M)$ given by

$$A_p^v(x) = -(D_X V)^T$$

is a well defined linear symmetric transformation. Here $(\ )^T$ denotes orthogonal projection of $T_p(N)$ onto $T_p(M)$. Then there exists an orthonormal basis $\{E_1, \dots, E_m\}$ of $T_p(M)$ of eingenvectors of A_p^v with respective eigenvalues, say, λ_i. If we pick $-v$ instead of v, one has $A_p^{-v}(E_i) = -\lambda_i E_i$, so the eigenvectors and the eigenvalues of A_p^{-v} are, respectively, E_i and $-\lambda_i$.

We say that p is a *parabolic* point of the immersion f if $\det(A_p^v) = 0$. Since $\det(A_p^{-v}) = (-\lambda_1)\cdot\ldots\cdot(-\lambda_m) = (-1)^m \det(A_p^v)$, to be a parabolic point is independent of the choice of the direction of v. We say that p is an *umbilic* point of f if $\lambda_1 = \ldots = \lambda_m = \lambda$. If p is simultaneously parabolic and umbilic, we then say that p is *planar*.

Now we characterize parabolic, umbilic and planar points in terms of an adapted frame $\bar{p} \in B(M)$. Let $e_1, \dots, e_m, e_n$, $n = m+1$, be a local frame field on N such that, when restricted to M, the vectors $e_1, \dots, e_m$ are tangent to M and e_n is normal to M. Let w_A be the corresponding fiel of dual forms, and let w_{AB} be the induced connection forms. Writing $v = e_n$, one has

$$A_p^v(X) = -(D_X(e_n)(p))^T = -\sum_{i=1}^m \langle D_X(e_n), e_i\rangle\, e_i(p) =$$

$$= \sum_{i=1}^m \langle e_n, D_X(e_i)\rangle\, e_i(p) = \sum_{i=1}^m w_{in}(X)\, e_i(p)$$

for each X in $T_p(M)$. On M we have

$$w_{in} = \sum_j h_{ij}^n w_j, \qquad h_{ij}^n = h_{ji}^n,$$

so

$$h_{ij}^n(\bar{p}) = w_{in}(e_j) = \langle A_p^v(e_j), e_i\rangle.$$

Then the matrix of A_p^v relative to the basis $\{e_1, \dots, e_n\}$ is $(h_{ij}^n)_{1 \le i, j \le m}$. The following lemma arises from the above discussion.

Lemma 4.1. *For any immersion* $f : M \to N$, *we have*

(i) *p is a planar point of f if and only if* $(h_{ij}^n(\bar{p})) = 0$, *for all* $\bar{p}$ F_p;

(ii) *p is a parabolic point if and only if* $\det(h_{ij}^n(\bar{p})) = 0$, *for all* $\bar{p}$ F_p;

(iii) *p is an umbilic point of f if and only if, for each $\bar{p}$ in F_p, there exists a real number $\lambda = \lambda(e_n)$ such that* $h_{ij}^n(\bar{p}) = \delta_{ij}\lambda$.

It is sufficient to check the above conditions for one element $\bar{p}$ of F_p.

4-b *Planar and parabolic points.* Since we are considering codimension one immersions, Z may be identified with $\mathbb{R}^d$, $d = m(m+1)/2$. The subspace $K = \{0\}$ is, of course, a model singularity and, by Lemma 4.1-(i), it is clear that p is a planar point of an immersion $f : M^m \to N^{m-1}$ if and only if $\mu(F_p) \subset K$. According to Remark 1) of Section 3, the set $Im_K(M, N)$ is open and dense in $Im(M, N)$. Denote by $Im_{\hat{K}}(M, N)$ the family of immersions from M to N without planar points. If f is such an immersion, then $\mu(F_p) \cap K = \phi$, so f is also K-geometrically transversal. On the other hand, let f be in $Im_K(M, N)$ and suppose that k, the locus of K-singular points of f, is nonempty. Then $\operatorname{codim} k = \operatorname{codim} K = m(m+1)/2 > m$, which is impossible. Then $k = \phi$ and $f \in Im_{\hat{K}}(M, N)$. In summary, we have

Theorem 4.2. *The set of immersions $f : M^m \to N^{m+1}$ such that f has no planar points is open and dense in $Im(M, N)$.*

Consider now the algebraic subvariety P of Z given by

$$P = \{(a_{11}, \dots, a_{1m}, a_{22}, \dots, a_{2m}, \dots, a_{mm}) \in Z \text{ such that } \det(a_{ij}) = 0\}$$

and note that p is a parabolic point of M if and only if $\mu(\bar{p}) \in P$ for all $\bar{p}$ in F_p. In [5, pp. 9-11], Feldman studies the case $N = R^{m+1}$ and shows that P is a model singularity and that $Im_p(M, R^{m+1})$ is dense in $Im(M, R^{m+1})$. By using the notation of Section 2, it is not hard to carry on these facts for general N.

In the case $m = 2$, P has a very simple description, namely P is given by the zeros of the polynomial $\varphi(x, y, z) = xz - y^2$ defined on $Z \cong R^3$.

It can be directely shown that P is a model singularity. It is also clear that the locus $\mathscr{P}(f)$, of parabolic points of an immersion $f: M^2 \to N^3$, coincides with the locus of P-singular points of f. The result below is just statement II of the Introduction.

Theorem 4.3. *Let M and N be a differentiable 2-dimensional manifold and a Riemannian 3-dimensional manifold, respectively. The set $Im_p(M, N)$ is open and dense in $Im(M, N)$. For each f in $Im_p(M, N)$, the set $\mathscr{P}(f)$ is either empty or a closed one dimensional submanifold of M; if M is compact, then $\mathscr{P}(f)$ is a finite collection of disjoint circles.*

Proof. Note that $\Sigma(P) = \{0\}$ is a model singularity and that P-genericity implies $\Sigma(P)$-genericity. Since

$$3 = \operatorname{codim} \Sigma(P) > 2 = \dim M,$$

we have $\mu(B(M)) \cap \Sigma(P) = \phi$ for any P-generic immersion. By the first part of Theorem 3.3 and Remark 1), $Im_p(M, N)$ is open and dense in $Im(M, N)$. By the second part of Theorem 3.3, $\mathscr{P}(f)$ is a closed submanifold of M with codim $\mathscr{P}(f) = \operatorname{codim} P = 1$, for any P-generic immersion f.

4-c. *Umbilic points.* If p is a umbilic point, then Lemma 4.1 says that

$$h^n_{11}(\bar{p}) = \ldots = h^n_{mm}(\bar{p}) = \lambda,$$

$$h^n_{ij}(\bar{p}) = 0, \qquad \text{if} \qquad i \neq j,$$

for each $\bar{p}$ in F_p, where λ depends only on the choice of e_n. This indicates that, in order to study umbilic points of an immersion f, we must consider the 1-dimensional vector subspace U given by those elements of Z which satisfies $a_{11} = \ldots = a_{mm}$, and $a_{ij} = 0$ for $i \neq j$. A straightforward calculation will show that U is a geometric model singularity. Since U is also a differentiable submanifold of Z, the umbilic generic immersions $Im_U(M, N)$ are open and dense in $Im(M, N)$. It is also clear that the locus of U-singular points of any immersion $f: M \to N$ coincides with the locus $\mathscr{U}(f)$ of umbilic points of f. When $m \geq 3$, $\mathscr{U}(f)$ must be empty, otherwise we have

$$m < m(m + 1)/2 - 1 = \operatorname{codim} U = \operatorname{codim} \mathscr{U}(f),$$

which is impossible. So we have

Theorem 4.4. *If $m \geq 3$, the set of immersions $f: M^m \to N^{m+1}$ such that f has no umbilic points is open and dense in $Im(M, N)$.*

From now on we will restrict our study to the case $m = 2$. If f is an U-generic immersion, then codim $U = \dim M$, so $\mathscr{U}(f)$ consists of isolated points. Also we are able to give an estimative for the cardinality $\# \mathscr{U}(f)$, when M is compact. In order to do this we need some facts on r-fields and their properties. We follow [9, pp. 275-278] for this discussion.

A *r-cross* is a set of r unity vectors in the Euclidean plane such that their tips form a regular polygon. Note that C_r, the family of all r-crosses in the plane, is homeomorphic to S^1. Define the associate bundle $C_r(M) \xrightarrow{\pi} M$ of r-crosses over M as follows: $C_r(M) = \{(p, C_r(p)),$ where $p \in M$ and $C_r(p)$ is a r-cross in $T_p(M)\}$. A *r-cross field* is then defined as a cross section of this bundle, that is, a map $\sigma: M \to C_r(M)$ such that $\pi\sigma =$ identity of M. A point $p \in M$ where σ is not defined is called a *singularity* of the cross field. The index of an isolated singularity is defined on a completely analogous way to that of a vector field. As one may expect, r-cross fields have some of the properties of the vector fields. The proof of the lemma below is found in [9, p. 277].

Theorem 4.5. *Suppose that σ is a r-cross field over a compact orientable manifold M with a finite number $p_1, \ldots, p_s$ of singularities. Then*

$$\chi(M) = \sum_{i=1}^{s} \operatorname{Index}(\sigma, p_i).$$

Finally we present a criterion for the computation of the index of a singular point p_0 of a r-cross field σ. Let D be the disk about p_0 over which $TM \cong D \times R^2$, $C_r(M) \cong D \times C_r$, and such that D contains no other singular points of σ. Choose a tangent frame field $e_1 e_2$ on D and let ε_1 be one leg of σ. Define $\arg\sigma = \sphericalangle(e_1, \varepsilon_1)$ and note that $\arg\sigma$ is well defined modulo $2\pi/r$, up to the choice of the frame.

Lemma 4.6. *Let $\psi: D \to R^2$ be a differentiable map whose Jacobian is not singular at p_0, and assume that there exists a rational number q such that $q \cdot \arg\psi = \arg\sigma$ (modulo $2\pi/r$). Then Index $(\sigma, p_0) = \pm q$, where the sign agrees with the sign of det [Jac $\psi(p_0)$].*

For the proof see [1, Lemma 7].

We return now to the umbilic points of U-generic immersions $f: M^2 \to N^3$. Let $W \subset N$ be an orientable neighborhood of an isolated umbilic point p_0 such that $V = W \cap M$ is also orientable. Choose a local tangent frame field $e_1 e_2$ on V and complete this field with a unit normal vector field v pointwise chosen. Fix on V and on W the orientations given by $e_1 e_2$ and $e_1 e_2 v$, respectively. Let $\tau: V \to B(M)$ be the cross section of $B(M)$ which gives the last frame. For each p in $V - p_0$ we may select an orthonormal basis of $T_p(M)$, namely the basis given by the eigenvectors E_1, E_2 of A^v_p. Let λ_1, λ_2 be the associate eigenvalues and notice that $\lambda_1 \neq \lambda_2$ on $V - p_0$. But $A^v_p(-E_i) = \lambda_i(-E_i)$, $i = 1, 2$, so what we really have are four orthonormal bases of eigenvectors with associate eigenvalues λ_1, λ_2. Although the eigenvectors of A^v_p give not an unique choice of an orthonormal basis for $T_p(M)$, they arise a 4-cross E_1, E_2, $-E_1$, $-E_2$ in $T_p(M)$, for each p in $V - p_0$. In summary, we have a 4-cross field $\sigma : V \to C_4(M)$ with a singular point at p_0. We define the index of p_0 as an umbilic point of f as being the index of σ at p_0.

Theorem 4.7. *The set $Im_U(M, N)$, of umbilic generic immersions from M^2 to N^3, is open and dense in $Im(M, N)$. Moreover the index of any umbilic point of f in $Im_U(M, N)$ is $\pm 1/2$.*

Proof. Since $\Sigma(U) = \phi$, $Im_U(M, N)$ is open and dense in $Im(M, N)$. Let p_0 be an umbilic point of f and let V and defined as above. For each p in $V - p_0$ we choose a leg E_1 of the 4-field σ defined on V by the principal directions of A_p^ν. Then $\theta = \sphericalangle(e_1, E_1)$ is the argument of σ, which is defined modulo $\pi/2$. Let E_2 be another leg of σ such that $E_1 \perp E_2$. Then $E_1 = \cos\theta e_1 + \operatorname{sen}\theta e_2$ and $E_2 = -\operatorname{sen}\theta e_1 + \cos\theta e_2$ or $E_2 = \operatorname{sen}\theta e_1 - \cos\theta e_2$. In any case

$$h^3_{11} - h^3_{22} = (\lambda_1 - \lambda_2)\cos 2\theta,$$

$$2h^3_{12} = (\lambda_1 - \lambda_2)\operatorname{sen} 2\theta,$$

where λ_1, λ_2 are real functions on V such that $A_p^\nu(e_i) = \lambda_i(p)\, E_i$, for $i = 1, 2$. Thus on $V - p_0$ we have

$$\operatorname{tg} 2\theta = \frac{2h^3_{12}}{h^3_{11} - h^3_{22}},$$

and this equation determines θ, modulo $\pi/2$. This indicates that we must consider the map $\psi : V \to R^2$ given by

$$\psi(u, v) = (h^3_{11} - h^3_{22}, 2h^3_{12})(\tau(u, v)),$$

because for such ψ one has $\operatorname{tg}(\arg\psi) = \operatorname{tg} 2\theta$, which implies that

$$\tfrac{1}{2}\arg\psi = \arg\sigma (\text{modulo } \tfrac{\pi}{2})$$

on $V - p_0$. In order to apply Lemma 4.7 and finish the proof, we only need to show that $\det[\mathrm{Jac}(\psi(p_0))] \neq 0$. For this we observe that the model singularity U is given by the zeros of the polynomials

$$\varphi_1(z) = h^3_{11} - h^3_{22}, \qquad \varphi_2(z) = h^3_{12},$$

where $z = (h^3_{11}, h^3_{12}, h^3_{22}) \in Z$. From Lemma 3.2 it follows that the map

$$\varphi(u, v) = (h^3_{11} - h^3_{22}, h^3_{12})(\tau(u, v))$$

has non singular Jacobian at p_0. But

$$\det[\mathrm{Jac}(\psi(p_0))] = \left(\frac{\partial h^3_{11}}{\partial u} - \frac{\partial h^3_{22}}{\partial u}\right)\left(2\,\frac{\partial h^3_{12}}{\partial v}\right) - \left(\frac{\partial h^3_{11}}{\partial v} - \frac{\partial h^3_{22}}{\partial v}\right)\left(2\,\frac{\partial h^3_{12}}{\partial u}\right)_{p_0} = 2\det[\mathrm{Jac}(\varphi(p_0))],$$

which finishes the proof.

Assume now that M^2 and N^3 are orientable and M^2 is compact. Then $\mathscr{U}(f)$ is a finite set of points $\{p_1, \dots, p_s\}$ in M, for each U-generic f. Since $\chi(M)$ is an integer, we apply Theorem 4.7 and Lemma 4.6 to conclude that

$$|\chi(M)| \leq \sum_{i=1}^{s} |\mathrm{Index}(\sigma, p_i)| = \frac{\#\mathscr{U}(f)}{2}.$$

If M is not orientable, we may consider the orientable double covering $\tilde{M} \xrightarrow{\pi} M$ of M. The immersion $\tilde{f} = f\pi : \tilde{M} \xrightarrow{\pi} N$ is easily seeing to have the following properties:

1. $B(\tilde{M}) = B(M)$,
2. given $p \in M$ and $\tilde{p} \in \tilde{M}$ such that $\pi(\tilde{p}) = p$, then $p \in \mathscr{U}(f)$ if and only if $\tilde{p} \in \mathscr{U}(\tilde{f})$.

Thus $\tilde{f}$ is also U-generic and, since $\#[\pi^{-1}(p)] = 2$,

$$\#\mathscr{U}(f) = \tfrac{1}{2}\#\mathscr{U}(\tilde{f}) \geq |\chi(\tilde{M})| = 2|\chi(M)|.$$

We summarize this discussion as follows.

Corollary 4.8. *Let M^2 be a compact differentiable manifold and N^3 be an orientable Riemannian manifold. Then there exists an open and dense subset of $Im(M, N)$, namely $Im_U(M, N)$, such that*

$$2|\chi(M)| \leq \#\mathscr{U}(f) < \infty,$$

for each f in $Im_U(M, N)$. Moreover, if M is orientable, then $\#\mathscr{U}(f)$ is even.

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