# Periodic solutions of a nonlinear propagation equation via a fixed point argument

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**Abstract.**

In this work the initial value problem for the equation

$$u_t + \beta u_x + \gamma f(u)_x - \delta u_{xxt} = g, \qquad \forall\, x \in R, \qquad \forall\, t \in [0, T),$$

with periodic boundary conditions is interpreted in the sense of periodic distributions and studied via fixed point arguments. Weak solutions exist if $f \in C^0(R)$ and $g \in L^\infty(L^2(0, 1))$. Moreover, regularity in $f$, $g$ and the initial data implies regularity of solutions.

## 1. Introduction.

Let $\beta$ and $\gamma$ be real numbers and $\delta$ and $T$ be positive real constants (under proper conditions $T$ may be $+\infty$). We shall consider the following problem:

Given real functions $f$ and $u_0$ defined on $R$ and a real function $g$ defined on $R \times [0, T)$, find a real function $u$ defined on $R \times [0, T)$ such that

$$(1.1) \qquad u_t + \beta u_x + \gamma f(u)_x - \delta u_{xxt} = g, \qquad \forall\, x \in R, \qquad \forall\, t \in [0, T),$$

$$(1.2) \qquad u(x, 0) = u_0(x), \qquad \forall\, x \in R,$$

and

$$(1.3) \qquad u(x + 1, t) = u(x, t), \qquad \forall\, t \in [0, T).$$

The above Equation (1.1), is a generalization of

$$(1.1)' \qquad u_t + \beta u_x + \gamma u u_x - \delta u_{xxt} = 0,$$

which was proposed by T. B. Benjamin, J. L. Bona and J. J. Mahony in [1], as an alternative for the $KdV$ equation, to model the propagation of long waves in nonlinear dispersive media.

The homogeneous case of problem (1.1')-(1.3) was studied by L. A. Medeiros and G. P. Menzala in [2]. There they showed existence and

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uniqueness of classical solutions when the third derivative of $u_0$ is square integrable. Also M. M. Miranda studied this problem in [5] where theorems concerning the existence of weak periodic solutions are presented. Numerical algorithms for computing solutions of this problem were analysed by M. A. Raupp [9].

The generalized problem (1.1)-(1.3) was first studied by B. P. Neves [7], [8]. In [7] the author obtained theorems asserting existence of classical and weak solution but only for $\beta = 0$, $\gamma < 0$, $g = 0$, and $f$ two times differentiable with positive first derivative. These conditions are clearly not satisfied in equation (1.1)', where $f(s) = s^2/2$. In [8] the restriction that $f' \geq 0$ is relaxed for infinitely differentiable $f$. In [4] L. A. Medeiros and M. M. Miranda proved the existence of weak solutions of (1.1)-(1.2) with the more stringent boundary condition

$$(1.3)' \qquad u(0, t) = u(1, t) = 0, \qquad \forall\, t \in [0, T).$$

which are defined on $[0,1] \times [0, T)$. A numerical analysis of problem (1.1)-(1.3), by C. A. de Moura et al., can be found in [6]. The non-periodic problem (1.1)-(1.2) was studied by Fixed Point techniques by L. A. Medeiros and G. P. Menzala [3].

The aim of this work is to study the existence, uniqueness and regularity of solutions of the periodic initial value problem (1.1)-(1.3). The approach that will be taken is quite different from the above mentioned works on this and related problems. The existence theory results from an application of the Schauder fixedpoint theorem. As a consequence, solutions which are global in time are obtained directly, rather than by iteration. Moreover, existence of weak solutions is warranted provided the nonlinearity $f$ is continuous. Uniqueness of solutions, corresponding to a given initial data $u_0$ and $g$, is established when $f$ is a locally Lipschitz function.

In the following we interpret problem (1.1)-(1.3) in the sense of periodic distributions theory. Making the nonlinear term independent, we decompose equation (1.1) and solve the associated linear equation by standard Galerkin procedures (the Fourier series method is not appropriate for this linear equation). Then we obtain a fixed point by mean of Schauder's Fixed Point Theorem. A "bootstrap" argument is again used to study regularity.

It is shown that weak solutions exist when $f$ is continuous and are unique if, in addition, $f$ is Lipschitz. Moreover, the solution is as regular as $u_0$ and a bit more regular than $f$ and $g$.

In section 2 the framework and the weak form of problem (1.1)-(1.3) that will be used are introduced and it is shown that $f$ can be replaced by a slightly different but more convenient function. In section 3 existence and regularity of solutions of the associated linear equation are analyzed. Finally, in section 4, a fixed point argument is used to lift the results to the nonlinear equation.



## 2. The weak formulation.

Before presenting the weak form of problem (1.1)-(1.3), we need to define some notation and standard results that will be useful later. The following is a list of spaces for further reference:

$P_0 = H^0(0, 1) = L^2(0, 1)$, the space of square integrable functions on (0,1);

$H^k(0, 1)$, the space of functions in $P_0$ with (generalized) derivatives up to order $k$ belonging to $P_0$;

$P_k$, the closed subspace of $H^k(0, 1)$ whose functions $v$ are such that $D^i_x v(0) = D^i_x v(1)$ for $i = 0, \dots, k-1$;

$C^k(I)$, the space of continuous functions whose derivatives up to order $k$ are continuous on $I$, $I \subset R$;

$C^\infty(I)$, the space of infinitely differentiable functions on $I$.

All these spaces are endowed with their usual topologies. For subspaces, the topology of the larger space is induced. We denote by $D_x$ (resp. $D_t$) the derivative with respect to $x$ (resp. $t$). If $i$ is a non-negative integer, $D^i_x$ and $D^i_t$ are the $i$-th powers of $D_x$ and $D_t$. In the sequel $V$ will stand for a Banach space. Its dual space will be denoted by $V'$ and the duality pairing will be denoted by $\langle \cdot , \cdot \rangle$. The innerproduct and norm of $H^k(0, 1)$ (and $P_k$) are denoted respectively by

$$(2.1) \qquad (f, g)_k = \sum_{i=0}^{k} (D^i_x f, D^i_x g)$$

and

$$(2.2) \qquad |f|^2_k = \sum_{i=0}^{k} (D^i_x f, D^i_x f),$$

except for $k = 0$ when $(\cdot, \cdot) = (\cdot, \cdot)_0$.

We need also to consider the space of periodic test functions

$$\mathscr{D}_p(0, 1) = \{\varphi \in C^\infty([0, 1]) \mid D^i_x \varphi(0) = D^i_x \varphi(1), \ \forall\, i\},$$

and its dual $\mathscr{D}'_p(0, 1)$, the space of periodic distributions. A sequence $\{\varphi_n\}$ converges to $\varphi$ in $\mathscr{D}_p(0, 1)$ if and only if $D^i_x \varphi_n \to D^i_x \varphi$ uniformly for all $i \geq 0$ (note that $D^0_x \varphi = \varphi$). Its dual $\mathscr{D}'_p(0, 1)$ is endowed with the weak* topology. More information on these spaces can be found in [10, Chap. IV].

The spaces above are spaces whose elements are functions of a single variable (which will be the spatial variable $x$). To take account of the variable $t$ we need some spaces of strongly measurable functions from $[0, T)$ into $V$, a Banach space. The space $L^1(V)$ of integrable functions is normed by

(2.3) $$|u|_{L^1(V)} = \int_0^T |u(t)|_V \, dt.$$

Its subspace $L^\infty(V)$, of essentially bounded functions, is normed by

$$|u|_{L^\infty(V)} = \operatorname*{ess\,sup}_{t\in[0,T)} |u(t)|_V.$$

The subspaces of $L^\infty(P_0)$ defined by

$$K(m,k) = \{u \in L^\infty(P_k) \mid D_t^i u \in L^\infty(P_k)\ i = 1, \dots, m\},$$

are endowed with the product norms

$$|u|_{m,k} = \max_{0 \le i \le m} |D_t^i u|_{L^\infty(P_k)}$$

for all non-negative integers $m$ and $k$.

The following properties of these spaces will be used in the sequel. As $P_k$ is a closed subspace of $H^k(0,1)$ it follows that $P_{k+1}$ is compactly imbedded in $P_k$. Since $\mathscr{D}_p(0,1)$ contains the trigonometric basis and $P_k$ is a subspace of $C^{k-1}[0,1]$, $\mathscr{D}_p(0,1)$ is densely imbedded in all $P_k$. This last statement is a consequence of Fourier series convergence theorems [12]. Finally, by the compactness criterion of Lions-Aubin [11, Chap. *I*], the space $K(1,1)$ is compactly imbedded in $K(0,0) = L^\infty(P_0)$.

Next we point out in what sense the weak solutions of problem (1.1)-(1.3) are to be considered. Since $\mathscr{D}_p(0,1)$ is dense in $P_k$ and $P_0$, we identify $P_0$ with its dual and consider all the spaces $P_k$ and their duals as subspaces of $\mathscr{D}'_p(0,1)$. Then the *weak problem* is to find a function $u \in K(1,1)$ such that for almost every $t \in [0,T)$

(2.5) $$\langle u_t(t), \varphi\rangle - \beta\langle u(t), \varphi_x\rangle - \gamma\langle f(u(t)), \varphi_x\rangle + \delta\langle u_{xt}(t), \varphi_x\rangle = \langle g(t), \varphi\rangle, \qquad \forall \varphi \in \mathscr{D}_p(0,1),$$

(2.6) $$u(0) = u_0.$$

Since solutions of the weak problem should be in $K(1,1)$ (conditions under which such a solution exists will be made precise in section 4), equation (2.6) makes sense. Moreover, due to the density of $\mathscr{D}_p(0,1)$ as a subspace of $P_1$, (2.5) can be modified to read as follows

(2.5)′ $$\langle u_t(t), v\rangle - \beta\langle u(t), v_x\rangle - \gamma\langle f(u(t)), v_x\rangle + \delta\langle u_{xt}(t), v_x\rangle = \langle g(t), v\rangle, \qquad \forall v \in P_1, \qquad t \in [0,T) \text{ ae.}$$



From now on, and without any loss of generality, we shall take $\beta = \gamma = \delta = 1$.

The following lemmas show that under reasonable conditions on $f$ and $g$, $f$ can be replaced by another function $f^*$, as regular as $f$, which is bounded and constant outside a bounded interval. This will be necessary for some arguments of section 4.

**Lemma 2.1.** *If* $g \in L^1(P_0)$ *and* $f \in C^0(R)$ *then any solution of the weak problem is uniformly bounded. That is, there is a constant C depending only on g and* $u_0$ *such that*

(2.7) $$\operatorname*{ess\,sup}_{t\in[0,T)} \sup_{x\in[0,1]} |u(x,t)| \le C.$$

*Proof.* Choose $v = u(t)$ in (2.5)′, then it follows that

(2.8) $$(u_t(t), u(t)) + (u_{xt}(t), u_x(t)) = (u_x(t), u(t)) + (f(u(t)), u_x(t)) + (g(t), u(t))$$

almost everywhere in $t$. But if $v \in P_1$,

(2.9) $$(v, v_x) = \int_0^1 v\, v_x \, dx = \int_0^1 \left(\frac{v^2}{2}\right)_x dx = v^2|_0^1 = 0.$$

Generalizing this, if $F$ is any primitive of $f$, it follows that

(2.10) $$(f(v), v_x) = \int_0^1 f(v)\, v_x \, dx = \int_{v(0)}^{v(x)} D_x F(s)\, ds = F(v(1)) - F(v(0)) = 0,$$

for all $v \in P_1$, since $v(0) = v(1)$. Thus, for almost every $t \in [0,T)$,

$$(u(t), u_x(t)) = (f(u(t)), u_x(t)) = 0,$$

and so

$$\tfrac{1}{2} D_t(|u(t)|_0^2 + |u_x(t)|_0^2) = (g(t), u(t)) \le |g(t)|_0\, |u(t)|_0.$$

Integrating the above equation we obtain

$$|u(t)|_1^2 \le |u_0|_1^2 + 2\int_0^t |g(\tau)|_0\, |u(\tau)|_0 \, d\tau,$$

and from this it follows that

$$\operatorname*{ess\,sup}_{t\in[0,T)} |u(t)|_1^2 \le |u_0|_1^2 + 2\int_0^t |g(\tau)|_0\, |u(\tau)|_0 \, d\tau \le$$

$$\leq |u_0|_1^2 + 2 \operatorname*{ess\,sup}_{t\in[0,T)} |u(t)|_0 \int_0^t |g(\tau)|_0 \, d\tau.$$

Therefore, for any $\varepsilon < 1$,

$$(2.11) \qquad (1-\varepsilon) \operatorname{ess\,sup} |u(t)|_1^2 \leq |u_0|_1^2 + \tfrac{1}{\varepsilon} |g|^2_{L^1(P_0)},$$

(2.11) implies (2.7), as $\sup_x |v(x)| \leq c\,|v|_1$, if $v \in P_1$.

**Lemma 2.2.** *Let $f \in C^0(R)$ be Lipschitz on any bounded interval. Then there exists at most one weak solution satisfying (2.7).*

*Proof.* Let $u_1$, $u_2$ be two solutions of the weak problem. Then $w = u_1 - u_2 \in K(1, 1)$ satisfies

$$(2.12) \qquad (w_t(t), v) - (w(t), v_x) - (f(u_1(t)) - f(u_2(t)), v_x) + $$

$$+ (w_{xt}, v_x) = 0, \qquad \forall\, v \in P_1,$$

for almost every $t$. For $v = w(t)$, equation (2.12) becomes

$$\tfrac{1}{2} D_t(|w(t)|_0^2 + |w_x(t)|_0^2) = (w(t), w_x(t)) + (f(u_1(t)) - f(u_2(t)), w_x(t)),$$

which by (2.9) implies that

$$\tfrac{1}{2} D_t\, |w(t)|_1^2 \leq |f(u_1(t)) - f(u_2(t))|_0\, |w_x(t)|_0.$$

Let $L$ be the Lipschitz constant of $f$ on $[-C, C]$, where $C$ is the constant of (2.7). Then,

$$|f(u_1(t)) - f(u_2(t))|_0 \leq L\,|w(t)|_0$$

and

$$(2.13) \qquad D_t\,|w(t)|_1^2 \leq L(|w(t)|_0^2 + |w_x(t)|_0^2) = L\,|w(t)|_1^2$$

from which it follows that

$$\operatorname*{ess\,sup}_{t\in[0,T)} |w(t)|_1^2 \leq e^{LT}\,|w(0)|_1^2 = 0.$$

Therefore, $w = 0$.

**Lemma 2.3.** *Suppose that $f$ is Lipschitz continuous on bounded intervals and that the solution of the weak problem satisfies (2.7). Then there is a bounded function $f^*$, as regular as $f$, such that the solution of the weak problem with equation*



$$(2.5)'' \qquad \langle u_t(t), v\rangle - \langle u(t), v_x\rangle - \langle f^*(u(t)), v_x\rangle + \langle u_{xt}(t), v_x\rangle = \langle g(t), v\rangle$$

*which satisfies (2.6) is the same as that of (2.5)'.*

*Proof.* Take $C$ as the constant of (2.7). If $f$ is continuous let

$$(2.14) \qquad f^*(s) = \begin{cases} f(-C), & \text{if } s < -C, \\ f(s), & \text{if } |s| < C, \\ f(C), & \text{if } s > C. \end{cases}$$

Now let $u^*$ be a solution of the weak problem for equation (2.5)''. We can suppose that $u^*$ satisfies (2.7) with the same constant as the original solution $u$ (because, $u_0$ and $g$ are the same and then

$$(2.15) \qquad f^*(u^*(x,t)) = f(u^*(x,t))$$

almost everywhere in $x$ and $t$. Consequently $u^*$ also satisfies equation (2.5)', and by Lemma 2.2 $u^* = uu$.

Suppose now that $f$ has more regularity, say $f \in C^k(\mathbb{R})$ for $k \geq 1$. In this case we define $f^*$ to be

$$(2.16) \qquad f^*(s) = \begin{cases} f(-C-\varepsilon), & \text{if } -s > C+\varepsilon, \\ \theta_1, & \text{if } C < -s < C+\varepsilon, \\ f(s), & \text{if } |s| < C, \\ \theta_2, & \text{if } C < s < C+\varepsilon, \\ f(C+\varepsilon), & \text{if } s > C+\varepsilon, \end{cases}$$

where $\theta_1$, $\theta_2$ are generalized Hermite interpolants on $[-C-\varepsilon, -C]$ and $[C, C+\varepsilon]$, respectively, such that

$$\theta_1(-C-\varepsilon) = f(-C-\varepsilon),$$

$$(2.17) \qquad D_x^i\,\theta_1(-C-\varepsilon) = 0, \qquad i = 1, \dots, k,$$

$$D_x^i\,\theta_1(-C) = D_x^i\, f(-C), \qquad i = 0, \dots, k,$$

and

$$\theta_2(C+\varepsilon) = f(C+\varepsilon)$$

$$(2.18)\qquad D_x^i\,\theta_2(C+\varepsilon) = 0, \qquad i = 1, \dots, k,$$

$$D_x^i\,\theta_2(C) = D_x^i\, f(C), \qquad i = 0, \dots, k.$$

Then clearly $f^* \in C^k(\mathbb{R})$ and $u^*$ satisfies (2.15). Therefore, $u^* = u$.

## 3. The associated linear equation.

The study of solutions of the weak problem will be pursued in two steps. First, we reduce the non linear equation (2.5)″ treating the non linear term as an independent one, thus obtaining a linear problem whose solution depends upon a given function $w$. That is, given $w$ in a proper space, we study solutions of

$$(3.1)\qquad \langle u_t(t), v\rangle - \langle u(t), v_x\rangle + \langle u_{xt}(t), v_x\rangle =$$

$$= \langle f^*(w(t)), v_x\rangle + \langle g(t), v\rangle, \quad \forall\, v \in P_1, \;\; a.e.\;\; t \in [0, T),$$

$$(3.2)\qquad u(0) = u_0,$$

which belong to $K(1, 1)$. Secondly, the mapping that associates $u$, the solution of (3.1)-(3.2), with the function $w$ is shown to have a fixed point.

In this section we consider the linear problem derived from (3.1)-(3.2) which, given functions $g(x, t)$, $h(x, t)$ and $u_0(x)$, is to find a function $u \in K(1, 1)$ satisfying (3.2) such that

$$(3.3)\qquad \langle u_t(t), v\rangle - \langle u(t), v_x\rangle + \langle u_{xt}(t), v_x\rangle =$$

$$= \langle g(t), v\rangle + \langle h(t), v_x\rangle, \qquad \forall\, v \in P_1,$$

almost everywhere in $t$.

Two results are presented below. The first is concerned with existence and uniqueness of solutions of the linear problem. The second relates the regularity of solutions to the regularity of $h$, $g$ and $u_0$.

**Theorem 3.1.** *If $u_0 \in P_1$, $h$ and $g \in L^\infty(P_0) = K(0, 0)$, and $T < \infty$, then there exists a unique function $u \in K(1, 1)$ which satisfies (3.3) and (3.2).*

*Proof.* The argument follows the usual path of the energy method in which compactness theorems are used. So we shall have three steps:

1. Finite dimensional approximation in $x$,
2. A priori estimates,
3. Passage to the limit.



*Step 1.* If $i$ and $m$ are non-negative integers, let

$$w_i(x) = \begin{cases} c_i \cos(\pi i x) & \text{when } i \text{ is even},\\ c_i \sin(\pi(i-1)\,x) & \text{when } i \text{ is odd},\end{cases}$$

$$V_m = \text{span}\,\{w_0, \dots, w_m\}$$

and

$$V_\infty = \text{span}\,\{w_0, \dots, w_m, \dots\} = \bigcup_m V_m,$$

where $c_i$ are normalizing constants. Observe that

$$(3.4)\qquad V_m \subset V_n \subset V_\infty \subset \mathcal{D}_p(0, 1)$$

whenever $m < n$ and that $V_\infty$ is a dense subspace of $P_k$ for all $k$, since $\{w_i\}_{i \ge 0}$ is a basis of $P_k$. This last claim, which implies that $\mathcal{D}_p(0, 1)$ is dense in $P_k$, is a consequence of convergence theorems for Fourier Series [12]. We define the finite dimensional approximation problem to be the finding of a function $u_m : [0, T) \to V_m$ such that

$$(3.5)\qquad (u_{mt}(t), w_i) - (u_m(t), w_{ix}) + (u_{mxt}(t), w_{ix}) =$$

$$= (g(t), w_i) + (h(t), w_{ix}), \qquad 0 \le i \le m, \qquad t \in [0, T),$$

$$(3.6)\qquad u_m(0) = u_{0m},$$

where $u_{0m}$ is such that $u_{0m} \in V_m$ and $u_{0m} \to u_0$ in $P_1$. More precisely, we take $u_{0m}$ to be the truncated Fourier series of $u_0$ and recall that, as $u_0 \in P_1$, $u_{0m} \to u_0$ in $P_1$. Writing

$$u_m(x, t) = \sum_{i=0}^{m} \alpha_i(t)\, w_i(x),$$

$$u_{0m}(x) = \sum_{i=0}^{m} \alpha_{0i}\, w_i(x),$$

and putting

$$\alpha(t) = (\alpha_1(t), \dots, \alpha_m(t))^T,$$

$$\alpha_0 = (\alpha_{01}, \dots, \alpha_{0m})^T,$$

it is easy to see that (3.4)-(3.5) is equivalent to the problem

$$(3.5)' \qquad (A+B)\alpha' = C\alpha + F(t),$$

$$(3.6)' \qquad \alpha(0) = \alpha_0,$$

where the matrices $A$, $B$ and $C$ have as their elements the quantities

$$A_{ij} = (w_i, w_j),$$

$$B_{ij} = (w_{ix}, w_{jx}),$$

$$C_{ij} = (w_{ix}, w_j),$$

and the vector $F$ has the components

$$F_i(t) = (h(t), w_{ix}) + (g(t), w_i).$$

The existence of a unique measurable local solution of (3.5)-(3.6) in $[0, T)$ is assured, since $A + B$ in non-singular for all $m$.

*Step* 2. Since (3.5) is equivalent to

$$(3.7) \qquad (u_{mt}(t), v) - (u_m(t), v_x) + (u_{mxt}(t), v_x) = (g(t), v) + (h(t), v_x), \quad \forall\, v \in V_m,$$

choosing $v = u_m(t)$ leads to

$$\frac{1}{2}\frac{d}{dt}(|u_m(t)|_0^2 + |u_{mx}(t)|_0^2) =$$

$$= (u_m(t), u_{mx}(t)) + (g(t), u_m(t)) + (h(t), u_{mx}(t))$$

and

$$(3.8) \qquad \frac{1}{2}\frac{d}{dt}|u_m(t)|_1^2 \le |g(t)|_0\, |u_m(t)|_0 + |h(t)|_0\, |u_{mx}(t)|_0\,,$$

since $(u_m(t), u_{mx}(t)) = 0$. Integrating (3.8), we have

$$|u_m(t)|_1^2 \le |u_{0m}|_1^2 + 2\int_0^t |g(\tau)|_0\, |u_m(\tau)|_0\, d\tau + 2\int_0^t |h(\tau)|_0\, |u_{mx}(\tau)|_0\, d\tau,$$

and so

$$\operatorname*{ess\,sup}_{t\in[0,T)} |u_m(t)|_1^2 \le |u_{0m}|_1^2 + 2\int_0^T |g(\tau)|_0\, |u_m(\tau)|_0\, d\tau + 2\int_0^T |h(\tau)|_0\, |u_{mx}(\tau)|_0\, d\tau.$$



This, by the same reasoning used to arrive at (2.11),

$$(3.9) \qquad (1 - \varepsilon_1 - \varepsilon_2) \operatorname*{ess\,sup}_{t\in[0,T)} |u_m(t)|_1^2 \le$$

$$\le |u_{om}|_1^2 + \frac{1}{\varepsilon_1}|g|^2_{L^1(P_0)} + \frac{1}{\varepsilon_2}|h|^2_{L^1(P_0)}.$$

Then, for $\varepsilon_1 = \varepsilon_2$ small enough, it follows that

$$(3.10) \qquad \operatorname*{ess\,sup}_{t\in[0,T)} |u_m(t)|_1^2 \le |u_o|_1^2 + C(|g|^2_{L^1(P_0)} + |h|^2_{L^1(P_0)}) = K_o^2,$$

from (3.9) and the inequality $|u_{0m}|_1 \le |u_0|_1$.

Estimate (3.10) is enough to guarantee that we can take $t_m = T$ in Step 1 but, to pass to the limit, another estimate is needed and to obtain it we take $v = u_{mt}(t)$ in (3.7). Thus,

$$|u_{mt}(t)|_0^2 + |u_{mxt}(t)|_0^2 =$$

$$= (u_m(t), u_{mxt}(t)) + (g(t), u_{mt}(t)) + (h(t), u_{mxt}(t)).$$

Consequently,

$$|u_{mt}(t)|_1^2 \le |u_m(t)|_0^2 + \tfrac{1}{4}|u_{mxt}(t)|_0^2 +$$

$$+ \tfrac{1}{2}|g(t)|_0^2 + \tfrac{1}{2}|u_{mt}(t)|_0^2 + |h(t)|_0^2 + \tfrac{1}{4}|u_{mxt}(t)|_0^2,$$

and

$$(3.11) \qquad |u_{mt}(t)|_1^2 \le 2\,|u_m(t)|_0^2 + |g(t)|_0^2 + 2\,|h(t)|_0^2.$$

Therefore,

$$(3.11)' \qquad \operatorname*{ess\,sup}_{t\in[0,T)} |u_{mt}(t)|_1^2 = |u_{mt}|^2_{0,1} \le 2K_0^2 + |g|^2_{0,0} + 2\,|h|^2_{0,0}.$$

*Step* 3. Estimates (3.10), (3.11)′ mean that the sequence $u_m$ is bounded in $K(1,1)$. Hence, by the weak* compactness of bounded sets in $K(1,1)$, we can extract a subsequence of $u_m$ converging weakly* to a function $u \in K(1,1)$. Denote the subsequence by $u_m$ for convenience.

But, $K(1,1)$ can be identified with a closed subspace of $L^\infty(P_1) \times L^\infty(P_1)$. Also $L^\infty(P_1)$ is the dual of $L^1(P_1')$ and

$$P_1 \subset P_0 \subset P_1',$$

with dense imbedding. Then $L^1(P_1) \subset L^1(P_1')$ continuously and the weak* convergence of $u_m$ to $u$ in $K(1, 1)$ means that

$$(3.12) \qquad \int_0^T (D_x^i D_t^j u_m(t), v(t))\,dt \to \int_0^T (D_x^i D_t^j u(t), v(t))\,dt,$$

for $0 \le i, j \le 1$ and for all $v \in L^1(P_o)$. Thus, a fortiori, (3.12) holds for all $v \in L^1(P_1)$.

Therefore $u$ satisfies

$$(3.13) \qquad \int_0^T [(u_t(t), v(t)) - (u(t), v_x(t)) + (u_{xt}(t), v_x(t))]\,dt =$$

$$= \int_0^T [(g(t), v(t)) + (h(t), v_x(t))]\,dt, \qquad \forall\, v \in L^1(P_1).$$

In particular, (3.13) holds for $v(t) = \theta(t)\, w(x) \in L^1(P_1)$, where $w \in P_1$ and $\theta(t) = \mu(A)^{-1} \mathscr{X}_A(t)$, for $A$ an arbitrary measurable subset of $(0, T)$ with positive measure and $\mathscr{X}_A$ the characteristic function of $A$. Hence any solution of (3.13) is also a solution of (3.3).

Since (3.11)′ implies that $u$ is a continuous map of $[0, T)$ into $P_1$, $u$ satisfies (3.2). This accounts for existence.

To show uniqueness, note that, if $u_1$ and $u_2$ are two solutions of (3.3)-(3.2), then $w = u_1 - u_2$ satisfies the relation

$$(3.14) \quad (w_t(t), v) - (w(t), v_x) + (w_{xt}(t), v_x) = 0, \quad \forall\, v \in P_1, \quad \text{a.e.} \quad t \in [0, T),$$

and is such that $w(0) = 0$. Take $v = w(t)$. It follows that

$$(3.15) \qquad \frac{1}{2} \frac{d}{dt} |w(t)|_1^2 = 0,$$

and since $w(0) = 0$, $w = 0$ in $L^\infty(P_1) \supset K(1, 1)$.

The regularity theorem is as follows.

**Theorem 3.2.** *Suppose that* $u_o \in P_{k+1}$ *and that* $g$ *and* $h$ *belong to* $K(\ell, k)$ *for some positive integers* $\ell$ *and* $k$. *Then* $u$, *the solution of* (3.3)-(3.2), *belongs to* $K(\ell + 1, k + 1)$.

*Proof.* To prove that $u \in K(\ell + 1, k + 1)$ it is enough to show that $u_m$ is bounded in this space. To see this, note that the space $K(\ell, k)$ is continuously imbedded in $K(1, 1)$ and can be identified with a closed subspace of a power of $L^\infty(P_1)$. Then repeat the arguments in Step 3 of the proof of Theorem 3.1 to see that any subsequence that converges weakly* in $K(\ell + 1, k + 1)$ converges to the solution of (3.13).



To show the boundness of $u_m$ in $K(\ell + 1, k + 1)$, observe first that, as $g$ and $h$ belong to $K(\ell, k)$, then $D_t^j g$ and $D_t^j h$ belong to $K(0, 0)$ for $0 \le j \le \ell$. Next, since

$$D_t^j (f(t), w_i) = (D_t^j f(t), w_i), \qquad \forall\, i \ge 0,$$

when $D_t^j f \in L^\infty(P_0)$, it follows that (with $F$ as in (3.5)′)

$$D_t^j F(t) \in L^\infty(R^m).$$

Due to his fact we can differentiate equation (3.5)′ $j$ times ($j \le \ell$) to arrive at the relation

$$(3.16) \qquad D_t^{j+1}\alpha = (A + B)^{-1}\, C\, D_t^j \alpha + (A + B)^{-1}\, D_t^j F(t).$$

Now, by the Caratheodory Theorem, on the existence of solutions of Ordinary Differential Equations,

$$D_t^j \alpha \in L^\infty(R^m), \qquad 0 \le j \le \ell + 1.$$

Consequently,

$$D_t^j u_m \in L^\infty(V_m), \qquad 0 \le j \le \ell + 1.$$

Moreover, (3.16) is equivalent to

$$(3.17) \qquad (D_t^{j+1} u_m(t), v) - (D_t^j u_m(t), v_x) + (D_t^{j+1} D_x u_m(t), v_x) =$$

$$= (D_t^j g(t), v) + (D_t^j h(t), v_x)$$

for all $v \in V_m$, whenever $0 \le j \le \ell$. Thus if we take $v = D_t^{j+1} u_m(t)$ in (3.17),

$$|D_t^{j+1} u_m(t)|_0^2 + |D_x D_t^{j+1} u_m(t)|_0^2 =$$

$$= (D_t^j u_m(t), D_x D_t^{j+1} u_m(t)) + (D_t^j g(t), D_t^{j+1} u_m(t)) + (D_t^j h(t), D_x D_t^{j+1} u_m(t)).$$

Thus,

$$(3.18) \quad |D_t^{j+1} u_m(t)|_1^2 \le 2\,|D_t^j u_m(t)|_0^2 + |D_t^j g(t)|_0^2 + 2\,|D_t^j h(t)|_0^2, \qquad 0 \le j \le \ell$$

(note that for $j = 0$ we have (3.11)). Thus, recursively,

$$(3.19) \qquad |D_t^{j+1} u_m|_{0,1}^2 \le C(|u_0|_1^2 + |g|_{L^1(P_0)}^2 + |h|_{L^1(P_0)}^2 + |g|_{j,0}^2 + |h|_{j,0}^2)$$

for $0 \le j \le \ell$. The estimate (3.19) means that $u_m$ is bounded in $K(\ell + 1, 1)$.

Next we show that the spatial derivatives are bounded. To do this pick $v = D_x^{2i}u_m(t)$ in equation (3.17) and integrate by parts $i$ times. Then,

$$(3.20)\qquad \frac{1}{2}\frac{d}{dt}(|D_x^i u_m(t)|_0^2 + |D_x^{i+1}u_m(t)|_0^2) =$$

$$= (D_x^i u_m(t), D_x^{i+1}u_m(t)) + (D_x^i g(t), D_x^i u_m(t)) + (D_x^i h(t), D_x^{i+1}u_m(t)).$$

But since $D_x^i u_m \in P_1$, $(d_x^i u_m(t), D_x^{i+1}u_m(t)) = 0$, and thus (3.20) implies

$$\frac{d}{dt}(|D_x^i u_m(t)|_0^2 + |D_x^{i+1}u_m(t)|_0^2) \le$$

$$\le |D_x^i g(t)|_0 |D_x^i u_m(t)|_0 + |D_x^i h(t)|_0 |D_x^i u_m(t)|_0,$$

which integrated in time is

$$|D_x^i u_m(t)|_0^2 + |D_x^{i+1}u_m(t)|_0^2 \le |D_x^i u_{m0}|_0^2 + |D_x^{i+1}u_{m0}|_0^2 +$$

$$+ \int_0^t |D_x^i g(\tau)|_0 |D_x^i u_m(\tau)|_0 \, d\tau + \int_0^t |D_x^i h(\tau)|_0 |D_x^{i+1}u_m(\tau)|_0^2 \, d\tau.$$

Since $g$ and $h \in K(0, k)$ and $u_0 \in P_{k+1}$ we have, by the argument used for (3.10), that

$$(3.21)\qquad |D_x^i u_m|_{0,0}^2 + |D_x^{i+1}u_m|_{0,0}^2 \le$$

$$\le |D_x^i u_0|_0^2 + |D_x^{i+1}u_0|_0^2 + C(|g|_{L^1(P_k)}^2 + |h|_{L^1(P_k)}^2),$$

for $0 \le i \le k$. That is, $u_m$ is bounded in $K(0, k+1)$.

One is left now with the estimation of the mixed derivatives. For this purpose, for $0 \le j \le \ell$ and $0 \le i \le k$, take $v = D_t^{j+1} D_x^{2i}u_m(t)$ in (3.17) and integrate by parts $i$ times in $x$ to obtain

$$|D_t^{j+1}D_x^i u_m(t)|_0^2 + |D_t^{j+1}D_x^{i+1}u_m(t)|_0^2 = (D_t^j D_x^i u_m(t), D_t^{j+1}D_x^{i+1}u_m(t)) +$$

$$+ (D_t^j D_x^i g(t), D_t^{j+1}D_x^i u_m(t)) + (D_t^j D_x^i h(t), D_t^{j+1}D_x^{i+1}u_m(t)).$$

Then

$$(3.22)\qquad |D_t^{j+1}D_x^i u_m(t)|_0^2 + |D_t^{j+1}D_x^{i+1}u_m(t)|_0^2 \le$$

$$\le 2|D_t^j D_x^i u_m(t)|_0^2 + |D_t^j D_x^i g(t)|_0^2 + 2|D_t^j D_x^i h(t)|_0^2.$$



The above inequality for $j = 0$ is

$$|D_t D_x^i u_m(t)|_0^2 + |D_t D_x^{i+1}u_m(t)|_0^2 \le$$

$$\le 2|D_x^i u_m(t)|_0^2 + |D_x^i g(t)|_0^2 + 2|D_x^i h(t)|_0^2,$$

whose right-hand side is bounded for all $i$ as a consequence of (3.21). Therefore, by recurrence on $j$, (3.22) implies

$$(3.23)\qquad |D_t^{j+1}D_x^i u_m(t)|_0^2 + |D_t^{j+1}D_x^{i+1}u_m(t)|_0^2 \le$$

$$\le C(|D_x^i u_0|^2 + |D_x^{i+1}u_0|^2 + |g|_{L^1(P_k)}^2 + |h|_{L^1(P_k)}^2 + |D_x^i g|_{j,0}^2 + |D_x^i h|_{j,0}^2),$$

for $0 \le j \le \ell$, $0 \le i \le k$.

Estimates (3.19), (3.21) and (3.23) show that the sequence $u_m$ of solutions of (3.5) is bounded in $K(\ell+1, k+1)$, and therefore $u \in K(\ell+1, k+1)$.

**Remarks.**

1. We may take $T = \infty$ in Theorems 3.1 and 3.2 by requiring that $g$ and $h$ belong to $K(0,0) \cap L^1(P_0)$, $g$ and $h$ belong to $K(\ell, k) \cap L^1(P_k)$ respectively.
2. The constants appearing in estimates (3.19) and (3.23) depend on $\ell$, growing with it. Thus, Theorem 3.2 does not take care of the case $\ell = \infty$. However, this is due to term $(u, v_x)$ in equation (3.3) and, since this term can be embodied into $(f(u), v_x)$, this limitation will have no significance in the nonlinear case.

## 4. The non-linear equation.

This section is devoted to showing that, for functions $w$ belonging to a properly chosen space, the operator which associates the solution of (3.1)-(3.2) with $w$ is well defined and has a fixed point. Also, through a "bootstrap" argument, solutions of (2.5)″ are shown to be as regular as $u_0$, $f$ and $g$.

**Theorem 4.1.** *Let $u_0 \in P_1$ and $g \in K(0,0)$ and let $f \in C^0(R)$ be Lipschitz continuous on bounded intervals. Then if $T < \infty$, there exists a function $u \in K(1,1)$ satisfying (2,5)′-(2.6).*

*Proof.* After Lemmas 2.1-2.3, we can work with equation (2.5)″. Let $w \in K(0,0)$. Since $f^*$ is bounded, say

$$(4.1)\qquad \sup_s |f^*(s)| \le C^*,$$

then

$$(4.1)' \qquad |f^*(w(t))|_0 \leq C^*,$$

and $f^*(w) \in K(0,0)$. Therefore, by Theorem 3.1, for each $w \in K(0,0)$ there is a unique solution $u_w$ of (3.1)-(3.2). Moreover $u_w \in K(1,1) \subset K(0,0)$. Thus, the operator

$$(4.2) \qquad F: K(0,0) \to K(0,0)$$

$$w \to F(w) = u_w$$

is well defined.

Next we see that operator $F$ is continuous and maps $K(0,0)$ into a bounded subset of $K(1,1)$.

To show continuity of $F$ let $w_1$ and $w_2$ be in $K(0,0)$ and set $u_1 = F(w_1)$ and $u_2 = F(w_2)$. Then, $y = u_1 - u_2$ satisfies

$$(4.3) \qquad (y_t(t), v) - (y(t), v_x) + (y_{xt}(t), v_x) =$$

$$= (f^*(w_1(t)) - f^*(w_2(t)), v_x), \qquad \forall v \in P_1, \qquad t \in [0, T)$$

and $y(0) = 0$. Taking $v = y(t)$ above and using (2.9), equation (4.3) yields

$$\frac{1}{2}\frac{d}{dt}(|y(t)|_0^2 + |y_x(t)|_0^2) \leq |f^*(w_1(t)) - f^*(w_2(t))|_0 \, |y_x(t)|_0.$$

But since $f^*$ is Lipschitz continuous, with constant $L$,

$$|f^*(w_1(t)) - f^*(w_2(t))|_0 \leq L\,|w_1(t) - w_2(t)|_0.$$

And

$$\frac{d}{dt}|y(t)|_1^2 \leq L^2 |w_1(t) - w_2(t)|_0^2 + |y(t)|_1^2,$$

which implies, integrating and using Gronwall Lemma, that

$$(4.4) \qquad |y(t)|_1^2 \leq L^2 \exp(t) \int_0^t |w_1(\tau) - w_2(\tau)|_0^2 \, d\tau.$$

Therefore,

$$(4.5) \qquad |u_1 - u_2|_{0,0} \leq LT^{1/2} \exp(T/2)\, |w_1 - w_2|_{0,0},$$

so that $F$ is continuous.



To prove $F$ maps $K(0,0)$ into a bounded subset of $K(1,1)$ let $v = u(t) = F(w)(t)$ in (3.1). Then,

$$(u_t(t), u(t)) + (u_{xt}(t), u_x(t)) = (f^*(w(t)), u_x(t)) + (g(t), u(t)),$$

as a consequence of (2.9). This implies the inequality

$$\frac{1}{2}\frac{d}{dt}|u(t)|_1^2 \leq |f^*(w(t))|_0 \, |u_x(t)|_0 + |g(t)|_0 \, |u(t)|_0,$$

and so

$$|u(t)|_1^2 \leq |u_0|_1^2 + \int_0^t |f^*(w(\tau))|_0 \, |u_x(\tau)|_0 \, d\tau + \int_0^t |g(\tau)|_0 \, |u(\tau)|_0 \, d\tau.$$

Then, by the same arguments as for (2.11), (3.10) we arrive at

$$(4.6) \qquad |u|_{0,1}^2 \leq 2(|u_0|_1^2 + |g|_{L^1(P_0)}^2 + \left(\int_0^T |f^*(w(t))|_0 \, dt\right)^2,$$

and since, by (4.1)′,

$$\int_0^T |f^*(w(t))|_0 \, dt \leq TC^*,$$

inequality (4.6) yields the bound

$$(4.7) \qquad |u|_{0,1} \leq 2(|u_0|_1^2 + |g|_{L^1(P_0)}^2 + TC^*) = K_1,$$

where $K_1$ does not depend on $w$.

Also, taking $v = u_t(t)$ in (3.1), we have

$$|u_t(t)|_0^2 + |u_{xt}(t)|_0^2 =$$

$$= (u(t), u_t(t)) + (f^*(w(t)), u_{xt}(t)) + (g(t), u_t(t)).$$

Pursuing now the same steps that were used to obtain (3.11)′, the above equality implies that

$$|u_t(t)|_1^2 \leq 2\,|u(t)|_0^2 + |f^*(w(t))|_0^2 + 2\,|g(t)|_0^2$$

so that, using (4.1)′,

$$(4.8) \qquad |u_t|_{0,1} \leq 2K_1 + (C^*)^2 + 2\,|g|_{0,0}^2 = K_2,$$

where $K_2$ does not depend on $w$.

Estimates (4.7)-(4.8) show that $F(w)$ belongs to a bounded subset of $K(1,1)$ for all $w \in K(0,0)$. Hence $F$ is a continuous operator which maps the closed convex set $K(0,0)$ into itself in such a way that its range is a precompact subset of $K(0,0)$. Therefore Schauder's Fixed Point Theorem ensures the existence of a fixed point for $F$. That is, there is an element $u \in K(0,0)$ such that $F(u) = u$. This is obviously a solution of (2.5)″. Moreover $u$ belongs to the range of $F$ and thus $u \in K(1,1)$.

**Remark 3.** Theorem 4.1 gives solutions of (2.5)″ for $T < \infty$. If the hypothesis are changed to require that

$$g \in L^1(P_0) \cap L^\infty(P_0), \tag{4.9}$$

we may take $T = \infty$. This can be done in the following way. Let $T_1 < \infty$ be such that $nT_1 = T$. Theorem 4.1 says that there is a function $u_1 : [0, T_1) \to P_1$ solution of (2.5)′ and Lemma 2.1 implies that

$$|u(T_1)|_1^2 \le |u_0|_1^2 + C \int_0^{T_1} |g(t)|_0 \, dt.$$

Since the arguments of Theorem 4.1 are independent of the origin of time there is a function $u_2 : [T_1, T_2) \to P_1$ solution of (2.5)′ such that $u_2(T_1) = u_1(T_1)$. Therefore

$$u(t) = \begin{cases} u_1(t), & \text{if } t \in [0, T_1) \\ u_2(t), & \text{if } t \in [T_1, T_2) \end{cases}$$

is a solution of (2.5)′ on $[0, T_2)$ satisfying

$$u(0) = u_0$$

and

$$|u(T_2)|_1^2 = |u_2(T_2)|_1^2 \le |u_1(T_1)|_1^2 + C \int_{T_1}^{T_2} |g(t)| \, dt \le |u_0|_1^2 + C \int_0^{T_2} |g(t)| \, dt.$$

Repeating the process, we can define $u$ on $[0, T)$. Clearly this procedure can be repeated indefinitely, and $u$ may be extended to $[0, \infty)$.

The next theorem tell us about the regularity of solutions of (2.5)″. First note that equations (1.1) and (2.5) are invariant when $f$ (or $f^*$) are replaced by $f - f(0)(f^* - f^*(0))$. Thus we can suppose, without loss of generality, that $f(0) = 0$. In this case, using a result of Sobolev [5], it is easy to see that $f(w) \in K(m,k)$ if $w \in K(m,k)$ and $f \in C^\ell(\mathbb{R})$, where $\ell = \max(m,k)$. Then we have:



**Theorem 4.2.** *In the case that $f \in C^\ell(\mathbb{R})$, $g \in K(m,k)$, $u_0 \in P_{k+1}$ and $T < \infty$, the solution $u$ of* (2.5)″ (or(2.5)) *is such that*

$$u \in K(m+1, k+1). \tag{4.10}$$

*Proof.* Since $K(m,k) \subset K(0,0)$ and $P_{k+1} \subset P_1$ there exists a function $\nu \in K(1,1)$ which is a solution of

$$(u_t(t), v) - (u(t), v_x) + (u_{xt}(t), v_x) = (f(u(t)), v_x) + (g(t), v), \quad \forall v \in P_1 \tag{4.11}$$

If $m = k = 0$ there is nothing to do. If $m, k > 0$, let $h = f(u(t))$. Since $u \in K(1,1)$ so is $h$. Then, by Theorem 3.2, $u \in K(2,2)$. This reasoning can be repeated until $h \in K(m,k)$, showing that $u \in K(m+1, k+1)$.

**Remark 4.** If $g \in K(m,k) \cap L^1(P_k)$ the result of Theorem 4.2 remains true for $T = \infty$ because Theorem 3.2 can be used when $T = \infty$ for $f(w) \in L^1(P_k)$ and, if $f(0) = 0$ and $f \in C^k(\mathbb{R})$, $w \in L^1(P_k)$ implies that $f(w) \in L^1(P_k)$.

**Remark 5.** With a slight modification in the argument of Theorem 3.2, one can weaken the hypothesis on $g$, in both regularity theorems, by requiring that $g \in K(m, k-1)$.

Concluding, we observe that the weak solutions for $m > 0$ and $k > 1$ are indeed classical solutions.

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