Embedding Lens spaces in a homotopy type of CP(2)

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1. Introduction.

In this work we give partial results on embeddings of Lens spaces in M^4 where M^4 denotes a four dimensional manifold having the homotopy type of CP(2).

The results presented here were motivated by Epstein's work [2], where he shows that $L(2k, q)$ does not embed in S^4.

All maps and manifolds involved here are C^∞, and M^4 always denotes a connected closed four dimensional manifold having the homotopy type of CP(2).

2. Statements of Results.

Basically we obtain non-embeddability results of $L(n, q)$ in M^4.

Two cases will be considered. The first case concerning n odd, and the second $n \equiv 2 \bmod 4$.

In the first case we make use of the μ-invariant, wich is defined for every $Z/2Z$ — homology 3-sphere.

In the second case we use results obtained by Rocklin [5].

The main results are contained in:

Theorem 2.1. *If n is odd and $L(n, q)$ embeds in M^4 then $\mu(L(n, q))$ is zero.*

Theorem 2.2. *If $n \equiv 2 \bmod 4$, then $L(n, q)$ does not embed in M^4.*

3. Outline of the Proofs.

A $Z/2Z$ homology 3-sphere X, bounds an orientable fourdimensional manifold Y with the properties:

1) $H_1(Y; Z)$ has no 2-torsion
2) The intersection form in $H_2(Y; Z)$ is even.

So the μ-invariant of X is defined by $\mu(X) = -\tau(Y)/16 \bmod 1$, as an element of Q/Z, where τ denotes the signature of Y.

Let us suppose now that $L(n, q)$ embeds in M^4.

Since $H_3(M^4; Z) = 0$, $L(n, q)$ separates M^4 in two components.

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So we have $M^4 = A_1 \cup A_2$ and $A_1 \cap A_2 = L(n, q)$.

Lemma 3.1. *If n is odd and L(n, q) embedds in M^4, then L(n, q) bounds an orientable four manifold Y such that:*

1) $H_1(Y; Z)$ *has no 2-torsion*
2) $H_2(Y; Z) = 0$.

Proof. The proof follows easily by applying the Mayer-Vietoris sequence to the decomposition $M^4 = A_1 \cup A_2$.

Now the proof of theorem 2.1 follows from this lemma; since $H_2(Y; Z) = 0$ we have $\tau(Y) = 0$ and so $\mu(L(n, q)) = 0$.

When n is odd and q even, there is a recipe to compute $\mu(L(n, q))$; see [3, p. 51]; which can be described in the following way.

There is a unique expansion

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{\cdot}{\cdot - \cfrac{1}{bs}}}}$$

where $|b_i| \geq 2$.

Denoting by P^+ the number of positive b's and P^- the number of negative ones then we have $\mu(L(n, q)) = (P^+ - P^-)/16 \bmod 1$.

With respect to canonical orientations we have that $L(n, q) = -L(n, n-q)$, and so this recipe can also be used to compute $\mu(L(n, q))$ for q odd.

Lemma 3.2. *Let n be an odd number: then* $\mu(L(n, 1)) = -\dfrac{n-1}{16} \bmod 1$

Proof. Induction on n.

Corollary 3.3. *If $n \not\equiv 1 \bmod 16$, n odd, L(n, 1) does not embedd in M^4.*

Comment: The result of corollary 3.3 can be used to show that the classes $n\alpha$ in $H_2(M^4; Z)$, n odd $n \not\equiv \pm 1 \bmod 8$ can not be realized by S^2.

If the class $n\alpha$ could be realized by S^2, the Euler number of the normal bundle of S^2 in M^4 would be n^2, and so the associated sphere bundle would be $L(n^2, 1)$, which is impossible by corollary 3.3.

So in this case corollary 3.3 gives the same sort of information as a theorem of Kervaire-Milnor [4, p. 1652].

Now we outline the proof of theorem 2.2.

Let U_h denote a non-orientable surface of genus h, and $N(2k, q) = \min \{h / U_h \text{ embeds in } L(2k, q)\}$.

Bredon and Wood [1] have shown that U_h embeds in $L(2k, q)$ iff $h = N(2k, q) + 2i$ for some integer $i \geq 0$.

In the same work Bredon and Wood presents an alternative recursion formula $N(2k, q)$:

$$N(2, 1) = 1$$

$$N(2k, q) = 1 + N(2(k-q), q') \quad \text{for } 0 < q < K$$

$$q' \equiv \pm \bmod 2(k-q);\ 0 < q' \leq k - q.$$

Lemma 3.4. *N(2k, q) and k have the same parity.*

Proof. Using induction and the recursive formula above it is easy to show that $N(2k, q) + k$ is even for all k.

From now on we consider just the case $n = 2k$ where k is odd.

By the above results we have U_h embedded in $L(2k, q)$ for some odd integer h.

If $L(2k, q)$ embeds in M^4 we have U_h embedded in M^4. Since the normal bundle of $L(2k, q)$ in M^4 is trivial, the normal bundle of U_h in M^4 has a non-zero cross-section.

Thus the class represented by U_h in $H_2(M^4; Z_2)$ is the zero class and the normal Euler number (twisted) of the normal bundle of U_h in M^4 is zero.

Using Rocklin's results [5, p. 47-48] we can construct a ramified double covering

$$\begin{array}{c} Y \\ \downarrow \pi \\ M^4 \end{array}$$

ramifield exactly over U_h where

$$\text{rank } H_2(Y; Z) = 2 \text{ rank } H_2(M^4; Z) + h$$

and $\tau(Y) = 2\,\tau(M^4) - a/2$ where τ is the signature and a the Euler number. So rank $H_2(Y; Z)$ is odd and $\tau(Y) = 2$.

This contradiction proves theorem 2.2.

Comment: It is known that the class 2α in $H_2(\mathrm{CP}(2), Z)$ can be represented by S^2 and so using the same argument of the last comment we have $L(4, 1)$ embedded in CP(2).

We do not know in general if $L(4k, q)$ embeds CP(2).

References

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