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Analysis of a *RLC* circuit in the dielectric breakdown limit

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Abstract.

The problem of a non-linear circuit consisting of a resistor, an inductor, and a capacitor allowing for electric arcs is studied in the framework of Convex Analysis. The differential inequalities governing the circuit are shown to yield a unique stable solution which can be computed through standard schemes.

1. Introduction. This paper is concerned with the use of Convex Analysis concepts to study non linear circuits. Specifically, we shall consider the problem of the circuit represented in Figure 1. Non linearity here comes from the possibility of disruptive currents associated to the phenomenon of dielectric breakdown. The circuit consists of: i) a resistance R, ii) an inductance L and iii) a capacitance C in parallel with an "electric arc" of critical voltage V_a.

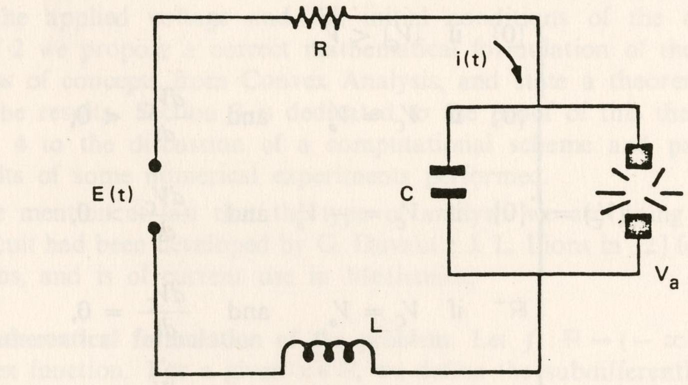

Figure 1

With t standing for time, the applied voltage will be indicated by $E(t)$, the total charge in the circuit by $q(t)$, the total current by $i(t)$, the charge of the capacitor by $q_C(t)$, the capacitor's voltage by $V_C(t)$ and the electric arc current by j.

Sub-circuit (iii) is to simulate the capacitor's dielectric in the breakdown vicinity, so that we ask the capacitor's voltage to satisfy the condition

(1.1) $$-V_a \leq V_c(t) \leq V_a, \text{ any } t \geq 0,$$

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and the eletric arc element to have the characteristic function $j = j(V_C)$ shown in Figure 2. This characteristic function can be achieved with a couple of Zenner diodes, for example, and in fact it should be pointed out that to describe it we need more than the usual concept of function. The

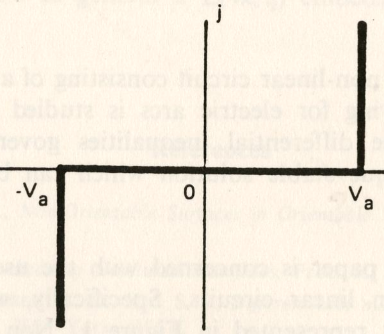


Figure 2

appropriate notion is that of a set valued function, or a graph. In analytical terms, thinking of V_C as a function of time, we have

(1.2) $$j(V_C) = \begin{cases} \{0\} & \text{if } |V_C| < V_a, \\ \{0\} & \text{if } V_C = V_a \text{ and } \dfrac{dV_C}{dt} < 0, \\ \{0\} & \text{if } V_C = -V_a \text{ and } \dfrac{dV_C}{dt} > 0, \\ \mathbb{R}^+ & \text{if } V_C = V_a \text{ and } \dfrac{dV_C}{dt} = 0, \\ \mathbb{R}^- & \text{if } V_C = -V_a \text{ and } \dfrac{dV_C}{dt} = 0. \end{cases}$$

Inside the limits defined by (1.1), the characteristic function of the capacitor is the usual linear relation (Figure 3):

(1.3) $$q_C(t) = C\,V_C(t).$$

Finally, balance between the circuit variables is given by Kirchoff's laws:

(1.4) $$E(t) - L\,\frac{di}{dt}(t) - Ri(t) - V_C(t) = 0, \qquad t > 0,$$

(1.5) $$i(t) - \frac{dq_C}{dt}(t) - j(V_C(t)) = 0, \qquad t > 0.$$

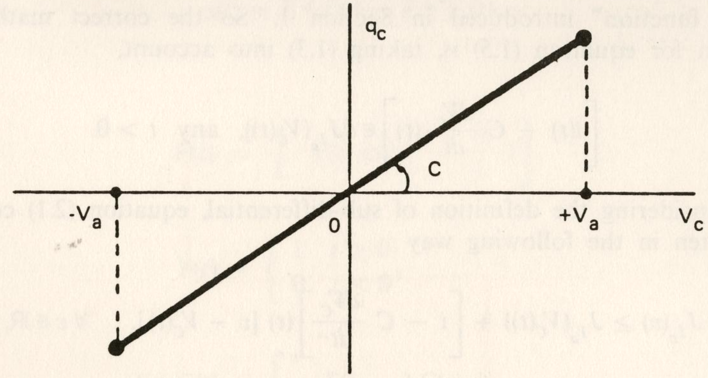


Figure 3

The problem we want to discuss in this piece is the compatibilization of equations (1.1)-(1.5), plus initial conditions. We shall show that this set of equations represents a well posed mathematical problem, in the sense that it yields a unique and stable vector state-function $(q(t),\ i(t),\ V_C(t))$, given the applied voltage and the initial conditions of the circuit. In Section 2 we propose a correct mathematical formulation of the problem, in terms of concepts from Convex Analysis, and state a theorem summarizing the results. Section 3 is dedicated to the proof of this theorem, and Section 4 to the discussion of a computational scheme and presentation of results of some numerical experiments performed.

We mention at last that the type of analysis we are going to do for this circuit had been developed by G. Duvaut e J. L. Lions in [2] for antenna problems, and is of current use in Mechanics.

2. Mathematical formulation of the problem. Let $f : \mathbb{R} \to (-\infty, +\infty]$ be a convex function. For a given $x \in \mathbb{R}$, we define the subdifferential of f at x as the set $\partial f(x) \subset \mathbb{R}$ defined by the criterion

$$\zeta \in \partial f(x) \text{ if and only if } f(y) \geq f(x) + \zeta(y - x),\ \forall\, y \in \mathbb{R}.$$

Let now $I \subset \mathbb{R}$ be a closed interval. The indicator function

$$J_I : \mathbb{R} \to (-\infty, +\infty]$$

of this interval is defined as

$$J_I(x) = \begin{cases} +\infty & \text{if } x \notin I, \\ 0 & \text{if } x \in I. \end{cases}$$

Then, if we call $I_a = [-V_a, V_a]$, it can be verified [1, 3] that J_{I_a} is a convex function and that $\partial J_{I_a} = j$, j being the electric arc element "characteristic function" introduced in Section 1. So the correct mathematical notation for equation (1.5) is, taking (1.3) into account,

$$\left[i(t) - C\frac{dV_C}{dt}(t) \right] \in \partial J_{I_a}(V_C(t)), \text{ any } t > 0. \tag{2.1}$$

Considering the definition of sub-differential, equation (2.1) could still be written in the following way

$$J_{I_a}(v) \geq J_{I_a}(V_C(t)) + \left[i - C\frac{dV_C}{dt} \right](t)\,[v - V_C(t)], \quad \forall v \in \mathbb{R}, \tag{2.2}$$

or

$$\left[C\frac{dV_C}{dt} - i \right](t)\,[v - V_C(t)] \geq 0, \quad \forall v \in I_a, \tag{2.3}$$

since $v \in \mathbb{R} - I_a$ would imply $J_{I_a}(v) = +\infty$ and (2.2) trivially satisfied, and for $v \in I_a$, $V_C \in I_a$, we have $J_{I_a}(v) = J_{I_a}(V_C) = 0$.

Collecting all the significant relations we would have then the following mathematical problem to represent the state of our circuit: "To find functions $i(t)$ and $V_C(t)$ such that

$$\begin{cases} \text{(i)} & V_C(t) \in I_a, \ \forall t \geq 0, \\ \text{(ii)} & E(t) = L\dfrac{di}{dt}(t) + Ri(t) + V_C(t), \ \forall t > 0, \\ \text{(iii)} & \left[C\dfrac{dV_C}{dt} - i \right](t)\,[v - V_C(t)] \geq 0, \ \forall t > 0, \ \forall v \in I_a, \\ \text{(iv)} & i(0) = 0, \ V_C(0) = 0. \end{cases} \tag{2.4}$$

In fact the current $i(t)$ can be eliminated in this problem (2.4), resulting an initial value problem for an integrodifferential inequality in V_C. This is done as follows. From (2.4) (ii), (iv),

$$\left(\frac{d}{dt} + \frac{R}{L} \right) i(t) = \frac{1}{L}(E(t) - V_C(t)),$$

$$i(0) = 0,$$

hence

$$i(t) = F * E(t) - F * V_C(t),$$

where

$$F(t) = \frac{1}{L}\, Y(t) \exp\left[-\frac{R}{L}t \right],$$

$$Y(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases},$$

$$F * G(t) = \int_{-\infty}^{\infty} F(t - s)\, G(s)\, ds,$$

any G integrable. Thus (2.4) is equivalent to

$$\begin{cases} \text{(i)} & V_C(t) \in I_a, \quad \forall t \geq 0, \\ \text{(ii)} & \left[C\dfrac{dV_C}{dt} + F * V_C \right](t)\,[v - V_C(t)] \geq F * E(t)\,[v - V_C(t)], \\ & \qquad \forall t > 0, \ \forall v \in I_a, \\ \text{(iii)} & V_C(0) = 0. \end{cases} \tag{2.5}$$

In respect to the inequality system (2.4), or (2.5), we observe first that the initial conditions were taken to be the simplest ones in view of the fact they do not affect the structure of the problem. We are not loosing any generality in our results. Second, the case of a "stable" dieletric, where the classical relation (1.3) holds true for any field V_C, corresponds to the limit situation $V_a \to \infty$. This would imply the same set of equations as in (2.5) with I_a replaced by $\mathbb{R}$. Hence, taking $v = V_C \pm 1$, we would have in this particular case the following integro-differential equation problem to be satisfied by V_C:

(i) $\left[C\dfrac{dV_C}{dt} + F * V_C \right](t) = F * E(t), \quad \forall t > 0,$

(ii) $V_C(0) = 0.$

This is, clearly, the classical linear RLC circuit equation.

One last bit of notation is needed. For any given $T > 0$,

$C^0(0, T)$ = space of continuous functions on $[0, T]$ with the supremum norm;

$L^p(0, T)$ = space of functions $f(t)$ on $[0, T]$ for which $|f(t)|^p$ is Lebesgue integrable, with norm $|f|_p = [\int_0^T |f(t)|^p \, dt]^{1/p}$, $p \geq 1$;

$L^\infty(0, T)$ = space of all Lebesgue measurable functions $f(t)$ on $[0, T]$ which are bounded, except possibly on a set of measure zero, with norm

$$|f|_\infty = \inf_{g(t) = f(t) \text{ a.e.}} \left\{ \sup_{0 \leq t \leq T} |g(t)| \right\}$$

A result describing properties of system (2.4) is the following.

Theorem 2.1. *Let $\infty \geq T > 0$ and $E \in L^2(0, T)$ be given. Assume also $dE/dt \in L^2(0, T)$. Then there exists a unique pair of functions $(i(t), V_C(t))$ such that*

$$i \in L^\infty(0, T) \cap L^2(0, T) \cap C^0(0, T), \tag{2.6}$$

$$V_C \in L^\infty(0, T) \cap C^0(0, T),$$

$$\frac{di}{dt} \in L^\infty(0, T) \cap L^2(0, T), \quad \frac{dV_C}{dt} \in L^\infty(0, T), \tag{2.7}$$

$$i(0) = 0, \qquad V_C(0) = 0, \tag{2.8}$$

which verifies relation (2.4) (i)-(iii) *for almost every $t \in (0, T]$. Furthermore, a constant K can be found such that*

$$\{|i|_\infty + |i|_2 + |V_C|_\infty\} \leq K|E|_2. \tag{2.9}$$

And $i(T) \to 0$ as $T \to \infty$.

Remark. The mechanical system in analogy to the circuit we are considering is a particle of mass L in a bed consisting of an elastoplastic string, stiffness C and limiting force V_a, and a dashpot R. The particle is subjected to an external force E.

3. Proof of Theorem 2.1. The proof, based on compactness arguments, consists of four steps:

A. Uniqueness;

B. Regularization: approximated solutions are defined by relaxing condition (2.4) and penalizing the "deviation" from it;

C. A priori estimates: energy type inequalities are established which are independent of the regularization parameters;

D. Passage to the limit: accumulation points of the approximated solutions are shown to be the required solution of the problem.

We start then with

Step A

Assume there are two solutions (i_1, V_C^1) and (i_2, V_C^2). We have,

$$L\frac{di}{dt} + Ri + V_C = 0, \tag{3.1}$$

$$\left[C \frac{dV_C^1}{dt} - i_1\right][v - V_C^1] \geq 0, \qquad \forall v \in I_a, \tag{3.2}$$

$$\left[C \frac{dV_C^2}{dt} - i_2\right][v - V_C^2] \geq 0, \qquad \forall v \in I_a, \tag{3.3}$$

$$i(0) = V_C(0) = 0, \tag{3.4}$$

where $i = i_1 - i_2$ and $V_C = V_C^1 - V_C^2$.

Take $v = V_C^2$ in (3.2), $v = V_C^1$ in (3.3) and add to obtain

$$V_C\left[i - C\frac{dV_C}{dt}\right] \geq 0. \tag{3.5}$$

Multiplying (3.1) by i and combining with (3.5), we get

$$\frac{C}{2}\frac{d}{dt}V_C^2 + \frac{L}{2}\frac{d}{dt}i^2 + Ri^2 \leq 0,$$

which implies, in view of (3.4),

$$\frac{C}{2}V_C^2(t) + \frac{L}{2}i^2(t) + R\int_0^t i^2(\tau)\,d\tau = 0,$$

for an arbitrary $t \in (0, T]$. That is,

$$V_C(t) \equiv 0, \ i(t) \equiv 0.$$

Step B

Let us define the function

$$\Pi_a : \mathbb{R} \to I_a$$

$$x \to \Pi_a(x) = \begin{cases} x & \text{if } x \in I_a, \\ V_a & \text{if } x > V_a, \\ -V_a & \text{if } x < -V_a, \end{cases}$$

and consider $j_\mu(x) = \frac{1}{\mu}(x - \Pi_a(x))$, any $\mu > 0$. We pose then the new problem

$$(3.6) \quad \begin{cases} \text{(a)} \quad L\dfrac{di_\mu}{dt} + Ri_\mu + V_C^\mu = E(t),\ t > 0, \\ \text{(b)} \quad i_\mu(t) = C\dfrac{dV_C^\mu}{dt} + j_\mu(V_C^\mu(t)),\ t > 0, \\ \text{(c)} \quad i_\mu(0) = 0, \quad V_C^\mu(0) = 0, \end{cases}$$

for the definition of approximations (candidates!) (i_μ, V_C^μ) to the eventual solution (i, V_C) of the original problem. Since, for a fixed μ, J_μ is Lipschitz continuous, the ordinary differential system (3.6) well defines the sequence (i_μ, V_C^μ). The objective is to show that any cluster point of $\{(i_\mu, V_C^\mu)\}_{\mu > 0}$ is a solution of (2.4), and for this the next a priori estimations of (3.6) are crucial.

Step C

The first set of estimates derive from the energy identity for system (3.6). We multiply (a) by $i_\mu(t)$, (b) by $V_C^\mu(t)$ and eliminate common factors to get

$$\frac{L}{2}\frac{d}{dt}i_\mu^2 + Ri_\mu^2 + \frac{C}{2}\frac{d}{dt}(V_C^\mu)^2 + j_\mu(V_C^\mu(t)\,V_C^\mu =$$

$$= E(t)\ i_\mu(t), \qquad \forall t > 0.$$

Integrating from 0 to t and taking (c) into account:

$$(3.7) \quad \frac{L}{2}i_\mu^2(t) + \frac{C}{2}V_C^{\mu^2}(t) + R\int_0^t i_\mu^2(\tau)\,d\tau + \int_0^t j_\mu(V_C^\mu(\tau))V_C^\mu(\tau)\,d\tau =$$

$$= \int_0^t E(\tau)\,i_\mu(\tau)\,d\tau.$$

Given any $T > 0$, identity (3.7) allows us to conclude that, for $\mu > 0$,

I. (a) i_μ remains in a bounded set of $L^\infty(0, T) \cap L^2(0, T)$,
(b) V_C^μ remains in a bounded set of $L^\infty(0, T)$,

in view of the fact that j_μ is monotone with $j_\mu(0) = 0$, and of the arithmetic inequality $a \cdot b \le (R/2)a^2 + (1/2R)b^2$, valid for numbers a and b. In fact we have

$$(3.8) \quad \frac{L}{2}|i_\mu|_\infty^2 + \frac{C}{2}|V_C^\mu|_\infty^2 + \frac{R}{2}|i_\mu|_2^2 \le \frac{1}{2R}|E|_2^2.$$

Next, take derivatives of (3.6) (a) and (b), and multiply by di_μ/dt and dV_C^μ/dt, respectively:

$$\frac{L}{2}\frac{d}{dt}\left(\frac{di_\mu}{dt}\right)^2 + R\left(\frac{di_\mu}{dt}\right)^2 + \frac{dV_C^\mu}{dt}\cdot\frac{di_\mu}{dt} = \frac{dE}{dt}\cdot\frac{di_\mu}{dt},$$

$$\frac{di_\mu}{dt}\cdot\frac{dV_C^\mu}{dt} = \frac{C}{2}\frac{d}{dt}\left(\frac{dV_C^\mu}{dt}\right)^2 +$$

$$+ \lim_{\Delta t \to 0}\frac{1}{\Delta t}[j_\mu(V_C^\mu(t + \Delta t)) - j_\mu(V_C^\mu(t))]\frac{dV_C^\mu}{dt}.$$

Now eliminate the common factor and observe that

$$\lim_{\Delta t \to 0}\frac{1}{(\Delta t)^2}\{[j_\mu(V_C^\mu(t + \Delta t)) - j_\mu(V_C^\mu(t))][V_C^\mu(t + \Delta t) - V_C^\mu(t)]\} \ge 0,$$

by the monotonicity of j_μ, to conclude, after integration on time,

$$\frac{L}{2}\left[\frac{di_\mu}{dt}(t)\right]^2 + \frac{C}{2}\left[\frac{dV_C^\mu}{dt}(t)\right]^2 + R\int_0^t\left[\frac{di_\mu}{d\tau}(\tau)\right]^2 d\tau \le$$

$$\le \frac{1}{2L}E^2(0) + \int_0^t \frac{dE}{d\tau}\frac{di_\mu}{d\tau}\,d\tau.$$

From the inequality we deduce as before that, for $\mu > 0$,

II. (a) $\dfrac{di_\mu}{dt}$ remains in a bounded set of $L^\infty(0, T) \cap L^2(0, T)$,
(b) $\dfrac{dV_C^\mu}{dt}$ remains in a bounded set of $L^\infty(0, T)$.

Finally we go into equation (3.6) (b) to conclude, after I and II, for $\mu > 0$,

III. $j_\mu(V_C^\mu(\cdot))$ remains in a bounded set of $L^\infty(0, T)$.

We observe the most important point that the constants in all the obtained estimates are allways independent of μ. They do depend only on the data of the original problem.

Step D

For any t_1, t_2 in $[0, T]$, we have, in consequence of II,

$$|i_\mu(t_1) - i_\mu(t_2)| = \left|\int_{t_1}^{t_2} \frac{di_\mu}{d\tau}(\tau)\, d\tau\right| \le C\,|t_1 - t_2|,$$

$$|V_C^\mu(t_1) - V_C^\mu(t_2)| = \left|\int_{t_2}^{t_1} \frac{dV_C^\mu}{d\tau}(\tau)\, d\tau\right| \le C\,|t_1 - t_2|,$$

where the C stands for different constants in different places. This shows that the families $\{i_\mu\}_{\mu>0}$ and $\{V_C^\mu\}_{\mu>0}$ are not only bounded in $C^0(0, T)$, but also equicontinuous. So, by Arzelà-Ascoli, there exist functions i, $V_C \in C^0(0, T)$ which are limits of sub-sequences $\{i_\nu\} \subset \{i_\mu\}_{\mu>0}$ and $\{V_C^\nu\} \subset \subset \{V_C^\mu\}_{\mu>0}$, respectively, as $\nu \to 0$.

We have also from I and II, and the compacity properties of bounded sets in L^p spaces [6], taking sub-sequences if necessary,

i_ν converges weakly to i in $L^2(0, T)$,

i_ν converges weakly-star to i in $L^\infty(0, T)$,

V_C^ν converges weakly-star to V_C in $L^\infty(0, T)$,

$\dfrac{di_\nu}{dt}$ converges weakly to $\dfrac{di}{dt}$ in $L^2(0, T)$,

$\dfrac{di_\nu}{dt}$ converges weakly-star to $\dfrac{di}{dt}$ in $L^\infty(0, T)$,

$\dfrac{dV_C^\nu}{dt}$ converges weakly-star to $\dfrac{dV_C}{dt}$ in $L^\infty(0,T)$,

when $\nu \to 0$.

Hence, (i, V_C) satisfies (2.6), (2.7). And if we take the limit $\nu \to 0$ in (3.6) (a), (3.6) (c), (3.8), we obtain (2.4) (ii), (2.8), (2.9), respectively. Since the last statement of Theorem 2.1 is a consequence of $i(t)$ being in $L^2(0, \infty)$ (if $E \in L^2(0, \infty)$) and uniformly continuous, it is left for the conclusion of the proof the checking of relations (2.4) (i) and (2.4) (iii).

For that we introduce the continuous functions

$$J_{I_a}^\nu(x) = \frac{1}{2\nu}\,[x - \Pi_a(x)]^2,$$

and

$$M(x) = \tfrac{1}{2}\,[x - \Pi_a(x)]^2, \qquad x \in \mathbb{R}.$$

We have

$$J_{I_a}^{\prime\nu}(x) = j_\nu(x), \tag{3.9}$$

$$M(x) \ne 0 \Leftrightarrow x \in \mathbb{R} - I_a, \tag{3.10}$$

$$M(x) = \nu\, J_{I_a}^\nu(x). \tag{3.11}$$

Now estimates I and III imply

$$|J_{I_a}^\nu(V_C^\nu(\cdot))|_\infty \le \tfrac{1}{2}\,|j_\nu(V_C^\nu(\cdot))|_\infty \{|V_C^\nu|_\infty + |\Pi_a \circ V_C^\nu|_\infty\} \le K, \tag{3.12}$$

where K is independent of ν. From this we deduce

$$0 \le M(V_C^\nu(t)) \le K\,\nu,$$

and taking the limit $\nu \to 0$,

$$0 = \lim_{\nu\to 0} M(V_C^\nu(t)) = M(V_C(t)).$$

Hence, from (3.10), $V_C(t) \in I_a$, for all $t \ge 0$, which is (2.4) (i).

The function $J_{I_a}^\nu(x)$ is convex, so that

$$J_{I_a}^\nu(v) - J_{I_a}^\nu(V_C^\nu(t)) - j_\nu(V_C^\nu(t))\,[v - V_C^\nu(t)] \ge 0, \qquad \forall v \in \mathbb{R}, \qquad t > 0.$$

Hence, by (3.6) (b) and (3.10), if we restrict ourselves to $v \in I_a$, we get

$$\left[C\,\frac{dV_C^\nu}{dt} - i_\nu\right][v - V_C^\nu(t)] \ge J_{I_a}^\nu(V_C^\nu(t)).$$

Taking the limit $\nu \to 0$ in the appropriate sense defined before,

$$\left[C\,\frac{dV_C}{dt} - i\right](t)\,[v - V_C(t)] \ge \tag{3.13}$$

$$\ge \lim_{\nu\to 0} J_{I_a}^\nu(V_C^\nu(t)), \qquad \forall v \in I_a, \quad \text{a.e. } t \text{ in } [0, T].$$

Now we define

$$\theta_a(x) = \frac{2}{3}|x - \Pi_a(x)|,$$

$$\theta_a^v(x) = \begin{cases} J_{I_a}^v(x) & \text{for} \quad x \in [-V_a - v, \quad V_a + v], \\ \frac{v}{2} + \frac{2}{3}(x - V_a - v) & \text{for} \quad x > V_a + v, \\ \frac{v}{2} - \frac{2}{3}(x + V_a + v) & \text{for} \quad x < -V_a - v. \end{cases}$$

It is clear that θ_a^v converges uniformly to θ_a and that

$$J_{I_a}^v(x) \geq \theta_a^v(x), \qquad x \in \mathbb{R}.$$

Hence,

$$J_{I_a}^v(V_C^v(t)) \geq \theta_a(V_C^v(t)) + [\theta_a^v(V_C^v(t)) - \theta_a(V_C^v(t))],$$

so that

$$\lim_{v \to 0} J_{I_a}^v(V_C^v(t)) \geq \theta_a(V_C(t)) = 0,$$

by virtue of (2.4) (i). This fact, together with (3.13), implies (2.4) (iii).

4. Numerical results. The theorem proving process of last section suggests natural schemes for the computation of a solution of our circuit problem: just discretize system (3.6) in any of the standard ways. For a fixed μ, j_μ is a nice monotone function, in condition to satisfy the convergence and stability requirements for the usual numerical processes [4]. The a priori estimates I, II and III, independent of μ, transfer those properties to the limit $\mu \to 0$.

We have chosen for our simulation essays a predictor-corrector scheme defined in the following way:

$$t_n = nH, \qquad 0 \leq n \leq N, \qquad H = \frac{T}{N},$$

$$\theta_n(t) = \begin{cases} 1 \text{ if } t \in [t_n, t_{n+1}), \\ 0 \text{ otherwise}, \end{cases}$$

$$I(t) = \sum_{n=0}^{N-1} \{I^n + (t - t_n)(I^{n+1} - I^n) H^{-1}\} \theta_n(t),$$

$$V_C(t) = \sum_{n=0}^{N-1} \{V^n + (t - t_n)(V^{n+1} - V^n) H^{-1}\} \theta_n(t),$$

$$Q(t) = \int_0^t I(\tau)\, d\tau = \sum_{n=0}^{N-1} H\left(\frac{I^{n+1} + I^n}{2}\right) =$$

= charge in the capacitor + added discharges at the arc,

where $(I^n, V^n)_{0 \leq n \leq N}$ are calculated through the algorithm

(S_0) $\quad I^0 = 0, \quad V^0 = 0;$

$(\hat{S}_n)$ I^n, V^n known, $n \geq 0$, compute predictors $\hat{I}^{n+1}$, $\hat{V}^{n+1}$ by

$$LH^{-1}(\hat{I}^{n+1} - I^n) + \frac{R}{2}(\hat{I}^{n+1} + I^n) + \frac{1}{2}(\hat{V}^{n+1} + V^n) = $$

$$= \frac{1}{2}(E(t_{n+1}) + E(t_n)),$$

$$\frac{1}{2}(\hat{I}^{n+1} + I^n) = CH^{-1}(\hat{V}^{n+1} - V^n) + j_\mu(V^n);$$

(S_n) I^n, V^n, $\hat{I}^{n+1}$, $\hat{V}^{n+1}$ known, compute correctors I^{n+1}, V^{n+1} by

$$LH^{-1}(I^{n+1} - I^n) + \frac{R}{2}(I^{n+1} + I^n) + \frac{1}{2}(V^{n+1} + V^n) =$$

$$= \frac{1}{2}(E(t_{n+1}) + E(t_n)),$$

$$\frac{1}{2}(I^{n+1} + I^n) = CH^{-1}(V^{n+1} - V^n) + j_\mu\left(\frac{\hat{V}^{n+1} + V^n}{2}\right);$$

(S_T) stop at $n = N - 1$.

Calculations were performed for different but all of them academic examples, with parameters values $\mu = 10^{-2}$, $T = 3$, $H = 10^{-2}$ and $R = L = C = 1$.

Example 1. Applied voltage was taken to be $E(t) = 5\,e^t$ and the critical voltage $V_a = 2.5$. Resulting curves are shown in Figure 4.

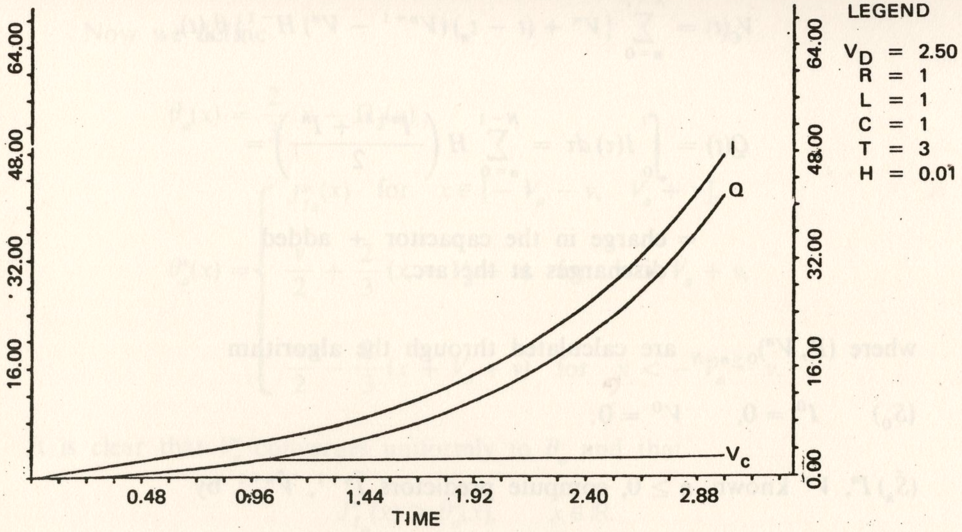


Figure 4

Example 2. Applied voltage was taken to be $E(t) = 20 \sin 10t$ and the critical voltage $V_a = 1.0$. Resulting curves are shown in **Figure** 5.

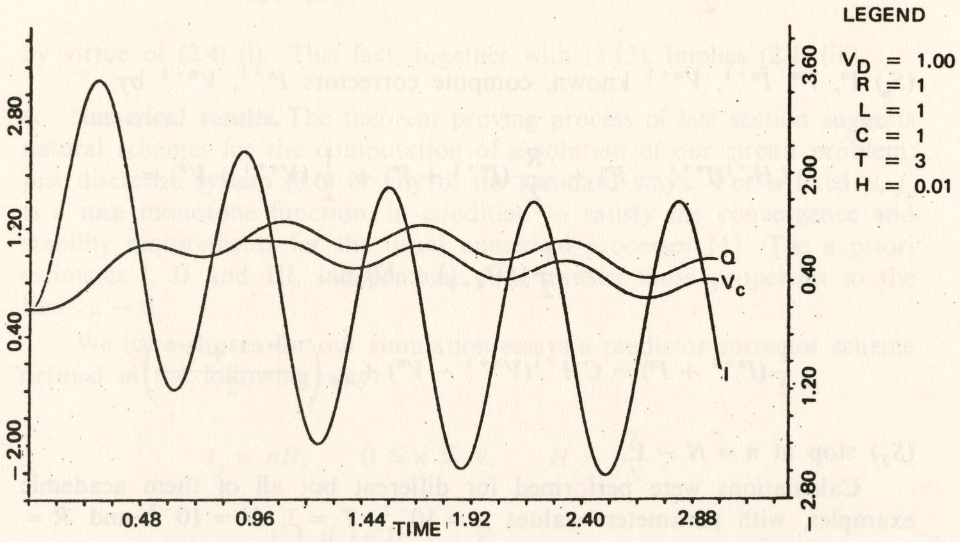


Figure 5

Example 3. Applied voltage was taken to be $E(t) = 10^2 \sin 10t$ and the critical voltage $V_a = 1.0$. Resulting curves are shown in Figure 6.

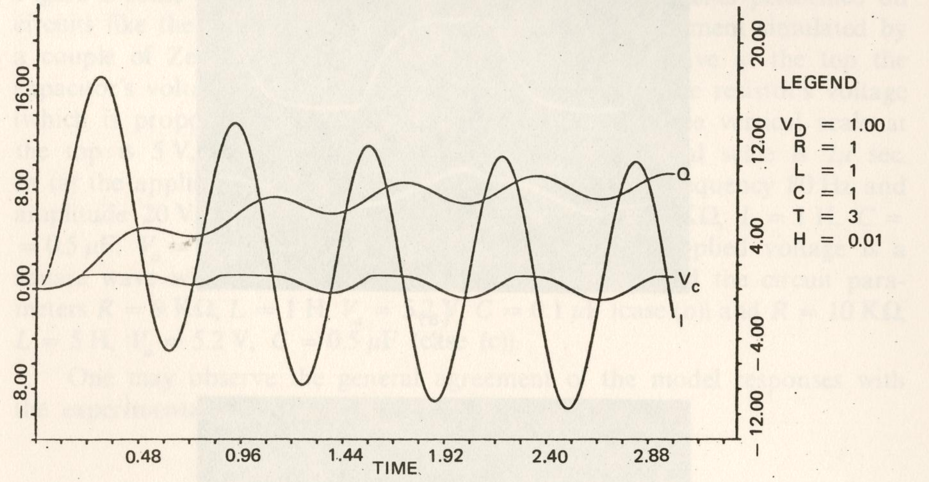


Figure 6

Example 4. Applied voltage was taken to be

$$E(t) = \begin{cases} +10^2 & t \in [j, j+1), \quad j = 0, 2, 4, \\ -10^2 & t \in [j, j+1), \quad j = 1, 3, 5, \end{cases}$$

the critical voltage $V_a = 1.0$, and $T = 6$. Resulting curves are shown in Figure 7.

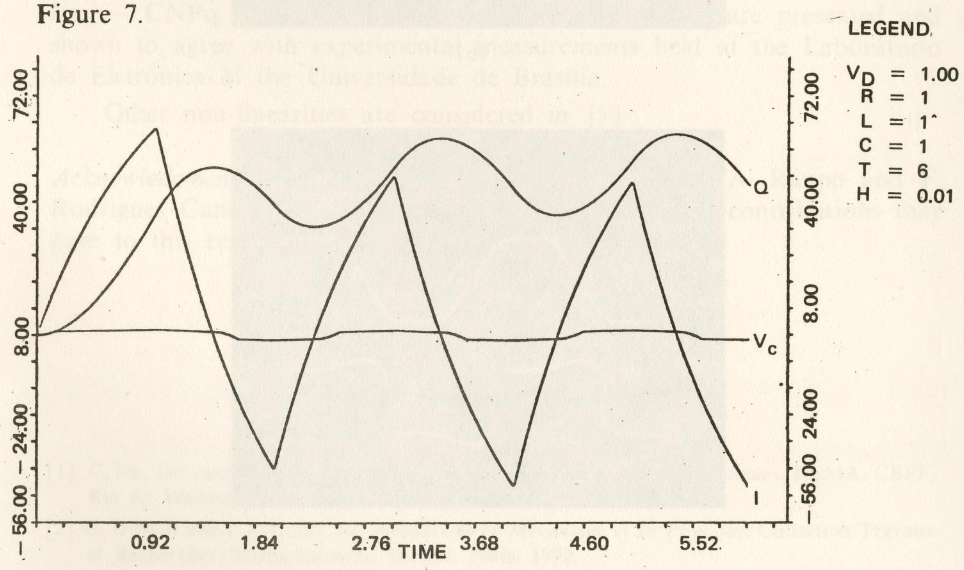
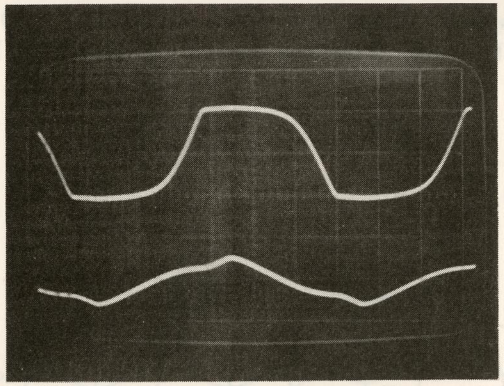
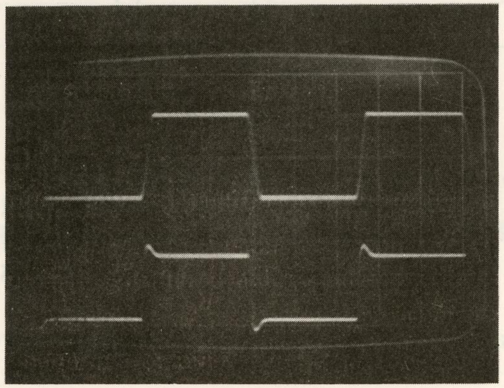


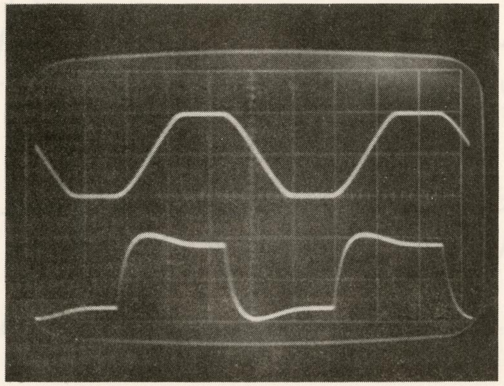
Figure 7



(a)



(b)



(c)

Figure 8

In order to allow the reader a qualitative physical evaluation of the theoretical and computational results presented here, we finally show in Figure 8 some osciloscope displays of voltage measurements performed on circuits like the one in Figure 1, with the electric arc element simulated by a couple of Zenner diodes. In each photograph we have at the top the capacitor's voltage V_C against time and at the bottom the resistor's voltage (which is proportional to the current) against time. The vertical scale at the top is 5 V, at the bottom 20 V, and the horizontal scale is 2μ sec. In (a) the applied voltage E is a triangular ware with frequency 80 Hz and amplitude 20 V, and the circuit parameters are $R = 2\,K\Omega$, $L = 1$ H, $C = = 0.5\,\mu F$, $V_a = 5.2$ V. In the other photographs the applied voltage is a square wave with frequency 100 Hz, amplitude 20 V, and the circuit parameters $R = 9\,K\Omega$, $L = 1$ H, $V_a = 5.2$ V, $C = 0.1\,\mu F$ (case (b)) and $R = 10\,K\Omega$, $L = 5$ H, $V_a = 5.2$ V, $C = 0.5\,\mu F$ (case (c)).

One may observe the general agreement of the model responses with the experimental measurements.

Conclusions.

A mathematical model (equation (2.4)) is proposed for the description of the response of an electric circuit consisting of elements of resistance, inductance, capacitance and an electrical arc represented in Figure 2, to an applied voltage $E(t)$. The well-posedness of this response problem is demonstrated and model consistent numerical calculations were performed in the IBM 370/145 computer at the Laboratório de Computação Científica — CNPq. A number of typical numerical results are presented and shown to agree with experimental measurements held at the Laboratório de Eletrônica of the Universidade de Brasilia.

Other non-linearities are considered in [5].

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References

[1] C. Do, *Les inequations variationelles en Mécanique des Millieux Continues*, I EMA, CBPF, Rio de Janeiro, 1978.

[2] G. Duvaut and J. L. Lions, *Les inequations en Mécanique et en Physique*, Collection Travaux et Recherches Mathematiques, Dunod, Paris, 1972.

[3] I. Ekeland and Temam, R., *Convex Analysis and Variational Problems North-Holland*, Amsterdam, 1976.

[4] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, John Wiley & Sons, New York, 1962.

[5] M. A. Raupp and O. R. Baiocchi, *Non linear Circuit transients*, to appear in Proceedings of the Seminar in Numerical Analysis and its Applications to Continuum Physics, held at CBPF, Rio de Janeiro, March 1980.

[6] K. Yosida, *Functional Analysis*, Springer, Berlin, (3rd ed.), 1971.

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