A characterization of tori with constant mean curvature in space form

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1. Introduction.

1.1. The purpose of this paper is to present complete proofs of some results stated in [11]. Let M^2 be a surface endowed with a Riemannian structure and $E^3(c)$ be the space form with curvature c. Let $x : M^2 \to E^3(c)$ be an isometric immersion, and denote by $H(x(p))$ the mean curvature of x at $p \in M$; the following definitions simplify the statement of the results below, and will be useful on the whole paper.

An H-deformation of x is a continuous map $F : (-\varepsilon, \varepsilon) \times M \to E^3(c)$ *such that, denoting* $x_t(p) = F(t, p)$, *we have:* i) x_t *is an isometric immersion,* ii) $x_0 = x$, iii) $H(x_t(p)) = H(x_0(p))$, *for each* $t \in (-\varepsilon, \varepsilon)$ *and* $p \in M$. An H-deformation is said to be *trivial* if, for each t, there exists an isometry L of $E^3(c)$, such that $x_t = L \circ x_0$. An isometric immersion x is *H-deformable* if it admits a non trivial H-deformation. X is said to be *locally H-deformable* if each point of M has a neighborhood restricted to which x is H-deformable. X is said to be *away from umbilics* if there exists a real number r such that $K - c - H^2 \leq r < 0$.

The following result gives a characterization of tori with constant mean curvature. In fact it is more general than that in [11].

1.2. Theorem. *Let M be a surface homeomorphic to the torus* T^2 *and let* $x : M \to E^3(c)$ *be a locally H-deformable isometric immersion. Then* $H(x) =$ $=$ *const.*

1.3. Remark. The converse is true for all surfaces by a Lawson result of Lawson [8].

If the immersion is analytic, then the hypothesis on x can be weakened.

1.4. Theorem. *Let M be a surface homeomorphic to the torus* T^2 *and let* $x : M \to E^3(c)$ *be an analytic isometric immersion. If there exists an open subset of M restricted to which x is H-deformable, then* $H(x) =$ *const.*

1.5. Remark. It follows from the proof of (1.2) that, if M is homeomorphic to T^2 and $H : M \to \mathbb{R}$ is a non constant function, then there exist at most two isometric immersions of M into $E^3(c)$ with mean curvature H.

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Klotz and Osserman in [7] classified the immersions into $\mathbb{R}^3$ of complete surfaces with $H = \text{const} \neq 0$ and Gaussian curvature K having constant sign. Observe that in the case $K \leq 0$, the hypothesis $H = \text{const} \neq 0$ implies that the immersion is away from umbilics. The next theorem generalizes the result of Klotz and Osserman for immersions into hyperbolic space ($c < 0$).

1.6. Theorem. *Let M be a complete surface and let $x : M \to E^3(c)$, $c < 0$, be an isometric immersion with $H(x) = \text{const}$. Assume that the Gaussian curvature K of M does not change sign; if $K \leq 0$, assume further that x is away from umbilics. Then $x(M)$ is either a geodesic sphere in $E^3(c)$ or else $x(M)$ is the set of equidistant point from a complete geodesic of $E^3(c)$.*

1.7. Remark. Hoffman in [4], generalized the result of [7] for immersions into 4-dimensional space form $E^4(c)$ with $c \geq 0$. Some related result was also obtained by Yau (see Remark 4.12).

1.8. Remark. The result of 1.6 can be extended for immersions into $E^4(c)$ with parallel mean curvature vector in the normal connection, using the Theorem 4 of Yau [12] to reduce the codimension.

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2. Symmetric Forms.

2.1. In what follows differentiable means C^∞. All manifolds will be differentiable as well as its maps vector fields and exterior forms.

Let E be a n-dimensional manifold, $\{\varepsilon_1, \dots, \varepsilon_n\}$ a frame defined on an open subset of E, $\{w_1, \dots, w_n\}$ its coframe and $\{w_{ij}\}$, $i, j = 1, \dots, n$ the associated connection forms. We have the following relations:

$$dw_i = \sum_k w_k \wedge w_{ki}, \qquad w_{ij} = -w_{ji}, \tag{2.2}$$

$$dw_{ij} = \sum_k w_{ik} \wedge w_{kj} + \Omega_{ij}, \tag{2.3}$$

where Ω_{ij}, $i, j = 1, \dots, n$ are the curvature forms of E. We say E has *constant curvature* $c \in \mathbb{R}$ if $\Omega_{ij} = -c w_i \wedge w_j$ everywhere on E. In this paper $E^3(c)$ will be a complete simply connected riemannian 3-dimensional manifold with constant curvature c, that is, it is a 3-dimensional space form. We will consider surface *immersions* in $E^3(c)$. All surfaces are oriented and have a riemannian *structure*.

Let M be a surface and $x : M \to E^3(c)$ an isometric immersion. For each $p \in M$ we may define a frame $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ on a neighbourhood of $x(p)$ adapted to the immersion. Since x is locally an embedding, we may identify M with $x(M)$ (locally). Under this identification $\{\varepsilon_1, \varepsilon_2\}$ is a local frame on M. The induced forms x^*w_i and x^*w_{ij} (which we will denote by w_i and w_{ij}) satisfy (2.2) and (2.3).

Using Cartan's Lemma and $w_3 = 0$ (from where it follows

$$0 = dw_3 = \sum_k w_k \wedge w_{k3})$$

we get

$$w_{i3} = \sum_j h_{ij} w_j, \qquad h_{ij} = h_{ji} \tag{2.4}$$

where h_{ij}, $i, j = 1, 2$, are differentiable functions. The *second fundamental form* h of the immersion is defined as $h = \sum_{i,j} h_{ij} w_i w_j$, and $H = \frac{1}{2} \sum_i h_{ii}$ is the *mean curvature* of the immersion.

The *gaussian* curvature K of M satisfies $dw_{12} = -K w_1 \wedge w_2$. Putting $w_{12} = a w_1 + b w_2$ we get

$$dw_{12} = \{-\varepsilon_2(a) + \varepsilon_1(b) + a^2 + b^2\} w_1 \wedge w_2;$$

then we have

$$K = \varepsilon_2(a) - \varepsilon_1(b) - a^2 - b^2 \tag{2.5}$$

If follows from (2.3) and (2.4) that

$$dw_{12} = \{-\det(h) - c\} w_1 \wedge w_2;$$

so

$$K = c + \det(h), \tag{2.6}$$

which is known as the Gauss equation.

By taking derivatives in (2.4), using (2.2) and comparing with (2.3), we get

$$-\varepsilon_2(h_{11}) + \varepsilon_1(h_{12}) + a(h_{11} - h_{22}) + 2bh_{12} = 0, \tag{2.7}$$

$$\varepsilon_1(h_{22}) - \varepsilon_2(h_{12}) + 2ah_{12} - b(b_{11} - b_{22}) = 0. \tag{2.8}$$

Equations (2.7) and (2.8) are called the *Codazzi equations*.

2.9. We say that a bilinear form T of M is differentiable if, its local expression in a frame is given by differentiable functions.

2.10. Definition. We say that a symmetric bilinear form T defined on M satisfies the Codazzi equations if

$$-\varepsilon_2(T_{11}) + \varepsilon_1(T_{12}) + a(T_{11} - T_{22}) + 2bT_{12} = 0 \tag{2.11}$$

and

$$\varepsilon_1(T_{22}) - \varepsilon_2(T_{12}) + 2aT_{12} - b(T_{11} - T_{22}) = 0 \tag{2.12}$$

where T_{ij} are the components of T in a local frame ε_1, ε_2 of M.

2.13. Consider the complex quadratic form $\psi(T)$ built from local components T_{ij} of a symmetric bilinear form T on M

$$\psi(T) = \{T_{11} - T_{22} - 2iT_{12}\}(w_1 + iw_2)^2.$$

2.14. Lemma. *The complex form $\psi(T)$ does not depend on the chosen frame: therefore, it is globally defined.*

A proof of this Lemma may be found in [3].

2.15. Remark. Let us denote by $U(\theta)$ the θ-rotation on each tangent space of M (in the positive sense of the orientation), and let $\bar{T}$ be another symmetric bilinear form on M. Suppose that the matrices of T, $\bar{T}$ and $U(\theta)$ on the basis $\{\varepsilon_1, \varepsilon_2\}$ (denoted by $[T]$, $[\bar{T}]$ and $[U(\theta)]$ respectively) satisfy $[\bar{T}] = [U(\theta)]\,[T]\,[U(\theta)]^{-1}$. An easy computation give us:

$$\psi(\bar{T}) = e^{2i\theta}\psi(T).$$

2.16. For each $p \in M$ there exists a neighbourhood V of p with a conformal parametrization, that is, there exists an open subset $U \subseteq \mathbb{R}^2$ and a diffeomorphism $\varphi : U \to V$ satisfying

$$\left|\frac{\partial\varphi}{\partial x}\right| = \left|\frac{\partial\varphi}{\partial y}\right| \quad \text{and} \quad \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y} \right\rangle = 0,$$

where x and y are $\mathbb{R}^2$-coordinates. A proof of this may be found in [2].

2.17. Definition. A complex quadratic form ψ defined on M is *holomorphic* if its expression in a conformal coordinate system is given by $\psi = f dz^2$, where $z = x + iy$ and f is a holomorphic function on U (for simplicity, we identify ψ and $\varphi^*(\psi)$, where $\varphi : U \to M$ is a chart.

2.18. Lemma. *Let T be a simmetric bilinear form on a connected surface M. If any two of the conditions below are satisfied, then all of them hold*

(i) *the trace of T is constant.*

(ii) *T satisfies Codazzi equations.*

(iii) *$\psi(T)$ is holomorphic.*

Proof. Let $\varphi : U \to V \subseteq M$ be a conformal parametrization. We take $\varepsilon_1 = 1/\lambda \times \partial\varphi/\partial x$ and $\varepsilon_2 = 1/\lambda \times \partial\varphi/\partial y$, where $\lambda = |\partial\varphi/\partial x| = |\partial\varphi/\partial y|$. The corresponding coframe is given by $w_1 = \lambda dx$ and $w_2 = \lambda dy$. Easy computations show that

$$w_{12} = -\frac{1}{\lambda}\varepsilon_2(\lambda)\,w_1 + \frac{1}{\lambda}\varepsilon_1(\lambda)\,w_2.$$

Therefore the Codazzi equations are written as

$$-\varepsilon_2(T_{11}) + \varepsilon_2(T_{12}) - \frac{1}{\lambda}\varepsilon_2(\lambda)(T_{11} - T_{22}) + \frac{2}{\lambda}\varepsilon_1(\lambda)\,T_{12} = 0, \tag{2.19}$$

$$\varepsilon_1(T_{22}) - \varepsilon_2(T_{12}) - \frac{2}{\lambda}\varepsilon_2(\lambda)\,T_{12} - \frac{1}{\lambda}\varepsilon_1(\lambda)(T_{11} - T_{22}) = 0. \tag{2.20}$$

Since $(w_1 + iw_2)^2 = \lambda^2 dz^2$, $\psi(T) = (T_{11} - T_{22} - 2iT_{12})\,\lambda^2 dz^2$. Let us find under what conditions the function $f = (T_{11} - T_{22} - 2iT_{12})\,\lambda^2$ satisfies Cauchy-Riemann equations. For that, we write $f = U + iV$, where $U = (T_{11} - T_{22})\,\lambda^2$ and $V = -2T_{12}\,\lambda^2$. Since φ is conformal, f satisfies Cauchy-Riemann equations if and only if

$$\varepsilon_1(U) - \varepsilon_2(V) = \varepsilon_2(U) + \varepsilon_1(V) = 0.$$

A straighforward computation give us

$$\varepsilon_1(U) - \varepsilon_2(V) =$$

$$= 2\lambda^2\left\{\frac{1}{2}[\varepsilon_1(T_{11}) - \varepsilon_1(T_{22})] + \varepsilon_2(T_{12}) + \frac{1}{\lambda}\varepsilon_1(\lambda)(T_{11} - T_{22}) + \frac{2}{\lambda}\varepsilon_2(\lambda)\,T_{12}\right\}$$

and

$$\varepsilon_2(U) + \varepsilon_1(V) =$$

$$= 2\lambda^2\left\{\frac{1}{2}[\varepsilon_2(T_{11}) - \varepsilon_2(T_{22})] - \varepsilon_1(T_{12}) + \frac{1}{\lambda}\varepsilon_2(\lambda)(T_{11} - T_{22}) - \frac{2}{\lambda}\varepsilon_1(\lambda)T_{12}\right\}.$$

Therefore, f is holomorphic if and only if the equations below are satisfied:

$$\text{(2.21)}\quad \frac{1}{2}\{\varepsilon_1(T_{11}) - \varepsilon_1(T_{22})\} + \varepsilon_2(T_{12}) + \frac{1}{\lambda}\varepsilon_1(\lambda)(T_{11} - T_{22}) + \frac{2}{\lambda}\varepsilon_2(\lambda)T_{12} = 0$$

$$\text{(2.22)}\quad \frac{1}{2}\{\varepsilon_2(T_{11}) - \varepsilon_2(T_{22})\} - \varepsilon_1(T_{12}) + \frac{1}{\lambda}\varepsilon_2(\lambda)(T_{11} - T_{22}) - \frac{2}{\lambda}\varepsilon_2(\lambda)T_{12} = 0.$$

We suppose now that (i) holds and we will show that (ii) is equivalent to (iii). Since the T-trace is constant, we have $\varepsilon_i(T_{11}) = -\varepsilon_i(T_{22})$, $i = 1, 2$, from where (2.19) is equivalent to (2.22), and (2.20) to (2.21).

Now, if (ii) and (iii) are true then (i) holds. In fact, from (2.20) and (2.21) we get $\varepsilon_1(T_{11}) + \varepsilon_1(T_{22}) = 0$. From (2.19) and (2.22) we get $\varepsilon_2(T_{11}) + \varepsilon_2(T_{22}) = 0$. But, M is connected, so $T_{11} + T_{22}$ is constant.

2.23. Let $f : M \to \mathbb{R}$ be a differentiable function. Using a local frame and the associated forms w_i, w_{ij}, $i, j = 1, 2$, we may write $df = \sum_i f_i w_i$ and define $f_{i;j}$ by the equations

$$\sum_j f_{i;j} w_j = df_i + \sum_j w_{ji}, \qquad i, j = 1,2.$$

If we define the *laplacian* of f by $\Delta f = \sum_i f_{i;i}$ and write $w_{12} = aw_1 + bw_2$ we get

$$\text{(2.24)}\qquad \Delta f = \varepsilon_1\varepsilon_1(f) + \varepsilon_2\varepsilon_2(f) - a\varepsilon_2(f) + b\varepsilon_1(f).$$

2.25. We say $p \in M$ is *umbilic* with respect to a given simmetric bilinear form T if there exists $c \in \mathbb{R}$ such that $T(x, x) = c$, for every $x \in T_pM$ with $|x| = 1$. We remark that in those points the matrix (T_{ij}) of T (in any basis) satisfies $T_{ij} = c\delta_{ij}$, where δ_{ij} is the *Kronecker symbol*. It is easy to verify that $p \in M$ is umbilic with respect to T if and only if 4 det T = {Trace of $T\}^2$ in p.

In what follows T will be a symmetric bilinear form on a surface M satisfying the Codazzi equations and with constant *trace t*. D will denote the determinant of T.

2.26. Lemma. *For each non umbilic point of M (with respect to T) one has*

$$\Delta D = (4D - t^2)\left\{K + \frac{4\,|dD|^2}{(4D - t^2)^2}\right\}.$$

Proof. On a neighbourhood of each non umbilic point there exists a frame $\{\varepsilon_1, \varepsilon_2\}$ that diagonalizes T, since T is symmetric and varies differentiably. The Codazzi equations give us

$$-\varepsilon_2(T_{11}) + a(T_{11} - T_{22}) = 0$$

$$\varepsilon_1(T_{22}) - b(T_{11} - T_{22}) = 0.$$

As $D = T_{11}T_{22}$ and $dT_{11} = -dT_{22}$, we get from the above equations

$$\text{(2.27)}\qquad \varepsilon_1(D) = b(T_{11} - T_{22})^2$$

and

$$\text{(2.28)}\qquad \varepsilon_2(D) = -a(T_{11} - T_{22})^2.$$

By using (2.27), (2.28), the Codazzi equations and $dT_{11} = -dT_{22}$, we get

$$\varepsilon_1\varepsilon_1(D) = (T_{11} - T_{22})^2\{\varepsilon_1(b) - 4b^2\}$$

and

$$\varepsilon_2\varepsilon_2(D) = -(T_{11} - T_{22})^2\{\varepsilon_2(a) + 4a^2\},$$

from where, and by (2.24), we get

$$\Delta(D) = (T_{11} - T_{22})^2\{\varepsilon_1(b) - \varepsilon_2(a) - 3b^2 - 3a^2\}.$$

As $(T_{11} - T_{22})^2 = t^2 - 4D$ and

$$a^2 + b^2 = \frac{|dD|^2}{(t^2 - 4D)^2},$$

using (2.5), we finally get

$$\Delta(D) = \{4D - t^2\}\left\{K + \frac{4\,|dD|^2}{(t^2 - 4D)^2}\right\}.$$

2.29. Lemma. *If* $t = 0$, *we have*

$$\Delta\{\log(-D)\} = 4K$$

in points where $D \neq 0$.

Proof. If follows easily from the preceding lemma.

2.30. Lemma. *Suppose* $t^2 - 4D \geq \alpha > 0$. *Then, the metric* $\sqrt{t^2 - 4D}\, ds^2$, *where* ds^2 *is the metric of M, has zero curvature. Besides, if M is complete, then M is parabolic.*

This is essentialy lemma 4 from [7].

2.31. Lemma. *The set of symmetric bilinear forms on M that satisfy the Codazzi equations is a subspace of the vector space formed by the symmetric bilinear forms on M.*

3. Proof of Theorem 1.2.

3.1. First of all we will consider local immersions that admit non-trivial H-deformations in the neighborhood of non-umbilic points. Scherrer in [9] has a characterization of these immersions. The method and result to be described here are different from [9].

Let $x : M \to E^3(c)$ be an immersion. For each point $p \in M$, there exists a neighborhood V of p and a conformal parametrization $\varphi : U \to V$, where U is open in $\mathbb{R}^2$. The quadratic form $\psi(h)$ of Lemma 2.14 can be described in this neighborhood by $\psi(h) = f(z)\,dz^2$, where $z = x + yi$ and $(x, y) \in \mathbb{R}^2$.

We wil consider the following operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

By the Cauchy-Riemann equations, a complex function $h(z)$ is holomorphic if and only if $\partial h/\partial \bar{z} = 0$ (or $\partial \bar{h}/\partial z = 0$).

3.2. Proposition. *Let $V \subset M$ be open and simply connected, let $\varphi : U \to V$ be a conformal parametrization and let $x : V \to E^3(c)$ be an isometric immersion without umbilic points. Then, x admits a non-trivial H-deformation if and only if the complex function f defined in U by $\psi(h) = f(z)\,dz^2$, satisfies*

$$\Delta^0 (\log f) = 4\left|\frac{\partial}{\partial \bar{z}}(\log f)\right|^2,$$

where Δ^0 is the Laplacian in the canonical metric of $\mathbb{R}^2$ given by

$$\Delta^0 = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}.$$

Proof. For each real function θ, let $U(\theta(p))$ be the rotation of angle $\theta(p)$ in T_pM in the positive direction of the given orientation. We point out that if there exists an H-deformation x_t of x, the matrices of h and h_t are similar, where h_t is the second fundamental form of the immersion x_t.

The reason is that h and h_t are symmetric, with the same trace and the same determinant. It is not difficult to show that for all $t \in (-\varepsilon, \varepsilon)$ there exists a function θ_t such that

$$[h_t] = [U(\theta_t)]\,[h]\,[U(\theta)]^{-1},$$

where $[h_t]$, $[U(\theta_t)]$ and $[h]$ are matrices of h_t, $U(\theta_t)$ and h respectively.

By the Remark (2.15), it follows

$$\psi(h_t) = e^{2i\theta_t}\,\psi(h). \tag{3.3}$$

Since by 2.31, $h - h_t$ satisfies Codazzi equations and its trace is zero, it follows from Lemma 2.18 that $\psi(h - h_t)$ is holomorphic.

Since $f(z)\,dz^2$ is the expression of $\psi(h)$ in U, it follows from (3.3) that $f(1 - e^{i2\theta_t})\,dz^2$ is the expression of $\psi(h - h_t)$. Therefore the function $f(1 - e^{i2\theta_t})$ is holomorphic.

We will now consider the following families of functions: $\eta_t = 2\theta_t$ and $g_t = e^{i\eta_t}$. The functions g_t satisfy $\partial g_t/\partial \varphi_t = i g_t$, $\bar{g}_t = g_t^{-1}$ and $\partial \bar{g}_t/\partial \varphi_t = -i\bar{g}_t$. To simplify the notation, we will write in the computations $\eta = \eta_t$ and $g = g_t$. Since $f(1 - g)$ is holomorphic, then

$$0 = \frac{\partial\{f(1-g)\}}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}}(1 - g) - ifg\frac{\partial \eta}{\partial \bar{z}}.$$

Therefore,

$$\frac{\partial \eta}{\partial \bar{z}} = -i(\bar{g} - 1)\frac{\partial}{\partial \bar{z}}(\log f). \tag{3.4}$$

Since $f(1 - g)$ is holomorphic, we have

$$0 = \frac{\partial\{\bar{f}(1-\bar{g})\}}{\partial z} = \frac{\partial \bar{f}}{\partial z}(1 - \bar{g}) + i\bar{f}\bar{g}\frac{\partial \eta}{\partial z}.$$

Hence

$$\frac{\partial \eta}{\partial z} = i(g - 1)\frac{\partial}{\partial z}(\log \bar{f}) = 0. \tag{3.5}$$

By the Schwarz theorem,

$$\frac{\partial^2 \eta}{\partial z\,\partial \bar{z}} = \frac{\partial^2 \eta}{\partial \bar{z}\,\partial z}$$

Then

$$\frac{\partial}{\partial z}\left\{(\bar g - 1)\frac{\partial}{\partial \bar z}(\log f)\right\} + \frac{\partial}{\partial \bar z}\left\{(g - 1)\frac{\partial}{\partial \bar z}(\log \bar f)\right\}.$$

By derivation, using (3.4) and (3.5), and the fact that

$$\frac{\partial f}{\partial \bar z} = \overline{\left(\frac{\partial \bar f}{\partial z}\right)}$$

and

$$\log \bar f = \overline{(\log f)},$$

we obtain

$$\frac{\partial}{\partial z}\left\{(\bar g - 1)\frac{\partial}{\partial \bar z}(\log g)\right\} = (\bar g - 1)\left\{\frac{1}{4}\Delta^{o}(\log f) - \left|\frac{\partial}{\partial \bar z}(\log f)\right|^2\right\}$$

and

$$\frac{\partial}{\partial \bar z}\left\{(g - 1)\frac{\partial}{\partial z}(\log \bar f)\right\} = (g - 1)\left\{\frac{1}{4}\Delta^{o}(\log \bar f) - \left|\frac{\partial}{\partial z}(\log f)\right|^2\right\}.$$

Then

$$(\bar g - 1)\left\{\frac{1}{4}\Delta^{o}(\log f) - \left|\frac{\partial}{\partial \bar z}(\log f)^2\right|^2\right\} +$$

$$+ (g - 1)\left\{\frac{1}{4}\Delta^{o}(\log \bar f) - \left|\frac{\partial}{\partial \bar z}(\log f)\right|^2\right\} = 0.$$

Multiplying this equation by g, we get

$$\left\{\frac{1}{4}\Delta^{o}(\log f) - \left|\frac{\partial}{\partial \bar z}(\log f)\right|^2\right\} - \tag{3.6}$$

$$- g\left\{\frac{1}{4}\Delta^{o}(\log f) + \Delta^{o}(\log \bar f) - 2\left|\frac{\partial}{\partial \bar z}(\log f)\right|^2\right\} +$$

$$+ g^2\left\{\frac{1}{4}\Delta^{o}(\log \bar f) - \left|\frac{\partial}{\partial \bar z}(\log f)\right|^2\right\} = 0.$$

We point out that the left hand side of equation (3.6) is a complex polynomial P of degree 2 which vanishes in g. Therefore, if for a point in U the family of functions g assume more than two values, then the coeficients of P are zero in this point. If this happens in a dense subset of U, then $P \equiv 0$. In this case

$$\Delta^{o}(\log f) = 4\left|\frac{\partial}{\partial \bar z}(\log f)\right|^2.$$

Therefore, we just have to show that there exists such a dense set. If, for each isometry L of $E^3(c)$, $x_{t_1} \neq L \circ x_{t_2}$, then it is not difficult to show that there exist an open and dense subset U_{12} in U, where $g_{t_1} \neq g_{t_2}$. This can be done by using Lemma 2.18 for $h_{t_1} - h_{t_2}$ and the fact that the zeroes of holomorphic functions are isolated. Let t_k, $k = 1,2,3$, satisfy the conditions above and take $V = \cap (U_{ij})$. Then V is dense in U and g_t assumes more then two values.

Suppose now that

$$\Delta^{o}(\log f) = 4\left|\frac{\partial}{\partial z}(\log f)\right|^2.$$

We will show that there exists a non-trivial H-deformation of x.

We will use the next result, to obtain the existence of functions η_t, satisfying the equations (3.4) and (3.5).

3.7. Proposition. *The system of equations*

$$\frac{\partial w}{\partial z} = A(z, w)$$

$$\frac{\partial w}{\partial \bar z} = B(z, w).$$

where w is any real function defined in an open subset of the complex plane, has local solution if and only if $\partial A/\partial \bar z = \partial B/\partial z$. The solution is unique if we have the initial condition $w(z_0) = t$.

This proposition follows from the existence and uniqueness of solutions of ordinary differential equations, and from the differentiability of solutions with respect to initial conditions.

In the system (3.4) (3.5) we have:

$$A = i(g - 1)\frac{\partial}{\partial z}(\log f),$$

$$B = -i(\bar{g}-1)\frac{\partial}{\partial \bar{z}}(\log f).$$

As we have seen, $\partial A/\partial \bar{z} - \partial B/\partial z = 0$ is equivalent to the equation (3.6), and (3.6) follows from

$$\Delta^o(\log f) = \left|\frac{\partial}{\partial \bar{z}}(\log f)\right|^2.$$

By proposition 3.2, for each $t \in \mathbb{R}$ there exists a function η_t satisfying (3.4), (3.5) and $\eta_t(p_0) = t$, $p_0 \in U$. Let $\theta_t = \frac{1}{2}\eta_t$ and consider the bilinear form h_t given by

$$[h_t] = [U(\theta_t)]\,[h]\,[U(\theta_t)]^{-1}.$$

The forms h_t will be the second fundamental forms of immersions x_t. To show that, we will show that h_t satisfy the Gauss and Codazzi equations. Since h_t and h have the same trace, the forms $h - h_t$ have trace equal to zero. By Lemma 2.28, $h - h_t$ satisfy the Codazzi equations if and only if $\psi(h - h_t)$ are holomorphic. Since the forms h_t were obtained by functions η_t satisfying (3.4) and (3.5), $\psi(h - h_t)$ are holomorphic. Therefore, $h - h_t$ satisfy the equations of Codazzi. Since h satisfies the Codazzi equations, then by Lemma 2.31 the same hold for h_t.

Now, by taking initial conditions in $p_0 \in U$, namely $x_t(p_0) = x(p_0)$ and $dx_t(p_0) = dx(p_0)$, we will obtain by the theorem of existence and uniqueness for immersions the H-deformation that we want.

3.8. We will give now the proof of Theorem 1.2. Let $x : M \to E^3(c)$ a locally H-deformable isometric immersion, where M is a surface isometric to the torus T^2. We will show that $H(x) = $ constant. Let $\varphi : \mathbb{R}^2 \to M$ be a conformal covering. It is easy to see that the conclusions of Propositions (3.2) are still true in this case. That is,

$$\Delta^o(\log f) = \left|\frac{\partial}{\partial \bar{z}}(\log f)\right|^2,$$

where f is defined in $\mathbb{R}^2$. Therefore, the imaginary part of $\log f$ is harmonic, and the real part of $\log f$ is subharmonic. As $|f|$ is bounded, $\log|f|$ is bounded from above. The points where $\log f$ is not defined, that is, the umbilic points, are isolated. This follows using Lemma (2.18) for $h - h_t$, for each t of the deformation, and using the fact that the zeroes of non-constant holomorphic functions are isolated. Then, we can conclude that $\log|f|$ is constant [1]. Therefore, $\Delta^o \log|f| = 0$ and

$$0 = \Delta^o(\log f) = \left|\frac{\partial}{\partial \bar{z}}(\log f)\right|^2$$

It follows that $\log f$ and hence f is holomorphic. Since h satisfies the Codazzi equations and $\psi(h)$ is holomorphic, it follows from Lemma 2.18 that $H(x) = $ constant.

3.9. Theorem 1.3 is a consequence of proposition 3.2 using the same argument as before and the analiticity of x.

4. Proof of Theorem 1.6.

4.1. The idea is to show first that immersions in the hypothesis come from surfaces with $K = $ constant. Then, we show that these surfaces immersed in $E^3(c)$ with $H(x) = $ constant are either umbilic, or have $K \equiv 0$. Finally, we describe surfaces with $K \equiv 0$, immersed in $E^3(c)$ with $H = $ = constant.

4.2. Proposition. *Let M be a complete surface and $x : M \to E^3(c)$ an isometric immersion with $H(x) = $ constant. If $K \geq 0$, or $K \leq 0$ and $K - c - H^2 \leq$ const < 0, then $K = $ constant.*

Proof. Suppose $K \geq 0$. We first observe that if x is umbilic, then $K = $ = const. This follows from the Gauss' equation $K = c + H^2$. Suppose x is not umbilic. Then, since $\psi(h)$ is holomorphic (Lemma 2.18) and the zeroes of $\psi(h)$ are precisely the umbilic points, these points are isolated. By Lemma (2.26),

$$\Delta K = \Delta(K - c) = 4(K - c - H^2)\left(K + \frac{1}{4}\,\frac{|d|^2}{(K - c - H^2)^2}\right) \leq 0$$

at the non-umbilic points. By continuity, $\Delta K \leq 0$.

Since M is complete and $K \geq 0$, it follows from [6] that M is compact or parabolic. As K is superharmonic and bounded from below, then $K \equiv$ constant.

Suppose now that $k \leq 0$ and $K - c - H^2 \leq$ const < 0. Consider the form $T = h - Hds^2$. Since h and Hds^2 satisfy the Codazzi equations by (2.31), T satisfies them too. Let $D = \det(T)$.

Then

$$D = K - c - H^2 \leq \text{const.} < 0.$$

Since trace $(T) = 0$, it follows from Lemma 2.30 that M is parabolic. Moreover, the function $\log(-D)$ is defined and bounded from below.

It follows from Lemma 2.29 that $\Delta\{\log(-D)\} \leq 0$. Hence $D =$ constant. Since by 2.26, $0 = \Delta D = 4KD$ and $D \neq 0$, then $K \equiv 0$.

4.3. Observe that the immersed surfaces in the hypothesis of Theorem 1.6 have $K =$ const ≥ 0.

4.4. Proposition. *Let M be a surface of constant curvature K and let $x : M \to E^3(c)$ be an isometric immersion with $H(x) =$ const. Then, either x is umbilic or K equals 0.*

Proof. Let $D = \det(h) = K - c$. Then D is constant and hence $\Delta D = 0$. By Lemma (2.26) if $p \in M$ is non-umbilic then

$$\Delta D(p) = 4(D - H^2) K = 0.$$

Moreover, as $D \neq H^2$ at non-umbilic points, $K(p) = 0$. Hence, since K is constant, if there exist a non-umbilic point in M, K equals 0.

4.5. Proposition. *Let M and N be complete surfaces with the same curvature $K =$ constant, and let $x : M \to E^3(c)$ and $y : N \to E^3(c)$ be isometric immersions, with the same mean curvature $H =$ constant. Then, there exists an isometry L of $E^3(c)$, such that $y(N) = L \circ x(M)$.*

Proof. We can suppose without lost of generality that M and N are simply connected. In fact, if it is not the case we can use the universal covering. By a theorem of Cartan, there exists an isometry $f : M \to N$. Consider the immersion $\tilde{x} = y \circ f$ of M in $E^3(c)$. If x is umbilic, apply 2.18 to $h - \tilde{h}$ to conclude that there exists an isommetry L of $E^2(c)$ such that $\tilde{x}(M) = L \circ x(M)$. Hence, $y(N) = \tilde{x}(M)$ coincides with $x(M)$ except for an isometry of $E^3(c)$. If x is not umbilic, it follows from Proposition (4.4) that $K \equiv 0$. We shall use $\tilde{x}$ to build another immersion $\hat{x}$ whose second fundamental form $\hat{h}$ coincides with h.

4.6. The forms h hand $\tilde{h}$ determine in M two pairs of line fields $\{v_1, v_2\}$ and $\{\tilde{v}_1, \tilde{v}_2\}$ which are the principal directions line-fields. Since M is simply connected, we may orient these line fields. We shall then have the frames $\{\varepsilon_1, \varepsilon_2\}$ and $\{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2\}$ obtained from the line fields $\{v_i\}$ and $\{\tilde{v}_i\}$. It follows easily that these frames may be taken to be positive. Let $t(p)$ be the angle between ε_1 and $\tilde{\varepsilon}_1$ in T_pM and $U(t(p))$ the rotation in T_pM of angle $t(p)$. It is easy to show that $\tilde{h}(p) = U(t(p))\, h(p)\, U(t(p))^{-1}$. From 2.15 we have

$$\psi(\tilde{h}(p)) = e^{2it(p)}\, \psi(h(p)).$$

Since $\psi(\tilde{h})$ and $\psi(h)$ are holomorphic forms and $|e^{it(p)}| = 1$, we conclude that t is a constant function.

4.7. Suppose there exists an isometry $g : M \to M$ such that $dg_p = U(t)(p)$ for each $p \in M$. Then $dg\, \varepsilon_i = \tilde{\varepsilon}_i$, $i = 1,2$. Hence if $\hat{x} = x \circ g$, then $\hat{x}$ is an isometric immersion of M whose 2^{nd} fundamental form $\hat{h}$ is diagonalized by $\{\varepsilon_1, \varepsilon_2\}$. Apply the same argument in 4.6 now to x and $\hat{x}$ to conclude that $\hat{h} = h$. By the theorem of uniqueness of immersions, x and $\hat{x}$ differ by an isometry in $E^3(c)$ [10]. Then $y(N) = \hat{x}(M)$ coincides with $x(M)$ except for an isometry in $E^3(c)$. It suffices to show that there exists an isometry g in the hypotheses above. But this follows from the fact that M is isometric to $\mathbb{R}^2$.

4.8. We are now able to prove theorem 1.6. Let $x : M \to E^3(c)$ $c < 0$, an immersion in the hypothesis of 1.6. By Proposition 4.2 and 4.4, x is umbilic or $K \equiv 0$. Without loss of generality, we suppose $H \geq 0$.

If $K > 0$, it is not difficult to show that $x(M)$ is a geodesic sphere. To do so, we prove that for each $t > 0$ there exists a geodesic sphere with mean curvature $H = t$ and apply Proposition 4.5.

Suppose $K \equiv 0$. It follows from the Gauss equation that $H \geq \sqrt{-c}$. The horospheres are complete surfaces with $K \equiv 0$ contained in $E^3(c)$ with $H = \sqrt{-c}$. The inclusion of the horospheres are umbilic ([10] vol. IV, pg. 170) and hence do not satisfy $K - c - H^2 <$ const < 0. For each $t > \sqrt{-c}$, we shall show that there exist a surface equidistant from a geodesic with $H = t$.

For each $\rho > 0$, let $S(\rho)$ be the surface of the points at a distance ρ from a complete geodesic in $E^3(c)$. The principal curvatures in $S(\rho)$ are given by:

$$k_1 = \sqrt{-c}\ \operatorname{cotgh}(\rho\sqrt{-c})$$

and

$$k_2 = \sqrt{-c}\ \operatorname{tgh}(\rho\sqrt{-c}).$$

The mean curvature H of the inclusion of $S(\rho)$, given by

$$H = H(\rho) = \frac{\sqrt{-c}}{2}\{\operatorname{cotgh}(\rho\sqrt{-c}) + \operatorname{tgh}(\rho\sqrt{-c})\} \tag{4.9}$$

is constant. Moreover by Gauss equation we have $K \equiv 0$.

Given $t > \sqrt{-c}$, it follows from (4.9) that we can choose ρ_t such that $H = t$ for $S(\rho_t)$. By Proposition 4.5, $S(\rho_t)$ is the only surface in the conditions of theorem 1.6 which satisfies $H = t$.

4.10. Corolary. *Let M be a complete surface and let $x : M \to E^3(c)$, $c < 0$, be an isometric immersion with $H(x) =$ constant. If $K \leq c$, then $x(M)$ is a minimal surface in $E^3(c)$.*

4.11. Corollary. *Let M be a complete surface with $K \leq 0$ and $K \not\equiv 0$. If $x : M \to E^3(c)$ is an isometric immersion with $H(x)$ = constant then $c \leq$ $\leq -H^2$.*

The proofs of these corollaries are immediate from 4.2 and 4.3.

4.12. Remark. The fact that if $K \geq 0$ then x is either umbilic or $K \equiv 0$ was also proved by Yau [12]. Actually the diverse pieces of the proof of Theorem 1.6 are more or less known. However we could not find in the literature its statements.

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