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Regular sequences of minors, II

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§1. Introduction and Notations.

In order to answer partially a question posed by Lazard [3, Remarque 6] concerning regular sequences of minors of an mxn $(m \leqslant n)$ matrix, we have studied in [1] the regular sequences of 2x2 minors of the 2xn generic matrix. In this paper we will establish some results concerning regular sequences of 2x2 minors of the mxn generic matrix.

We begin by fixing some notations: Let $M = (X_{ij})$ be the mxn generic matrix (X_{ij} are indeterminates over $\mathbb{Z}$ and $2 \leqslant m \leqslant n$) and let $R = \mathbb{Z}[X_{ij}]$. $I_2(M)$ is the ideal generated by all the 2x2 minors of M. We know that grade $I_2(M) = (m\text{-}1)\ (n\text{-}1)$ [4, page 171]. From now on, by a minor we mean a 2x2 minor.

We will see that only if M is the 2xn, 3x3 or 3x4 generic matrix, we can find a regular sequence of (2x2) minors of length grade $I_2(M)$ (cf. Proposition 1). The regular sequences of minors of the 2xn generic matrix were studied in [1]. In §§3 and 4 we give necessary and sufficient conditions for a sequence of minors of the 3x3 and 3x4 matrices to be regular.

Let $h(m,n)$ be the integer $mn/2$ if mn is even and $(mn\text{-}1)/2$ if mn is odd. We will see that $h(m,n)$ is the upper bound for the length of a regular sequence of minors (if $m,n \leqslant 3$). In §5 we will prove that this bound is reached.

We denote by D_{ij}^{kl} the minor $X_{ik}X_{jl} - X_{il}X_{jk}$ of M. If m=3, to simplify the notation, we take $X_{1j} = X_j$, $X_{2j} = Y_j$, $X_{3j} = Z_j$, $D_{12}^{ij} = D_Z^{ij}$, $D_{13}^{ij} = D_Y^{ij}$, $D_{23}^{ij} = D_X^{ij}$, $i,j = 1,\dots,n$. M_{ij} (resp. M^{ij}) is the ideal generated by the minors of the 2xn (resp. mx2) submatrix of M formed by its i-th and j-th rows (resp. columns). A row (resp. column) of a minor is the row (resp. column) of the matrix associated to this minor. If S is a sequence of minors, (S) stands for the ideal generated by the elements of S. Finally, we observe that, since the minors are homogeneous elements of R, if a sequence S of minors is regular, then every permutation of S is also a regular sequence.

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§2. A basic result.

Now we state our first result:

Proposition 1. *If m=3, $n \geqslant 5$ or if $m \geqslant 4$ there does not exist a regular sequence of length $(m$-1) $(n$-1) among the minors of M.*

Proof. Let S be a regular sequence of minors of M. Since grade $(x_{1l}, \ldots, x_{ml})$ = $= m$ the l-th column of M appears, at most, in m minors of S. We have n columns in M, but two columns appear in each minor. Thus we must have

$$\text{grade } I_2(M) = (m\text{-}1)\,(n\text{-}1) \leqslant \frac{mn}{2}. \tag{1}$$

If m=2, (1) holds for any value of n. If $m > 2$ we have

$$\left[m \leqslant n \leqslant \frac{2m-2}{m-2} \right]. \tag{2}$$

Solving (2), yields the proposition.

Remark 1. It follows from the proof of Proposition 1 that the integer $h(m,n)$ defined above is an upper bound for the length of a regular sequence of minors.

We often use the following elementary result:

Lemma 1. *Let T be an integral domain and $X_1, \ldots, X_n$ indeterminates over T. If $a_i \in T[X_1, \ldots, X_{i-1}]$, i=1, ... , n, then the ideal $(X_1\text{-}a_1, \ldots, X_n\text{-}a_n)$ of $T[X_1, \ldots, X_n]$ is prime.*

§3. The 3x3 matrix.

In this section, unless otherwise stated, M denotes the 3x3 generic matrix. We have grade $I_2(M)$ = 4. The following theorem caracterize the regular sequences of minors of M.

Theorem 1. *Up to renumbering of variables, $D_X^{12}, D_Y^{12}, D_Y^{13}, D_Z^{23}$ is the only regular sequence of minors of M of lenght four.*

Before proving Theorem 1 we will establish a lemma.

Lemma 2. *Let M be the 3x4 generic matrix. Let $J_1 = J_1(M) = (M^{12}, D_Z^{13}, D_Z^{23})$, $J_2 = J_2(M) = (J_1, D_Y^{14}, D_Y^{24}, D_X^{34})$ and $J_3 = J_3(M) = (M_{12}, M^{12}, M^{34})$. Then J_1 is a prime ideal, grade J_1 = 3 and grade $J_i \leqslant 5$ for i = 2,3.*

Proof. By [4, Theorem 1], J_1 is a prime ideal. It follows at once from [1, Proposition 1] that $\{D_X^{12}, D_Y^{12}, D_Z^{13}\}$ is an R-sequence; thus grade $J_1 \geqslant 3$.

Let $P = (X_2, X_3, X_4, Y_1, \ldots, Y_4, Z_1, \ldots, Z_4)$. Since $J_i \subset P$, i = 1,2,3 it is enough to show that grade $J_1 R_p \leqslant 3$ and grade $J_i R_p \leqslant 5$, i = 2,3 [2, Theorem 134]. In R_p we may suppose that our matrix is

$$M' = \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \\ Z_1 & Z_2 & Z_3 & Z_4 \end{vmatrix}$$

where $a_i \in R_p$, i = 2,3,4. Let $M_2 = M' \, e_{1,2}^{-a_2}$ and $M_1 = M_3 = e_{2,1}^{-Y_1} \cdot e_{3,1}^{-Z_1} \cdot M_2$ where $e_{i,j}^{a}$ is the standard elementary matrix with a in the i,j position. So, we have

$$M_2 = \begin{vmatrix} 1 & 0 & a_3 & a_4 \\ Y_1 & Y_2' & Y_3 & Y_4 \\ Z_1 & Z_2' & Z_3 & Z_4 \end{vmatrix}, \quad M_1 = M_3 = \begin{vmatrix} 1 & 0 & a_3 & a_4 \\ 0 & Y_2' & Y_3' & Y_4' \\ 0 & Z_2' & Z_3' & Z_4' \end{vmatrix}$$

where $Y_i' = Y_i - a_i Y_1$, $Z_i' = Z_i - a_i Z_1$, i = 2,3,4.

Now, we note that $J_i(M)R_p = J_i(M') = J_i(M_i)$, i = 1,2,3. Since $J_1(M_1) = (Y_2', Z_2', Y_3')$, $J_2(M_2) = (Y_2', Z_2', Y_3', Z_4', Y_3 Z_4 - Y_4 Z_3)$ and $J_3(M_3) = (Y_2', Y_3', Y_4', Z_2', a_3 Z_4' - a_4 Z_3')$, the lemma follows.

Proof of Theorem 1. Let S be a regular sequence consisting of four minors of M. Then, the following conditions hold:

(a) A row (resp. column) of M appears, at least in two and at most in three minors of S.

(b) If two minors of S belong to a 2x3 (resp. 3x2) submatrix of M, then the other two minors of S do not belong to a 3x2 (resp. 2x3) submatrix of M.

In fact, if a row of M appears, at most in one minor of S, then three minors of S belong to a 2x3 submatrix M' of M and grade $I_2(M')$ = 2. If a row (the first, say) appears in all minors of S then $(S) \subset (X_1,X_2,X_3)$ and grade $(S) \leqslant 3$ (later on, we will take some reduction analogous to that, with no comments).

If the condition (b) is not satisfied, we may suppose that the 2x3 (resp. 3x2) submatrix is formed by the first two rows (resp. columns) of M. Then $(S) \subset J_1$ and grade $J_1 = 3$ (the ideal J_1 was defined in Lemma 2).

Now, let $S = \{x_1,x_2,x_3,x_4\}$ satisfy conditions (a) and (b) above. We may suppose that the first column appears in three minors of S (because of condition (a) it occurs for two columns of M). Thus, we may suppose that $x_1 = D_X^{12}$, $x_2 = D_Y^{12}$, $x_3 = D_r^{13}$, $x_4 = D_s^{23}$ and we must determine r and s. Because of condition (a) we must have $r = Z$ or $s = Z$. But if $r = s = Z$ condition (b) is not satisfied. Since we have not distinguished the first two rows and the first two columns, we may suppose $r = Y$ and $s = Z$. To complete the proof we have to show that $\{D_X^{12},D_Y^{12},D_Y^{13},D_Z^{23}\}$ is indeed an R-sequence: it is a consequence of the next proposition.

Proposition 2. *The ideal* $P = (D_X^{12},D_Y^{13},D_Z^{23})$ *is prime. Since* $D_Y^{12} \notin P$, *it follows that* $\{D_X^{12},D_Y^{12},D_Y^{13},D_Z^{23}\}$ *is an R-sequence.*

Proof. Let $x_1 = D_X^{12}$, $x_2 = D_Y^{13}$, $x_3 = D_Z^{23}$, let $P_i = (x_1, \ldots, x_i)$, $i = 1,2,3$, and let Σ_i be the multiplicative system of the powers Z_i^j, $j \geqslant 0$, $i = 1,2$. Repeating the following argument for $i = 1,2$, yields the proposition.

As P_i is a prime ideal and $x_{i+1} \notin P_i$, we have that the generators of P_{i+1} form an R-sequence. So it follows that the associated primes of P_{i+1} have grade $i+1$. Also Z_i is not a zero divisor modulo P_{i+1}. Hence, by Lemma 1, in $(R/P_i)_{\Sigma_i}$, the ideal $(Z_i^{-1}x_{i+1})(R/P_i)_{\Sigma_i}$ is prime. It then follows that (x_{i+1}) (R/P_i) is a prime ideal, i.e., P_{i+1} is a prime ideal.

§4. The 3x4 matrix.

In this section, M stands for the 3x4 generic matrix. We will study the regular sequences of minors of M. We have grade $I_2(M) = 6$.

Theorem 2. *Up to renumbering of variables,*

$$D_X^{12},D_Y^{12},D_Y^{13},D_X^{34},D_Z^{34},D_Z^{24}$$

$$D_X^{12},D_Y^{13},D_Z^{23},D_Z^{14},D_Y^{24},D_X^{34}$$

$$D_X^{12},D_Y^{13},D_Z^{23},D_X^{14},D_Z^{24},D_Y^{34}$$

are the only regular sequences of minors of M of length six.

Proof. Let S be a regular sequence consisting of six minors of M. Then, the following conditions hold:

(a) If M' is a 2x4 (resp. 3x3) submatrix of M, then exactly two (resp. three) minors of S belong to M', or equivalently, a row (resp. column) of M appears in exactly four (resp. three) minors of S.

(b) If there exist two 3x2 submatrices M_1 and M_2 of M such that two minors of S belong to M_1 and two other minors of S belong to M_2 then the two remaining minors of S do not belong to a 2x4 submatrix of M.

(c) Suppose that there exist two 2x3 submatrices M_1 and M_2 of M such that two minors of S belong to M_1 and two other minors of S belong to M_2. If the j-th (resp. k-th) column of M appears in the two minors that belong to M_1 (resp. M_2) then no other minor of S belongs to the 3x2 submatrix of M formed by the j-th and the k-th columns of M.

In fact, if condition (a) is not satisfied, there exists a 2x4 (resp. 3x3) submatrix M' such that at most one (resp. two) minor of S belongs to M'. We may suppose that M' is formed by the first two rows (resp. three columns) of M. Then there are five (resp. four) minors of S that belong to the ideal (Z_1,Z_2,Z_3,Z_4) (resp. (X_4,Y_4,Z_4)). Since grade $(Z_1,Z_2,Z_3,Z_4) = 4$ (resp. grade $(X_4,Y_4,Z_4) = 3$) S cannot be regular and we have a contradiction.

If condition (b) (resp. condition (c)) is not valid, bearing in mind condition (a), we may suppose $S = \{D_X^{12},D_Y^{12},D_X^{34},D_Y^{34},D_Z^{ij},D_Z^{kl}\}$ (resp. $S = \{D_Z^{13},D_Z^{23},D_Y^{14},D_Y^{24},D_X^{12},D_X^{34}\}$). But $(S) \subset J_3$ (resp. $(S) \subset J_2$) (the ideal J_3 (resp. J_2) was defined in Lemma 2) and grade $J_3 \leqslant 5$ (resp. grade $J_2 \leqslant 5$).

Now, let S be a sequence of six minors satisfying the conditions above. Because of condition (a), we know that there are exactly three minors in S (say x_1,x_2,x_3) that belong to the 3x3 submatrix of M formed by its first three columns. We may suppose that $\{x_1,x_2,x_3\}$ is a subsequence of the particular regular sequence $\{D_X^{12},D_Y^{12},D_Y^{13},D_Z^{23}\}$. In the last three minors of S, the fourth column of M appears. Using only the condition (a) we may suppose that S is equal to one of the following sequences:

$$\{D_X^{12},D_Y^{12},D_Y^{13},D_X^{34},D_Z^{34},D_Z^{24}\}$$

$$\{D_X^{12},D_Y^{13},D_Z^{23},D_Z^{14},D_Y^{24},D_X^{34}\}$$

$$\{D_X^{12},D_Y^{13},D_Z^{23},D_X^{14},D_Z^{24},D_Y^{34}\}$$

$$\{D_X^{12},D_Y^{13},D_Z^{23},D_Y^{14},D_Z^{24},D_X^{34}\}$$

$$\{D_X^{12},D_Y^{12},D_Z^{23},D_X^{34},D_Y^{34},D_Z^{14}\}$$

But only the first three sequences satisfy conditions (b) and (c). All we have to prove is that these sequences are regular. In the next proposition, we prove that the first is. The proof for the others is analogous.

Proposition 3. *The sequence* $S = \{D_X^{12}, D_Y^{12}, D_X^{34}, D_Z^{34}, D_Y^{13}, D_Z^{24}\}$ *is regular.*

Proof. It follows from [1, Proposition 1] that $S' = \{D_X^{12}, D_Y^{12}, D_X^{34}, D_Z^{34}\}$ is a regular sequence and the associated primes of (S') are: $Q_1 = (M^{12}, M^{34})$,

$Q_2 = (M^{12}, Y_3, Y_4)$, $Q_3 = (Z_1, Z_2, M^{34})$ and $Q_4 = (Z_1, Z_2, Y_3, Y_4)$. Since $D_Y^{13} \notin Q_i$, $i = 1, \ldots, 4$, $S'' = \{D_X^{12}, D_Y^{12}, D_X^{34}, D_Z^{34}, D_Y^{13}\}$ is a regular sequence.

Now we will study the associated primes of (S''). For, let P be a such prime. Then $P \supset (S'')$ and by [2, Theorem 130] grade $P = 5$. We will divide the proof in four steps.

a) First we suppose that $Z_1 \in P$ and $Y_3 \in P$.

Since $Z_1, D_X^{12}, D_Y^{12}, D_Y^{13} \in P$, it follows that $Y_1Z_2, X_1Z_2, X_1Z_3 \in P$.

Since $Y_3, D_X^{34}, D_Z^{34} \in P$, it follows that $Y_4Z_3, X_3Y_4 \in P$.

If $Z_2 \notin P$ and $Y_4 \notin P$ then $P \supset (X_1, Y_1, Z_1, X_3, Y_3, Z_3)$. Since grade $(X_1, Y_1, Z_1, X_3, Y_3, Z_3) = 6$, this possibility can not occur.

If $Z_2 \notin P$ and $Y_4 \in P$, then $P \supset (X_1, Y_1, Z_1, Y_3, Y_4) \supset (S'')$. Let $P_1 = (X_1, Y_1, Z_1, Y_3, Y_4)$. Since grade $P_1 = 5$, $P = P_1$.

If $Z_2 \in P$ and $Y_4 \notin P$, then $P \supset (Z_1, Z_2, X_3, Y_3, Z_3) \supset (S'')$. Let $P_2 = (Z_1, Z_2, X_3, Y_3, Z_3)$. Thus $P = P_2$.

If $Z_2, Y_4 \in P$ then $P \supset (Z_1, Z_2, Y_3, Y_4, X_1Z_3) \supset (S'')$. Let $P_3 = (X_1, Z_1, Z_2, Y_3, Y_4)$ and $P_4 = (Z_1, Z_2, Z_3, Y_3, Y_4)$. Thus $P = P_3$ or $P = P_4$.

b) If $Z_1 \in P$ but $Y_3 \notin P$ we determine as above the associated primes of (S''): We have $P = P_5 = (Z_1, Z_2, Z_3, Z_4, D_Z^{34})$ or $P = P_6 = (X_1, Y_1, Z_1, M^{34})$ or $P = P_7 = (X_1, Z_1, Z_2, M^{34})$.

c) If $Z_1 \notin P$ and $Y_3 \in P$ then $P = P_8 = (M^{12}, X_3, Y_3, Z_3)$ or $P = P_9 = (M^{12}, D_Y^{13}, D_Y^{23}, Y_3, Y_4)$. (It follows at once from Lemma 2 that P_9 is prime and grade $P_9 = 5$).

d) Finally if $Z_1Y_3 \notin P$, we consider the multiplicative system Σ of the powers $Z_1^iY_3^j$, $i,j \geqslant 0$. In R_Σ, $(S'')R_\Sigma = (Y_2 - Z_1^{-1}Y_1Z_2,\ X_2 - Z_1^{-1}X_1Z_2,\ X_3 - Z_1^{-1}X_1Z_3,\ X_4 - Y_3^{-1}X_3Y_4,\ Z_4 - Y_3^{-1}Y_4Z_3)$ is prime, by Lemma 1.

Since $D_Z^{24} \notin (S'')R_\Sigma, P_i$, $i = 1, \ldots, 9$, the sequence S is regular.

§5. The mxn matrix.

We have seen in Remark 1 that the integer $h(m,n)$ defined in the introduction is an upper bound for the length of a regular sequence of minors of the mxn generic matrix ($m,n \geqslant 3$). In this section we will see how to construct such a sequence of length $h(m,n)$.

Theorem 3. *Let M be the mxn generic matrix* ($3 \leqslant m \leqslant n$). *Then we can find a regular sequence of minors of M of length $h(m,n)$.*

Proof. We remark that if M has a regular sequence of minors of length s then the transpose ${}^tM$ has also such a sequence. We obtain the proof in two steps.

First we prove using the methods of the proof of Proposition 3 that:

i) if M is 3x5 then the sequence $\{D_X^{12}, D_Y^{12}, D_Y^{13}, D_Z^{23}, D_X^{45}, D_Z^{45}, D_X^{34}\}$ is regular.

ii) if M is 3x6 then the sequence $\{D_X^{12}, D_Y^{12}, D_Y^{13}, D_Z^{23}, D_X^{45}, D_Z^{46}, D_Y^{56}, D_X^{36}\}$ is regular.

iii) if M is 4x4 then the sequence $\{D_{12}^{12}, D_{23}^{12}, D_{34}^{12}, D_{12}^{34}, D_{14}^{34}, D_{34}^{34}, D_{12}^{23}, D_{34}^{14}\}$ is regular.

iv) if M is 4x5 then the sequence $\{D_{12}^{12}, D_{12}^{23}, D_{12}^{34}, D_{12}^{15}, D_{34}^{12}, D_{34}^{15}, D_{34}^{45}, D_{34}^{34}, D_{13}^{34}, D_{24}^{25}\}$ is regular.

v) if M is 5x5 then the sequence $\{D_{12}^{23}, D_{12}^{13}, D_{13}^{13}, D_{23}^{12}, D_{45}^{23}, D_{45}^{12}, D_{34}^{23}, D_{12}^{45}, D_{23}^{45}, D_{34}^{45}, D_{45}^{45}, D_{15}^{14}\}$ is regular.

vi) if M is 5x6 then the sequence $\{D_{12}^{23}, D_{12}^{13}, D_{13}^{13}, D_{23}^{12}, D_{45}^{23}, D_{45}^{12}, D_{34}^{23}, D_{12}^{56}, D_{12}^{46}, D_{13}^{46}, D_{23}^{45}, D_{45}^{56}, D_{45}^{45}, D_{35}^{56}, D_{45}^{14}\}$ is regular.

After that, we can suppose that if m is even (resp. odd) then $n \geqslant 6$ (resp. $n \geqslant 7$). We also suppose, by induction, that the statement is valid for the m'xn' matrices such that either $m' < m$ or $m' = m$ and $n' < n$.

If m is even (resp. odd) we consider the mx$(n-3)$ (resp. mx$(n-4)$) submatrix M_1 formed by the first $n-3$ (resp. $n-4$) columns of M and the mx3 (resp. mx4) submatrix M_2 formed by the remaining columns of M. By induction, since $m \geqslant 4$ and $n \geqslant 6$ (resp. $m \geqslant 3$ and $n \geqslant 7$) we can find a regular sequence S_1 of length $h(m,n-3)$ (resp. $h(m,n-4)$) among the minors of M_1 and a regular sequence S_2 of length $h(m,3)$ (resp. $h(m,4)$) among the minors of M_2. It follows that the sequence $S_1 \cup S_2$ obtained by joining the minors of S_2 to the ones of S_1 is regular and its length is just $h(m,n)$.

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