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Sharp existence results for a class of semilinear elliptic problems.

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Abstract.

In this paper a semilinear elliptic second-order problem is considered. Under very general assumptions we give a precise description of the number of solutions of the problem. These results extend in particular a result due to A. Ambrosetti and G. Prodi.

Introduction.

The problem considered here is of the following type: let Ω be a bounded regular domain in $\mathbb{R}^N$, we look for solutions u of

(1) $$-\Delta u = g(x,u) + f(x) \text{ in } \Omega,\ u \in C^2(\overline{\Omega}),\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega;$$

where ν is the unit outward normal to $\partial\Omega$, $f \in C^{0,\alpha}(\overline{\Omega})$ (for some $0 < \alpha < 1$) and $g(x,u)$ is a smooth nonlinearity satisfying essentially:

(2) $$\varlimsup_{t\to-\infty} \frac{g(x,t)}{t} < 0 < \varliminf_{t\to+\infty} \frac{g(x,t)}{t} \text{ (uniformly in } x \in \overline{\Omega});$$

and some appropriate growth condition at $+\infty$.

If $f(x) = t\varphi(x) + f_1(x)$, where $t \in R$, $\varphi \in C^{0,\alpha}(\overline{\Omega})$ with

(3) $$\varphi \geqslant 0 \text{ in } \overline{\Omega},\ \varphi \not\equiv 0$$

we prove (see Section I) that there exists $t_0 (= t_0(\varphi,f_1)) \in R$ such that

i) if $t > t_0$, there is no solution of (1);

ii) if $t = t_0$, there is at least a minimum solution of (1);

iii) if $t < t_0$, there is a minimum solution of (1) and there are at least two distinct solutions.

This result extends and sharpens many earlier results due to A. Ambrosetti and G. Prodi [2], M. S. Berger and E. Podolak [5], P. Hess and B. Ruf [9], J. L. Kazdan and F. W. Warner [11], H. Berestycki [4], H. Amann and P. Hess [1], E. N. Dancer [8]. The main assumption that we remove is the "at

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most linear growth at $+\infty$" and in addition we prove the existence for $t < t_0$ of two ordered solutions.

In Section II, we consider the special case of $f(x,t)$ convex in t and we give some results of a geometrical nature concerning the set of functions f for which (1) admits a solution. Our main concern is to extend the results of H. Berestycki [4] to the case in which we no longer assume that g grows at most linearly at $+\infty$.

I. The general case.

Let α be in (0,1) and let $f \in C^{0,\alpha}(\overline{\Omega})$. We assume that the nonlinearity $g(x,t)$ belongs to $C^{0,\alpha}(\overline{\Omega})$ (uniformly for t bounded) and $g(x,t)$ is Lipschitz continuous in t, uniformly for x in $\overline{\Omega}$. In addition, we restrict the growth of $g(x,t)$ for t large by the following assumption:

(4) $\quad \lim_{t\to+\infty} g(x,t)\, t^{-p} = 0$, uniformly in $x \in \overline{\Omega}$, for some $p < \dfrac{N}{N-2}$;

(if $N = 2$, $\dfrac{N}{N-2}$ may be replaced by any $p < \infty$; and if $N = 1$, we make no assumption at all). We then have

Theorem I.1. *Under assumptions* (2), (4) *and if* $f(x) = t\varphi(x) + f_1(x)$ *with* $\varphi \in C^{0,\alpha}(\overline{\Omega})$ *satisfying* (3), *there exists* $t_0 \in R$ $(t_0 = t_0(\varphi, f_1))$ *such that:*

i) *if* $t > t_0$, *there is no solution of* (1);

ii) *if* $t = t_0$, *there is at least a minimum solution of* (1);

iii) *if* $t < t_0$, *there is a minimum solution of* (1) *and there are at least two distinct solutions.*

Remark I.1. As it will be clear from an inspection of the proof, the same result holds if we replace $-\Delta$ by any uniformly elliptic second-order operator (with smooth coefficients) and if we suppose that g depends also on ∇u: $g = g(x,u,p)$ for $(x,u,p) \in \overline{\Omega} \times R \times R^N$; we then need to assume that $g(x,t,p)$ is bounded for $(x,p) \in \overline{\Omega} \times R^N$ and t bounded and that the limits in (3), (4) hold uniformly in $p \in R^N$. In addition, we may also replace (1) by

(1') $\quad -\Delta u = f(x,u,t)$ in Ω, $u \in C^2(\overline{\Omega})$, $\dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$;

assuming as in [1]:

(5) $\quad \forall m \in R$, $\exists \varphi \in C(\overline{\Omega})$ such that $\dfrac{\partial f}{\partial t}(x,\xi,t) \geq \varphi(x) > 0$,

for x in Ω, $\xi \geq m$ and $t \in R$.

Remark I.2. Assumption (4) is a technical assumption which insures that solutions of (1) are a priori bounded (cf. the proof of Theorem I.1 below).

We believe that the same result is true with $\dfrac{N}{N-2}$ replaced by $\dfrac{N+2}{N-2}$. For a similar reason, if we replace Neumann boundary condition by a more general one, then we need to replace $\dfrac{N}{N-2}$ by $\dfrac{N+1}{N-1}$ (we then use in the proof of Theorem I.1, the a priori estimates of H. Brezis and R. E. L. Turner [6]).

Proof of Theorem I.1. The proof is divided in several steps: we prove 1) there exist arbitrary negative subsolutions of (1), 2) the set of t such that (1) has a solution is of the form $(-\infty, t_0]$, 3) that (1) has always a minimum solution if $t \leq t_0$, and finally 4) that (1) has two distinct solutions for $t < t_0$.

1) Let $\psi \in C^{0,\alpha}(\overline{\Omega})$, then there exists $v \in C^2(\overline{\Omega})$ such that

$$-\Delta v \leq g(x,v) + f(x) \text{ in } \overline{\Omega},\ v \leq \psi \text{ in } \overline{\Omega},\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Indeed, because of (2) we have

(6) $\quad g(x,t) \geq -\alpha t - C$ for $t,x \in R \times \overline{\Omega}$ and for some $\alpha, C > 0$.

Then if we define v by $v = -\max\left\{\dfrac{1}{\alpha}(\|f\|_\infty + C), \|\psi\|_\infty\right\}$ we have obviously $v \leq \psi$ and

$$-\Delta v = 0 \leq -\alpha v - C + f(x) \leq g(x,v) + f(x).$$

2) We first prove that if t is bounded, all possible solutions of (1) are bounded in $C^{2,\alpha}(\overline{\Omega})$.

Indeed, because of (6), we deduce obviously from the maximum principle that if u is a solution of (1), one has: $u(x) \geq -\dfrac{1}{\alpha}(\|f\|_\infty + C)$. In particular u^- is bounded in $L^\infty(\Omega)$. Next, if we integrate (1) on Ω, we obtain

$$\int_\Omega g(x,u) = -\int_\Omega f(x) \leq Const.;$$

since g satisfies (2) and u^- is bounded in $L^\infty(\Omega)$, this implies:

$$\int_\Omega |g(x,u)|dx \leq Const.,\quad \int_\Omega |u|dx \leq Const..$$

In particular we have: $\|-\Delta u\|_{L^1}, \|u\|_{L^1} \leqslant Const.$. This implies by well-know regularity results: $\|u\|_{L^p} \leqslant Const., \forall p < \frac{N}{N-2}$. Since g satisfies (4), it is easy to obtain by a bootstrap argument:

$$\|u\|_{L^\infty} \leqslant Const.$$

Let us prove now that if (1) has a solution for some t, then for all $s \leqslant t$, (1) has a solution. Indeed let u be a solution of (1) for t and let $s < t$, obviously u is a supersolution of (1) (for s) i.e.:

$$-\Delta u = g(x,u) + t\varphi + f_1 \geqslant g(x,u) + s\varphi + f_1.$$

On the other hand, by step 1) above, we know there exists v satisfying

$$-\Delta v \leqslant g(x,v) + s\varphi + f_1, \; v \leqslant u.$$

Then by classical results on sub and supersolutions, this proves our claim.

Thus we know that the set of t such that (1) has a solution is either $(-\infty, t_0]$ (with $t_0 < +\infty$) or $(-\infty, +\infty)$ (it is necessarily closed in view of the a priori bounds proved above). We just need to prove that (1) cannot have a solution for all t: we argue by contradiction and we suppose (1) has a solution u_t for all t. Then we define u_1, u_2 by

$$\begin{cases} -\Delta u_1 + \alpha u_1 = \varphi \text{ in } \Omega, \; \dfrac{\partial u_1}{\partial \nu} = 0 \text{ on } \partial\Omega, \; u_1 \in C^2(\overline{\Omega}) \\ -\Delta u_2 + \alpha u_2 = f_1 - C \text{ in } \Omega, \; \dfrac{\partial u_2}{\partial \nu} = 0 \text{ on } \partial\Omega, \; u_2 \in C^2(\overline{\Omega}). \end{cases}$$

In view of (6), we have

$$u_t \geqslant t\, u_1 + u_2 \text{ in } \overline{\Omega}.$$

Since φ satisfies (3), we have $u_1 > 0$ in $\overline{\Omega}$ and thus for t large enough

$$u_t > 0 \text{ in } \overline{\Omega}.$$

Because of (2), we have: $g(x,t) \geqslant \alpha t - C$ for $t \geqslant 0$ for some $\alpha, C > 0$. Then integrating (1) on Ω and using the fact that u_t is positive, we obtain

$$\alpha \int_\Omega u_t dx + t \int \varphi \, dx \leqslant Const. \text{ (indep. of } t);$$

since $\int \varphi dx > 0$, we obtain a contradiction for t large enough.

3) Now let $t \leqslant t_0$, then (1) has always a minimum solution if $t \leqslant t_0$. We already know that (1) has a solution u and that all possible solutions of (1) satisfy: $u \geqslant -\frac{1}{\alpha}(\|f\|_\infty + C)$ (α, C given by (6)). But $v = -\frac{1}{\alpha}(\|f\|_\infty + C)$ is a subsolution of (1) (take $\psi = 0$ in Step 1)) and thus $u \geqslant v$. Then, by well-

known results, this implies that (1) has a minimum solution $\tilde{u}$ among all solutions satisfying: $w \geqslant v$ in $\overline{\Omega}$. Since all solutions w of (1) satisfy: $w \geqslant v$ in $\overline{\Omega}$, $\tilde{u}$ is in fact the minimum solution of (1).

4) Finally let $t < t_0$, and let us prove that (1) has two distinct solutions. We are going to use a topological degree argument (we refer to J. Leray and J. Schauder [12], or to L. Nirenberg [15] for a definition and the main properties of the Leray-Schauder degree).

Let us first introduce some notations, let u_{t_0} be the minimum solution of (1) where f is given by $t_0\varphi + f_1$. By Steps 1), 2), 3), we know there exists a strict subsolution v of

$$-\Delta v \leqslant g(x,v) + t\varphi + f_1, \; \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega$$

and a minimum solution u_t of (1) (with f given by $t\varphi + f_1$) satisfying:

$$v < u_t < u_{t_0} \text{ in } \overline{\Omega}.$$

We are going to prove the existence of a solution u of (1) which does not satisfy:

$$v < u < u_{t_0} \text{ in } \overline{\Omega}$$

and thus $u > u_t$ in $\overline{\Omega}$, $u \nless u_{t_0}$ in $\overline{\Omega}$.

By Step 2) and the a priori bounds, we may choose $C_1 > 0$ such that all solutions u of (1) satisfy: $\|u\|_{C(\overline{\Omega})} < C_1$, and we may assume

$$\|v\|_{C(\overline{\Omega})}, \|u_{t_0}\|_{C(\overline{\Omega})} < C_1.$$

Now in view of the smoothness of $g(x,t)$, there exists $\lambda > 0$ such that $g(x,t) + \lambda t$ is nondecreasing on $[-C_1, +C_1]$, for all x in $\overline{\Omega}$. Obviously u is a solution of (1) if and only if u is a fixed point of the compact operator K defined on $C(\overline{\Omega})$ by: $Kv = u$ is given by

$$\begin{cases} -\Delta u + \lambda u = g(x,v) + \lambda v + t\varphi + f_1 \text{ in } \overline{\Omega}, \; u \in W^{2,p}(\Omega) \; (p < \infty), \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases}$$

We first prove that if M is large enough, the degree of $I - K$ on $B_M = \{w \in C(\overline{\Omega}), \|w\|_{C(\overline{\Omega})} < M\}$ (with respect to 0) is well defined and $d(I\text{-}K, B_M, 0) = 0$.

In order to prove this, we define a family K_s of compact operators in $C(\overline{\Omega})$ defined by: $K_s v = u_s$ is given by

$$\begin{cases} -\Delta u_s + \lambda u_s = s(g(x,v) + \lambda v + f) + (1-s)(1 + v^+ + \lambda v) \text{ in } \Omega, \\ \dfrac{\partial u_s}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases}$$

The same proof as in Step 2) gives, that all solutions u_s of: $u_s = K_s u_s$ satisfy: $\|u_s\|_{C(\overline{\Omega})} < M$ (indep. of $s \in [0,1]$). We will also assume that $M > C_1$. Thus the degree of $I - K_s$ on B_M is well defined and independent of $s \in [0,1]$:

$$d(I - K, B_M, 0) = d(I - K_0, B_M, 0).$$

Now, if u_0 is a solution of: $u_0 = K_0 u_0$, we have

$$-\Delta u_0 = u_0^+ + 1, \frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial\Omega,\ u_0 \in C^2(\overline{\Omega}),$$

and thus $\int_\Omega (1 + u_0^+) dx = 0$, which is impossible; thus there is no fixed point of K_0 and $d(I - K_0, B_M, 0) = 0$.

We then prove that if $\mathcal{O} = \left\{ w \in C(\overline{\Omega}),\ v < w < u_{t_0} \text{ in } \overline{\Omega} \right\}$ then $d(I - K, \mathcal{O}, 0)$ is well defined and is equal to $+1$. Indeed let $\varphi \in \mathcal{O}$, and let us define $\tilde{K}_s v$ on $C(\overline{\Omega})$ by

$$\tilde{K}_s v = s K v + (1 - s)\varphi, \text{ for } s \in [0,1].$$

Because of the choice of v, u_{t_0} and λ we have obviously:

$$K : \overline{\mathcal{O}} \to \mathcal{O} \text{ and thus } \tilde{K}_s : \overline{\mathcal{O}} \to \mathcal{O}.$$

This implies that $d(I - \tilde{K}_s, \mathcal{O}, 0)$ is well defined and independent of $s \in [0,1]$, therefore we deduce

$$d(I - K, \mathcal{O}, 0) = d(I - \tilde{K}_s, \mathcal{O}, 0) = d(I - \tilde{K}_0, \mathcal{O}, 0).$$

Now $\tilde{K}_0 v$ is constant, equal to φ which belongs to $\mathcal{O}$, thus

$$d(I - \tilde{K}_0, \mathcal{O}, 0) = +1.$$

We are now able to conclude: indeed by the above arguments we have

$$d(I - K, B_M - \overline{\mathcal{O}}, 0) = -1;$$

and this means that (1) has a solution which does not belong to $\overline{\mathcal{O}}$.

II. The convex case.

We now consider the case where g is convex, more precisely we deal with the following problem:

$$(7) \qquad -\Delta u = \varphi(u) + f(x) \text{ in } \Omega,\ u \in C^2(\overline{\Omega}),\ u = 0 \text{ on } \partial\Omega,$$

where $f \in C^{0,\alpha}(\overline{\Omega})$ (for some $\alpha \in (0,1)$) and where φ satisfies

$$(8) \qquad \varphi \text{ is strictly convex on } R,\ \varphi \in C^1(R);$$

$$(9) \qquad \lim_{t \to -\infty} \frac{\varphi(t)}{t} < \lambda_1$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω, with Dirichlet boundary conditions.

It is well-known that if $\lim_{t \to +\infty} \frac{\varphi(t)}{t} \leqslant \lambda_1$, then (1) has a unique solution u for every $f \in C^{0,\alpha}(\overline{\Omega})$. In what follows, we will assume in addition to (8)-(9):

$$(10) \qquad \lim_{t \to +\infty} \frac{\varphi(t)}{t} > \lambda_1$$

We then define K to be the set of functions f in $C^{0,1}(\Omega)$ such that (7) has at least one solution. In addition we set

$A_m = \{ f \in C^{0,1}(\Omega)$, (7) has at least two distinct solutions$\} \subset K$

$A_1 = \{ f \in C^{0,1}(\Omega)$, (7) has exactly one solution$\} \subset K$;

obviously $K = A_m \cup A_1$.

Our first result states (setting $X = C^{0,1}(\Omega)$):

Theorem II.1. *Under assumptions (8)-(9)-(10), K is a convex set, unbouded, with $\mathring{K} \neq \phi$ and $E - K$ is nonempty, unbounded.*

Furthermore for all $f \in K$, there exists a minimum solution u of (7), such that the first eigenvalue of the operator $-\Delta - \varphi'(u)$ (with Dirichlet boundary conditions) is nonnegative.

In addition $A_m \subset \mathring{K}$ (and $\partial K \subset A_1$) and for all $f \in \mathring{K}$ then the first eigenvalue of the operator $-\Delta - \varphi'(u)$ is positive.

Remark II.1. This result may be extended to the case of more general elliptic operators and to more general boundary conditions (in particular Neumann conditions). In addition, we may assume that φ depends on x ((9), (10) being uniform in $x \in \overline{\Omega}$).

Remark II.2. This result is an extension of a result due to H. Berestycki [4], where it is assumed in addition that: $\lim_{t \to +\infty} \frac{\varphi(t)}{t} < \lambda_2$, where λ_2 is the second eigenvalue of $-\Delta$. However in that special case a more precise description of K may be given: indeed (see [4]) i) K is closed, ii) $A_m = \mathring{K} = \{ f \in C^{0,1}(\overline{\Omega})$, (7) has exactly two solutions$\}$. We will see below that if we relax the assumption: $\lim_{t \to +\infty} \frac{\varphi(t)}{t} < \lambda_2$, then we need some assumption to ensure that $A_m = \mathring{K}$, and that K is closed.

Let us for the moment indicate that in general for f in $\mathring{K}$ (7) may have more than two solutions (even an infinite number of solutions): take $\varphi(u) = \frac{2}{N-2} e^u$ and $f = 0, N < 10$ with Ω the unit ball in R^N – see D. D. Joseph and T. S. Lundgren [10]). Furthermore we do not know any other assumption than: $\lim_{t\to+\infty} \frac{\varphi(t)}{t} < \lambda_2$, to ensure that for f in $\mathring{K}$, (7) has exactly two solutions.

Remark II.3. For f in K, the minimum solution u of (7) depends continuously on f.

To simplify notations, we may assume without loss of generality: $\varphi(0) = 0$.

Before going into the proof, let us give two results which answer the questions raised above (in Remark II.2): (we assume for the sake of simplicity $N \geqslant 3$).

Theorem II.2. *Under assumptions* (8)-(9)-(10) *and if we assume:*

$$(11) \qquad \lim_{t\to+\infty} \left\{\Phi(t)\, h(t)^{-2/N} t^{-2}\right\} = 0, \ \lim_{t\to+\infty} \varphi(t) t^{\frac{N+2}{N-2}} = 0,$$

where $\Phi(t) = \int_0^t \varphi(s)ds$ *and* $h(t) = ½\varphi(t)t - \Phi(t) \geqslant 0$, *then* $A_m = \mathring{K}$ *and thus* $A_1 = \partial K$. *In addition, for* f *in* ∂K, *the first eigenvalue of the operator* $(-\Delta - \varphi'(u))$ *is zero, where* u *is the corresponding solution of* (7).

Remark II.4. Let us give a few examples where the (technical) condition (11) is satisfied:

i) if φ satisfies: $\theta\, \varphi(t)t - \Phi(t) \geqslant 0$ for $t \geqslant t_0$, and for some $\theta \in (0,½)$ then $h(t) \geqslant (½ - \theta)\, t\, \varphi(t)$, and if we know that

$$\lim_{t\to+\infty} \varphi(t)\, t^{-(N+2)/(N-2)} = 0$$

then $\Phi(t)\ h(t)^{-2/N}\ t^{-2} \leqslant C\, t\, \varphi(t)\ t^{-2/N}\ \varphi(t)^{-2/N}\ t^{-2} \leqslant C \frac{\varphi(t)^{\frac{N-2}{N}}}{t^{\frac{N+2}{N}}}$ and thus (11) is satisfied and soon as we have

$$(12) \begin{cases} \theta\varphi(t)t - \Phi(t) \geqslant 0 \text{ for } t \geqslant t_0 \text{ and for some } \theta \in (0,½) \\ \lim_{t\to+\infty} \varphi(t)t^{-(N+2)/(N-2)} = 0. \end{cases}$$

(12) has been introduced by A. Ambrosetti and P. H. Rabinowitz [3], and contains in particular $\varphi(t) = |t|^p$ for $1 < p < \frac{N+2}{N-2}$.

ii) if φ satisfies: $\lim_{t\to+\infty} \varphi(t)\ t^{-N/(N-2)} = 0$, then (11) is satisfied. Indeed since φ is convex, it is easy to prove that $h(t) \geqslant \alpha\varphi(t) - C$; and then

$$\Phi(t)h(t)^{-2/N}\ t^{-2} \leqslant C\, t\, \varphi(t)\, \varphi(t)^{-2/N}\ t^{-2} = C\frac{\varphi(t)^{\frac{N-2}{N}}}{t}.$$

If we consider the particular case $\varphi(t) = |t|^p$ (with $1 < p < \infty$) then (8)-(9)-(10) hold obviously and (11) holds if and only if $p < \frac{N+2}{N-2}$. The following example shows that such a restriction is needed and that $\frac{N+2}{N-2}$ is the critical exponent for A_m to be equal to $\mathring{K}$.

Example. We assume that Ω is starshaped ($N \geqslant 3$), $\varphi(t) = |t|^p$ with $p \geqslant \frac{N+2}{N-2}$, and we take $f = 0$. Then (7) is equivalent to

$$(7') \qquad -\Delta u = u^p \text{ in } \Omega,\ u \geqslant 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,\ u \in C^2(\overline{\Omega}).$$

Then in view of the results of S. I. Pohozaev [16], (7') has a unique solution $u \equiv 0$. Thus $0 \in A_1$. But by an obvious application of the implicit function theorem, for f in $C^{0,\alpha}(\overline{\Omega})$ small, (7') has still a solution and therefore $f \in K$. Hence $0 \in \mathring{K}$.

Finally concerning the question of the closedness of K, let us just indicate that problem is entirely similar to the following problem: let $(0,\lambda^*)$ be the maximal interval such that there exists a solution of

$$(13) \qquad -\Delta u = \lambda(\varphi(u) + f(x)),\ u \in C^2(\overline{\Omega}),\ u = 0 \text{ on } \partial\Omega;$$

where we assume $f \geqslant 0$, $\varphi(0) > 0$; then does there exist a solution of (13) for $\lambda = \lambda^*$? This question is answered in M. G. Crandall and P. H. Rabinowitz [7] (see also F. Mignot and J. P. Puel [14]) and just applying their results and techniques, we obtain:

Proposition II.1. *If one of the following conditions is satisfied*

$$(14) \qquad \begin{cases} t\,\varphi'(t) \geqslant \theta\, \varphi(t), \textit{ for } t \geqslant t_0 \textit{ and for some } \theta > 1,\ t_0 > 0, \\ \lim_{t\to+\infty} \varphi(t)t^{-(N+2)/(N-2)} = 0; \end{cases}$$

$$(15)\quad \begin{cases} \varphi(t) = at^m + \psi(t), \text{ for } t \geqslant 0 \text{ and for some } a > 0, \text{ where } \psi \text{ satisfies:} \\ \lim\limits_{t \to -\infty} \dfrac{\psi(t)}{t^m} = \lim\limits_{t \to +\infty} \dfrac{\psi'(t)}{t^{m-1}} = 0; \end{cases}$$

or

$$(16)\quad \begin{cases} \varphi \text{ is a class } C^2 \text{ and satisfies: } \beta\varphi'(t)^2 > \varphi(t)\,\varphi''(t) \geqslant \mu(\varphi'(t))^2 \\ \text{for } t \geqslant t_0; \text{with } 0 < \beta < 2 + \mu + \sqrt{\mu} \text{ and } N < 4 + 2\mu + 4\sqrt{\mu}\,; \end{cases}$$

where $t_0 > 0$; *then* K *is closed.*

Let us remark that the results of D. D. Joseph and T. S. Lundgren [10] show that these conditions are nearly optimal (see also [7], [14], for examples of nonlinearities φ satisfying (14), or (15), or (16)).

Let us now prove Theorems II.1 and Theorems II.2:

Proof of Theorem II.1. We only prove that $A_m \subset \mathring{K}$, since all the other statements follow directly from the proof of H. Berestycki [4]. Let $f_0 \in A_m$, there exist at least a minimum solution of (2) u and another distinct solution, say $\bar{u} > u$. Since we have

$$-\Delta(\bar{u} - u) = \varphi(\bar{u}) - \varphi(u) = \left\{ \frac{\varphi(\bar{u}) - \varphi(u)}{\bar{u} - u} \right\} (\bar{u} - u)\ ;$$

this implies that the first eigenvalue of the operator $-\Delta - \dfrac{\varphi(\bar{u}) - \varphi(u)}{\bar{u} - u}$ (this last function being extended by $\varphi'(u)$ on $\partial\Omega$) is 0. But since φ is strictly convex, we have

$$\frac{\varphi(\bar{u}) - \varphi(u)}{\bar{u} - u} > \varphi'(u) \text{ in } \Omega,$$

therefore the first eigenvalue of the operator $-\Delta - \varphi'(u)$ is positive. Then by an obvious application of the implicit function theorem, for f near f_0 in X, (7) has a solution i.e.: $f_0 \in \mathring{K}$

Proof of Theorem II.2. Let $f \in \mathring{K}$, we know (by Theorem II.1) there exists u minimum solution of (7) and that the first eigenvalue of $-\Delta - \varphi'(u)$ is positive. To prove that $f \in A_m$, we just need to show there exists a solution v of

$$(17)\quad \begin{cases} -\Delta v = \varphi(u(x) + v) - \varphi(u(x)) \text{ in } \Omega\ \ v \in C^2(\overline{\Omega}) \\ v > 0 \text{ in } \Omega,\ v = 0 \text{ on } \partial\Omega. \end{cases}$$

Since φ satisfies (11) and since the first eigenvalue of $-\Delta - \varphi'(u)$ is positive, we may apply the existence results of P. L. Lions [13] to conclude.

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