Ω-stability of plane vector fields

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The purpose of this paper is to give sufficient conditions for Ω-stability of plane vector fields, which also imply genericity. It thus solves the question posed by Mendes in [4]. I would like to express my thanks to Floris Takens for suggesting the problem and helpful conversations.

1. Definitions and statement of the result.

By $\mathfrak{X}^r(\mathbb{R}^2)$ $(r \geqslant 1)$ we denote the complete C^r-vector fields on the plane $\mathbb{R}^2$, endowed with the C^r-Whitney topology. We write X, X' for elements of $\mathfrak{X}^r(\mathbb{R}^2)$. X_t denotes the flow induced by X. Critical elements of X (singularities and closed orbits) are denoted by σ, and the subset of $\mathbb{R}^2$, consisting of all points on critical elements is $\cup\sigma_i$ $(i \in I)$.

For $x \in \mathbb{R}^2$ $\gamma_X(x)$, $\gamma_X^+(x)$, $\gamma_X^-(x)$ is the trajectory of x, the positive half-trajectory of x, the negative half-trajectory of x, respectively, under X_t.

$\Omega(X) \subset \mathbb{R}^2$ is the nonwandering set of X. (for a definition see Palis-de Melo [5] pp. 129-130)

For a vector field X and $x \in \mathbb{R}^2$ we define the first positive prolongation set $D^1_{1,X}(x)$ by:

$D^1_{1,X}(x) = \{y \in \mathbb{R}^2 \mid \exists\ x_n \to x,\ t_n > 0 \text{ such that } X_{t_n}(x_n) \to y\}$

and the first positive prolongation limit set $J^1_{1,X}(x)$ by

$J^1_{1,X}(x) = \{y \in \mathbb{R}^2 \mid \exists\ x_n \to x,\ t_n \to \infty \text{ such that } X_{t_n}(x_n) \to y\}$.

By induction we define for $k > 1$:

$D^k_{1,X}(x) = \{z \in \mathbb{R}^2 \mid z \in D^1_{1,X}(y) \text{ for some } y \in D^{k-1}_{1,X}(x)\}$

$J^k_{1,X}(x) = \{z \in \mathbb{R}^2 \mid z \in J^1_{1,X}(y) \text{ for some } y \in J^{k-1}_{1,X}(x)\}$.

By transfinite induction it is possible to define higher prolongation (limit) sets $D^1_{\alpha,X}(x)$, $J^1_{\alpha,X}(x)$ and $D^k_{\alpha,X}(x)$ for each ordinal number α by:

$D^1_{\alpha,X}(x) = \{y \in \mathbb{R}^2 \mid \exists\ x_n \to x, y_n \to y, k_n > 0, \text{ ordinals } \beta_n < \alpha \text{ with } y_n \in D^{k_n}_{\beta_n,X}(x_n)\}$

$J^1_{\alpha,X}(x) = \{y \in \mathbb{R}^2 \mid \exists\ x_n \to x, y_n \to y, k_n > 0, \text{ ordinals } \beta_n < \alpha \text{ with } y_n \in J^{k_n}_{\beta_n,X}(x_n)\}$

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Then $D^k_{\alpha,X}(x) = J^k_{\alpha,X}(x) \cup \overset{+}{\gamma}_X(x)$ (Bhàtia-Szegö [1]).

For $X \in \mathcal{X}^r(\mathbb{R}^2)$, $R(X) \subset \mathbb{R}^2$ is the subset consisting of those points x, for which there exists an ordinal number α and a positive k with $x \in J^k_{\alpha,X}(x)$. Now $\Omega(X)$ is the subset of $R(X)$ consisting of those points x for which $x \in J^1_{1,X}(x)$.

We will call a vector field $X \in \mathcal{X}^r(\mathbb{R}^2)$ Ω-(R-)stable if for every neighbourhood U of $\Omega(X)$ ($R(X)$), there exists a neighbourhood $\mathfrak{U}$ of X such that for each $X' \in \mathfrak{U}$:

i) $\Omega(X') \subset U$ ($R(X') \subset U$);

ii) if a connected component of U contains points of $\Omega(X)$ ($R(X)$), then it also contains points of $\Omega(X')$ ($R(X')$);

iii) there exists a homeomorphism $h_{X'}: \Omega(X) \to \Omega(X')$ ($R(X) \to R(X')$) transforming trajectories of X into trajectories of X'.

The usual definition of Ω-stability only states (iii); our definition is closer to the notion of "absolute stability" introduced by Guckenheimer [2].

We denote the Kupka-Smale vector fields in $\mathcal{X}^r(\mathbb{R}^2)$ by $\mathcal{X}^r_{k\text{-}s}$, i.e. for $X \in \mathcal{X}^r_{k\text{-}s}$:

i) every σ_i is hyperbolic;

ii) $i_1, i_2 \in I$, then $W^u(\sigma_{i_1})$, $W^s(\sigma_{i_1})$ (the unstable and stable manifolds of σ_{i_1} and σ_{i_2}) are in general position.

In this paper we prove the following:

Theorem. *Vector fields $X \in \mathcal{X}^r_{k\text{-}s}$ are R-stable if and only if $R(X) = \cup\sigma_i$ and these vectorfields are generic.*

Because always $\cup\sigma_i \subset \Omega(X) \subset R(X)$ we also have:

Corollary. *All vector fields $X \in \mathcal{X}^r_{k\text{-}s}$ with the property $R(X) = \cup\sigma_i$ are Ω-stable, and Ω-stability is a generic property in $\mathcal{X}^r(\mathbb{R}^2)$.*

2. Proof of the theorem.

We start with proving the sufficiency of the condition in the theorem. Here we need the following.

Remark. Let X be a C^r-vector field on a manifold M (not necessery two dimensional) with $\underline{R}(X) = \phi$. Then there is a C^1-function L on M with $X(L) < 0$.

Proof. By results of Auslander, (see e.g. [1] p. 132) there is a continuous function L on M with $L(X_t(x)) < L(x)$ for each $x \in M$ and each positive t. From the proof of this statement in [1], it is clear, that we may assume L to be a C^1-function, thus satisfying $X(L) \leqslant 0$. Now we can change L on flow boxes in such a way that the inequality becomes strict.

The above remark enables us to control vector fields outside neighbourhoods of recurrent points. To get local Ω-stability in the critical elements we need.

Proposition. *Let $X \in \mathcal{X}^r_{k\text{-}s}$, with critical elements $\{\sigma_i | i \in I\}$ and $R(X) = \cup\sigma_i$. For each compact set $K \subset \mathbb{R}^2$, the subset $I(K)$ of I, defined by $I(K) = \{i \in I | \sigma_i \cap K \neq \phi\}$, is finite.*

Proof. It is clear, that there is only a finite number of singularities in K. Now suppose there is an infinite subset I_1 of $I(K)$ such that for each $i \in I_1$, σ_i is a closed orbit which intersects K. Choose $x_i \in \sigma_i \cap K$ ($i \in I_1$), then after taking a subsequence we may assume $x_i \to x \in K$. Now $x \in \Omega(X) \subset R(X)$ and x does not belong to a closed trajectory, because these are all hyperbolic and isolated, so x is a singularity.

Again by hyperbolicity, x must be a saddle. Take a small neighbourhood U of x, not containing closed orbits and singularities, except x, in its closure. Then each σ_i has a point x_i' on the boundary of U for i large enough.

Again after taking a subsequence we may assume $x_i' \to x'$.

Now $x' \in \Omega(X) \subset R(X)$ but x' is not a singularity and x' lies not on a closed orbit.

This contradiction proves the proposition.

From this proposition it follows that for vector fields as above we may choose neighbourhoods U_i of each σ_i with mutually disjoint closures. R-stability for such vector fields then reduces to the existence of a neighbourhood $\mathfrak{U}$ of the vector field X with the property that $R(X') \subset \cup U_i$ and X' possesses a unique critical element σ_i', of the same kind (sink, saddle, closed orbit, etc.) as σ_i in each U_i, for every vectorfield X' in $\mathfrak{U}$. The existence of such a neighbourhood $\mathfrak{U}$ is proved in:

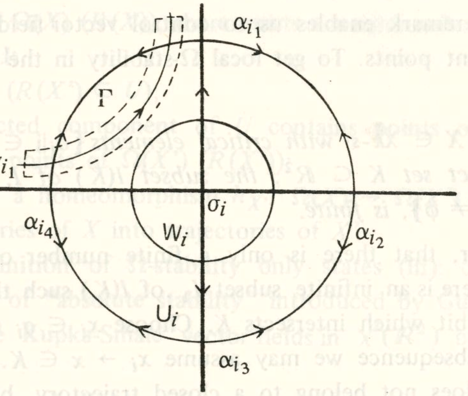
Lemma 1. *Every vector field $X \in \mathcal{X}^r_{k\text{-}s}$ with $R(X) = \underset{i}{\cup}\sigma_i$ is R-stable.*

Proof. We construct a neighbourhood $\mathfrak{U}$ of X as above.

Let L be a C^1-function on $\mathbb{R}^2 \text{-} \cup\sigma_i$ with $X(L) < 0$.

Choose neighbourhoods U_i of each σ_i with disjoint closures as follows: If σ_i is an attractor or a repellor then we choose U_i, with its boundary transverse to X.

If σ_i is a saddle we choose a neighbourhood U_i on which the vector field is topologically equivalent to its linear part. On the boundary of each saddle neighbourhood U_i we can choose small open arcs $\alpha_{i1}, \ldots, \alpha_{i4}$, transverse to X, each containing one point of a separatrix, (see fig.) with:
$\max\{L(x) | x \in \alpha_{i1} \cup \alpha_{i3}\} < \min\{L(x) | x \in \alpha_{i2} \cup \alpha_{i4}\}$.



Let $\gamma_{i1}, \ldots, \gamma_{i4}$ be trajectories of X, such that arcs of $\gamma_{i1}, \alpha_{i1}, \gamma_{i2}, \ldots, \alpha_{i4}$ constitute the boundary of a neighbourhood V_i of σ_i with $V_i \subset U_i$.

Let Γ_{ij} be a flow-box neighbourhood of $\gamma_{ij} \cap U_i$ (j = 1, ... , 4) with $\Gamma_{ij} \cap bd(U_i) \subset \bigcup_j \alpha_{ij}$

At last we define neighbourhoods W_i of each σ_i as follows. If σ_i is a saddle then $W_i \subset V_i$ and $W_i \cap \Gamma_{ij} = \phi$.

If σ_i is not a saddle then $W_i = U_i$.

We get a neighbourhood $\mathfrak{U}_1$ of X consisting of vector fields X', which satisfy on $\cup \operatorname{clos} U_i$ the following conditions:

1) for each X' in $\mathfrak{U}_1$ and for each i with σ_i not a saddle there is a unique critical element σ_i' of X' in U_i of the same kind as σ_i, and X' is transverse to $bd(U_i)$.

2) for each X' in $\mathfrak{U}_1$ and for each i with σ_i a saddle there is a unique saddle σ_i' of X' in W_i. Furthermore X' has trajectories $\gamma_{i1}, \ldots, \gamma_{i4}$ with $\gamma_{ij} \cap U_i \subset \Gamma_{ij}$ (j = i, ... ,4), and X' is transverse to the α_{ij}'s.

Outside the U_i's we can control perturbations of X by choosing a neighbourhood $\mathfrak{U}_2$ of X, only imposing conditions on $\mathbb{R}^2 - \bigcup_i W_i$, such that $X'(L) | \mathbb{R}^2 - \cup W_i < 0$ for each $X \in \mathfrak{U}_2$. Finally we show that $R(X^1) = \cup \sigma_i$ for each $X' \in \mathfrak{U} = \mathfrak{U}_1 \cap \mathfrak{U}_2$.

To this end, choose $X' \in \mathfrak{U}$ arbitrary.

By construction of $\mathfrak{U}_2$, L strictly decreases on trajectories of X', which are totally contained in $\mathbb{R}^2 - \bigcup_i W_i$, so each orbit in $R(X')$ intersects at least one W_i.

Using the assumed inquality about values of L on the boundary of saddle neighbourhoods, we have that $L(y) < L(x)$ for any two points $x,y \in \mathbb{R}^2 - \cup U_i$ with $y \in \gamma^+_{X'}(x)$. From this, using continuity of L, it is easy to see that for a point $z \in J^k_{\alpha,X'}(x)$, $x,z \in \mathbb{R}^2 - \cup U_i$, we have $L(z) \leqslant L(x)$.

Now if $R(X') \neq \cup \sigma_i'$ then there is a trajectory γ' in $R(X')$, crossing the boundary of a saddle neighbourhood W_{i_0}.

Assume first that γ' is not a separatrix of σ_{i_0}', then γ' intersects two of the $\alpha_{i0,j}$'s, succesively in x_1 and x_2, so $L(x_2) < L(x_1)$.

But because $\gamma' \subset R(X')$, $x_1 \in J^k_{\alpha,X'}(x_2)$ for some α and k, and thus $L(x_1) \leqslant L(x_2)$ which is a contradiction.

The other possibility, where γ' is a separatrix of σ_{i0}, is showed to be impossible with the same arguments, so $R(X') = \cup \sigma_i'$ and the lemma is proved.

In the following it is shown that there exists a residual subset of $\chi^r(\mathbb{R}^2)$ consisting of vector fields X with the property that $R(X) = \cup \sigma_i$. This, together with lemma 1 will give the theorem.

We denote by $\tilde{\chi}^r$ the subset of $\chi^r(\mathbb{R}^2)$ containing all vector fields X with the following four properties:

i) X is a Kupka-Smale vector field.
ii) $\Omega(X) = \cup \sigma_i$.
iii) X is not Ω-explosive.
iv) X is a max-tolerance stable.

Here we call X not Ω-explosive if for each neighbourhood U of $\Omega(X)$ and for each compact set $K \subset \mathbb{R}^2$ there exists a neighbourhood $\mathfrak{U}$ of X with $\Omega(X') \cap K \subset U$ for each $X' \in \mathfrak{U}$. A vectorfield X is called max-tolerance stable if for each positive ϵ and for each compact set $K \subset \mathbb{R}^2$ there is a neighbourhood $\mathfrak{U}$ of X such that for each $X' \in \mathfrak{U}$ the following holds:

a) for each X'-trajectory γ' with $\gamma' \cap K \neq \phi$ there is an orbit γ of X, such that $\gamma' \cap K$ is contained in the ϵ-neighbourhood of γ.

b) for each X-trajectory γ with $\gamma \cap K \neq \phi$ there is an orbit γ' of X', such that $\gamma \cap K$ is contained in the ϵ-neighbourhood of γ'.

Now $\tilde{\chi}^r$ contains a residual subset of $\chi^r(\mathbb{R}^2)$, ([3], [6] and [7]) and we will show that $R(X) = \bigcup_i \sigma_i$ for each $X \in \tilde{\chi}^r$.

The following lemma is just a slight extension of lemma 4 in [8]:

Lemma 2. *For each $X \in \tilde{\chi}^r$ we have $D^2_{1,X} = D^1_{1,X}$.*

Proof. Suppose there are two points p and q in $\mathbb{R}^2$, with $q \in D^2_{1,X}(p)$, $q \notin D^1_{1,X}(p)$. We first show that we may assume p and q to be regular points:

If p is on an attractor, then q is on the same attractor, so $q \in D^2_{1,X}(p) - D^1_{1,X}(p)$ is impossible.

If p is on a repellor σ_0, then there exists a neighbourhood V of σ_0, with its boundary transverse to X. Now if $q \in D^2_{1,X}(p) - D^1_{1,X}(p)$ then there is a point p' on bd(V) with $q \in D^2_{1,X}(p') - D^1_{1,X}(p')$.

Finally, in the case where p is a saddle point, there exists a point p' on an unstable separatrix of p with $q \in D^2_{1,X}(p') - D^1_{1,X}(p')$.

Thus p, and, with the same arguments, q, may assumed to be regular points. Let $m \in \mathbb{R}^2$ be a point with $q \in D^1_{1,X}(m)$ and $m \in D^1_{1,X}(p)$. If m is a regular point, then by definition of $\tilde{\chi}^r$ we have $p,q,m \notin \Omega(X)$. Now the arguments in the proof of lemma 4 in [8] apply to give a contradiction. If m is not a regular point, then by hyperbolicity m is a saddle and we may choose m_1, m_2 on separaties of m in such a way that $q \in D^1_{1,X}(m_2)$, $m_2 \in D^1_{1,X}(m_1)$, $m_1 \in D^1_{1,X}(p)$. Again $p,q,m_1,m_2 \notin \Omega(X)$ and applying the arguments of lemma 4 in [8] twice, gives a contradiction.

Lemma 3. *For $X \in \tilde{\chi}^r$ we have $R(X) = \cup\sigma_i$.*

Proof. We will show by transfinite induction, that for each ordinal number α and for each positive number k: $D^k_{\alpha,X} = D^1_{1,X}$.

Because $\Omega(X) = \{x \in \mathbb{R}^2 \,|\, x \in J^1_{1,X}(x)\}$ and $J^k_{\alpha,X}(x) = D^k_{\alpha,X}(x) - \gamma^+(x)$ for each α and k we then have $\Omega(X) = R(X)$.

By definition of $\tilde{\chi}^r$ then $R(X) = \cup\sigma_i$.

Induction:

$D^k_{1,X} = D^1_{1,X}$ follows from lemma 2.

Now assume $D^k_{\beta,X} = D^1_{1,X}$ for each ordinal number $\beta < \alpha$ and $y \in D^1_{\alpha,X}(x)$ for two points $x,y \in \mathbb{R}^2$.

Then there are sequences $x_n \to x$, $y_n \to y$ ordinal numbers $\beta_n < \alpha$ and positive numbers k_n with $y_n \in D^{k_n}_{\beta_n,X}(x_n)$.

By hypothesis $y_n \in D^1_{1,X}(x_n)$. Let ϵ be a small positive number, then there exists n_0 with x_{n_0}, y_{n_0} in the ½ϵ-neighbourhood of x,y respectively. Because $y_{n_0} \in D^1_{1,X}(x_{n_0})$ there is a trajectory γ of X intersecting the ½ϵ-neighbourhoods of both x_{n_0} and y_{n_0}.

ϵ was chosen arbitrary, so $y \in D^1_{1,X}(x)$.

Now $D^1_{\alpha,X} = D^1_{1,X}$ and thus also $D^k_{\alpha,X} = D^1_{1,X}$.

Because $R(X) = \cup\sigma_i$ holds for a residual and hence a dense set of vectorfields $X \in \chi^r(\mathbb{R}^2)$, this is a necessery condition for R-stability. The proof of the theorem is complete.

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