A survey of the geometric results in the classical theory of minimal surfaces

William H. Meeks III

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Section 1. Introduction.

In recent years there has been a beautiful flowering in the theory of minimal surfaces. Many fundamental conjectures have been solved and deep unexpected applications to other parts of mathematics and physics have been attained. The author has written this survey in an attempt to organize and present the new and classical results in this theory.

Since the quantity of the new material is rather overehelming and because the analytic material is already well presented in books and papers (see [3], [18], [30], [49], [57], [81], [86]), the author has emphasized those aspects of the subject which have an intuitive or geometrical flavor. Even with this limitation on the subject matter, the survey covers approximately seventy theorems in twenty four sections.

Because of lack of time and space the author has unfortunately been unable to include a discussion of the many fascinating unsolved problems in this theory. He refers the interested reader to his book "Lectures on Plateau's problem" for a discussion of more than fifty unsolved problems.

The author would like express his gratitude to Gudlaugur Thorbergsson who carefully read and criticized the first attempts at organizing this survey.

Section 2. The definition of a branched minimal surface.

The theory of minimal surfaces begins with the following variational formula in the calculus of variations.

Theorem 1 (first variation formula). *Let $f:M\to\mathbb{R}^3$ be a compact immersed surface with boundary. Let $f_t:M\to\mathbb{R}^3$ be a smooth variation of M for $t \in$ $\in (-1,1)$ such that $f_o = f$ and $f_t|\partial M = f|\partial M$. Let V be the variational vector field restricted to $f_o = f$. If $A(t)$ is the area of f_t then*

$$\frac{d}{dt}A(t)|_{t=0} = A'(0) = -2\int_M \langle V, H\cdot N\rangle dM$$

where H is the mean curvature of M, N is a unit normal vector field on M, and dM is the induced area form of the immersion f.

An immediate corollary of theorem 1 is that if the mean curvature H of M is identically equal to zero, then M is a critical point to area for smooth variations of M that fix the boundary of M. The converse of this statement, *if M is a critical point to area for smooth variations which fix the boundary, then the mean curvature H is identically zero*, also follows from the first variation formula by choosing appropriate variational vector fields V.

Immersed surfaces in $\mathbb{R}^3$ with mean curvature identically equal to zero are called *minimal surfaces*. For analytic reasons it is natural to generalize the definition of minimal surface to surfaces which have a finite number of singularities called *branch points*. In order to understand the concept of branch point on a minimal surface we make the following definitions.

Definition 1. A *Riemann surface M* is a surface with an atlas of conformal coordinates. If an orientable surface M has a Riemannian metric, then a classical theorem implies that there exists a system of conformal coordinates on M which is called a system of *isothermal coordinates* on M.

Definition 2. A smooth mapping $f:M\to\mathbb{R}^3$ of a Riemann surface M into $\mathbb{R}^3$ will be called *conformal* if in conformal coordinates (x,y) on M, $|f_x| = |f_y|$ and $\langle f_x, f_y\rangle = 0$ where $\langle , \rangle$ is the inner-product on $\mathbb{R}^3$.

Definition 3. If $f:M\to\mathbb{R}^3$ is conformal, then a point $p=(x,y)\in M$ is a *branch point* of f if the differential df_p is zero or equivalently if $|f_x| = |f_y| = 0$.

Definition 4. A smooth map $f:M\to\mathbb{R}^3$ from a Riemann surface is *harmonic* if the coordinate functions of f are harmonic functions on M. A function $g:M\to\mathbb{R}$ on a Riemann surface is called harmonic if in conformal coordinates the Laplacian $\Delta g = \dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2}$ is identically zero.

Theorem 2. *Suppose $f:M\to\mathbb{R}^3$ is an immersion of an oriented surface M and Δ is the Laplacian of M with respect to the induced metric. Then*

$$\Delta f = H\cdot N$$

where $\Delta f = (\Delta f_1, \Delta f_2, \Delta f_3)$, H is the mean curvature of $f(M)$ and N is the unit normal on $f(M)$.

The above theorem immediately implies that a conformal immersion $f:M \to \mathbb{R}^3$ of a surface is minimal if and only if f is harmonic. This property naturally leads to the following definition.

Definition 5. A conformal harmonic map $f:M \to \mathbb{R}^3$ of a Riemann surface M is called a *branched minimal immersion* and the image $f(M)$ a *branched minimal surface.*

Section 3. Minimal surfaces with boundary and boundary regularity.

In the previous section we gave the definition of a branched minimal surface as being a conformal harmonic map of a Riemann surface into $\mathbb{R}^3$. Since it is natural to consider examples of minimal surfaces which have boundary, we will consider examples of minimal surfaces $f:M \to \mathbb{R}^3$ where M has a boundary ∂M. We shall assume that $f(\partial M)$ is a collection of pairwise disjoint curves called the boundary of $f(M)$. We shall also assume that $f|\partial M$ is a homeomorphism with the boundary curves $f(\partial M)$.

In 1949 H. Lewy proved the following surprising boundary regularity theorem for minimal surfaces.

Theorem 3. *Let γ be an analytic Jordan curve in $\mathbb{R}^3$ and $f:M \to \mathbb{R}^3$ a branched minimal surface with boundary γ. Then f is analytic and $f(M)$ is contained in the interior of a larger branched minimal surface.*

Lewy's proof was based on an analytic reflection principle of which the following is a special case.

Theorem 4. *If α is a straight line segment on the boundary of the minimal surface M, then M can be continued analytically across α by reflection.*

Lewy's theorem was later improved by Hildebrandt [41] using elliptic methods (and later by others, see page 86 in [49]). Hildebrandt proved the following

Theorem 5. *If $f:M \to \mathbb{R}^3$ is a branched minimal surface with smooth boundary curves, then f is smooth.*

Once the minimal surface $f:M \to \mathbb{R}^3$ is known to be smooth one can consider the problem of the existence of branch points on the boundary of the minimal surface. Nitsche showed in [80] that there are at most a finite number of branch points for f when the boundary curves are smooth and M is compact.

Section 4. The maximum principle for minimal surfaces and a generalization of a theorem of Rado.

The word "minimal" is naturally associated to minimal surfaces for several reasons. The first reason being that a compact minimal surface with boundary is frequently a surface of least area having this given boundary. The second reason is that small pieces of an unbranched minimal surface always have least area with respect to their boundaries. Since all immersed surfaces are locally graphs over their tangent planes this local area property is a consequence of the following theorem [61] and the fact that we can choose a small piece of the surface to be a graph over a small planar disk. Actually, this special case is much easier to prove.

Theorem 6. *Let γ be a continous Jordan curve which has a monotonic parallel or central projection onto a convex Jordan curve $\hat{\gamma}$ in a plane* P. *Then*

1. *There exists a compact minimal surface M in $\mathbb{R}^3$ with $\partial M = \gamma$ and $\overset{o}{M}$ is a graph over the interior of the disk in P with boundary $\hat{\gamma}$.*
2. *M is the unique compact branched minimal surface with boundary γ.*
3. *M has least area with respect to all compact piecewise smooth surfaces with boundary γ.*

The proof of the above theorem is somewhat delicate in the generality stated and depends in an essential way on the maximum principle for minimal surfaces. The maximum principle for minimal surfaces which is given below follows from the more general Hopf maximum principle [91].

Theorem 7 (Maximum principle for minimal surfaces). *Suppose that M_1 and M_2 are two connected branched minimal surfaces such that for a point $p \in M_1 \cap M_2$, the surface M_1 locally lies one side of M_2 near p. Then the surfaces M_1 and M_2 coincide near p.*

We now give a sketch of the following important special case of theorem 6 using the above maximum principle (see [57]).

Special Case: *If γ is a continuous Jordan curve which has a one-to-one parallel projection onto a convex plane curve $\hat{\gamma}$ and $f:M \to \mathbb{R}^3$ is a compact* *branched minimal surface with boundary curve γ, then $f(M)$ is a graph. Furthermore, there is at most one such graph.*

Proof of the special case: The curve $\tilde{\gamma}$ is the boundary of a convex disk D in the plane P containing the curve $\tilde{\gamma}$. We shall assume the plane P is the xy plane. Geometrically the curve γ lies on the boundary of the convex cylinder $C = \{(x,y,z) | z \in \mathbb{R}$ and $(x,y) \in D\}$. Since C is convex and M is compact, the maximum principle for harmonic functions implies that the interior of the compact minimal surface $f{:}M \to \mathbb{R}^3$ is contained in the interior of the solid cylinder C.

If $f(M)$ is not a graph of a continuous function, then by elementary differential topology there is a point $q \in \overset{o}{D}$ with $\pi^{-1}(q) \cap f(M)$ consisting of at least two points where $\pi{:}\mathbb{R}^3 \to \mathbb{R}^2$ is orthogonal projection onto the xy-plane. Let $p_1, p_2 \in \pi^{-1}(q) \cap f(M)$ be two points with the z coordinate of p_2 greater than that of p_1. Then the surfaces $f(M)$ and $f(M) + (p_2 - p_1)$ intersect in the point p_2. Thus, there is a nontrivial vertical translation $(0,0,t)$ so that the intersection of $(f+(0,0,t))$ (M) and $f(M)$ is nonempty.

Let $T = \max \{t \in \mathbb{R} \mid (f + (0,0,t))\ (M) \cap f(M) \neq \emptyset\}$. Note that $T>0$ and exists by the compactness of $f(M)$. Now let p be an element in the intersection $(f+(0,0,T))$ $(M) \cap f(M)$. Since $(f+(0,0,t))$ (γ) is disjoint from γ for all $t>0$ and this curve is contained on the boundary of the cylinder C, p must correspond to two points $p_3, p_4 \in \overset{o}{M}$ with $f(p_3) = (f+(0,0,T))$ (p_4). By our choice of T, the immersed surfaces $f(M)$ and $(f+(0,0,T)$ (M) must locally lie on one side of each other near p. Otherwise, the surfaces $(f+(0,0,T+\epsilon))$ (M) would intersect $f(M)$ for some $\epsilon>0$.

By the maximum principle for minimal surfaces, $f(M)$ and $(f+(0,0,T))$ (M) must agree on a open set. However, as these surfaces are analytic, the unique continuation property implies that they must have the same image until one of their boundary curves. Since the boundary curves of the surfaces are disjoint, we arrive at a contradiction which proves the surface $f(M)$ is a graph.

If $f_1, f_2{:}M \to \mathbb{R}^3$ are two distinct compact minimal surfaces which are graphs, then there is a nontrivial vertical translation $(0,0,t)$ so that $(f_1+(0,0,t))$ $(M) \cap f_2(M)$ is nonempty. The argument used above using the maximum principle gives a contradiction and implies the special case of theorem 6.

Remark. Rado [94] was the first person to give a result in the direction of theorem 6. He proved: If a curve γ has a one-to-one parallel or central projection onto a convex plane curve, then γ is the boundary of a unique branched minimal disk and this disk is a graph over the plane (see also page 224 in [25]).

Section 5. Stable minimal surfaces and soap film examples.

An interesting class of minimal surfaces are those for which the second variation of area, A''(0) is positive. Such surfaces are well known to arise from physical experiments with soap film surfaces on bent circular wires. The soap film has the property that it is a local minimum of energy because it is physically stable. By formulas in the calculus of variations, surfaces which are local minimums to energy are also local minimums to area and vice-versa. Because of this relationship with energy it is natural to call a minimal surface *stable* if for every compact subsurface the second variation of area is positive.

The following wire configurations give rise to some famous classical examples of minimal surfaces which can be realized as soap film surfaces.

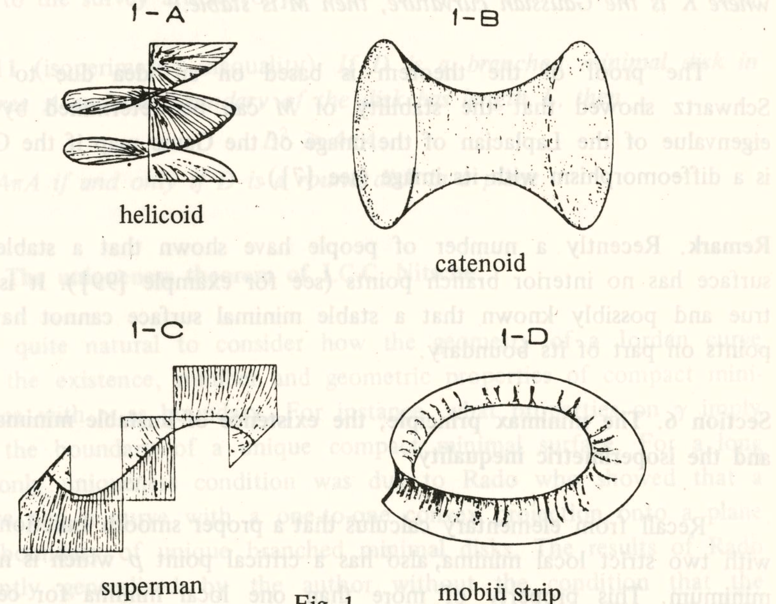


Fig. 1

The helicoid, whose boundary consist of three line segments and a helix, is ruled by straight lines. The boundary of the catenoid consists of two circles on parallel planes with the same axis. It is topologically an annulus which is the surface of revolution of the catenary curve. The superman surface S is the building block for a complete triply periodic minimal surface (see [56]). That is to say, there exists a lattice L such that $M = \{v+S | v \in L\}$ is a connected complete minimal surface which is invariant under translation by elements in the lattice L. Möbius strip surface shows that a least area compact surface with boundary a fixed Jordan curve may be non orientable.

Returning now to the theory of stable minimal surfaces in $\mathbb{R}^3$ we have the following global uniqueness theorem (see [22] and [27]).

Theorem 8. *If M is a complete minimally immersed surface in $\mathbb{R}^3$ which is stable, then M is a flat plane.*

The following formula by Manfredo do Carmo and Lucas Barbosa gives a very important criterion for a minimal surface in $\mathbb{R}^3$ to be stable.

Theorem 9. *Let $G:M \to S^2$ be the Gauss map for a minimal surface M. If the area of G(M) is less than 2π, then M is stable. In particular, if*

$$C(M) = \int_M KdA > -2\pi$$

where K is the Gaussian curvature, then M is stable.

The proof of the theorem is based on an idea due to Schwartz. Schwartz showed that the stability of M can be determined by the first eigenvalue of the Laplacian of the image of the Gauss map if the Gauss map is a diffeomorphism with its image (see [7]).

Remark. Recently a number of people have shown that a stable minimal surface has no interior branch points (see for example [99]). It is probably true and possibly known that a stable minimal surface cannot have branch points on part of its boundary.

Section 6. The minimax principle, the existence of unstable minimal surfaces and the isoperimetric inequality.

Recall from elementary calculus that a proper smooth function $f:\mathbb{R}^n \to \mathbb{R}$ with two strict local minima, also has a critical point p which is not a local minimum. This property of more than one local minima for certain real valued functions defined on certain topological spaces implying the existence of an unstable critical point is quite common in problems encountered in the calculus of variations. Sometimes it is possible to prove the existence of an unstable critical point from what is called "the minimax principle" (see [18]).

From physical experiments with soap films is not difficult to find curves which bound two stable minimal disks (see for example the figure 2 in section 10). An application of the minimax principle could therefore imply the existence of an unstable minimal disk for these curves. The spaces involved in the case of minimal disks are infinite dimensional and hence the

application of the minimax principle becomes quite delicate. In our situation the following application of a C^0 minimax principle has been proved by Morse-Tompkins [75] and Shiffman [104].

Theorem 10. *Let γ be a smooth Jordan curve which is the boundary of two geometrically distinct minimal disks of locally least area (energy) in the C^0 topology on piecewise smooth immersed disks with boundary γ. Then γ is the boundary of an unstable minimal disk which may have branch points.*

The unstable minimal disk given in the above theorem cannot have very large area compared to the length of its boundary curve. This follows from the isoperimetric inequality below for minimal surfaces proved by Carleman. For some interesting generalizations of the isoperimetric inequality we refer the reader to the survey article [85].

Theorem 11 (isoperimetric inequality). *If D is a branched minimal disk in $\mathbb{R}^3$ with area A and the boundary of the disk has length L, then*

$$L^2 \geqslant 4\pi A$$

and $L^2 = 4\pi A$ if and only if D is a round disk in a plane.

Section 7. The uniqueness theorem of J.C.C. Nitsche.

It is quite natural to consider how the geometry of a Jordan curve influences the existence, number and geometric properties of compact minimal surfaces with γ as boundary. For instance, what properties on γ imply that γ is the boundary of a unique compact minimal surface. For a long time the only uniqueness condition was due to Rado who showed that a plane curve and a curve with a one-to-one convex projection onto a plane were the boundary of unique branched minimal disks. The results of Rado were recently generalized by the author without the condition that the surface be simply connected. (Recall theorem 6 in section 4). In [61] an elementary proof is given that shows that for every smooth plane Jordan curve γ an $\epsilon > 0$ can be calculated geometrically so that any curve $\tilde{\gamma}$ ϵ-close to γ in the C^2-norm is the boundary of a unique compact minimal surface which is a graph over the plane containing γ.

The following theorem of Nitsche [78] presents one of the most beautiful global relationships between the geometry of a curve γ and the existence of minimal disks with boundary γ. The existence of at least one minimal disk follows from the solution of Plateau's problem 1 to be discussed in section 9.

Theorem 12. *If γ is a smooth Jordan curve with total curvature less than or equal to 4π, then γ is the boundary of a unique branched minimal disk and this minimal disk is an immersion.*

The idea of the proof is as follows. From a formula of Gauss-Bonnet type for branched minimal disks with boundary γ, it can be calculated that the minimal disk must be immersed.

By the Gauss-Bonnet formula or an immersed minimal disk D

$$\int_D KdA = 2\pi - \int_\gamma k_g(t)dt$$

where $k_g(t)$ is the geodesic curvature of $\gamma(t)$. Since the geodesic curvature is less than or equal to the curvature, the above formula yields the inequality

$$\int_D KdA \geqslant 2\pi - \int_\gamma k(t)dt \geqslant 2\pi$$

If $\int_D KdA > -2\pi$, then D is stable by theorem 9 of Barbosa and Carmo. Nitsche actually shows that the case $\int_D KdA = -2\pi$ and D unstable cannot occur so we may assume that D is stable.

The crucial point in the proof by Nitsche is to prove that the disk is a strict minimum in the C^0-topology. Theorem 10 in section 6 implies that if γ is the boundary of two disks, then γ is the boundary of an unstable minimal disk. Since all minimal disks with boundary γ are stable, γ must be the boundary of at most one minimal disk.

Section 8. A theorem of Shiffman.

Another uniqueness theorem that depends on the geometry of the boundary of the minimal surface is a theorem of Shiffman concerning the uniqueness of the catenoid. That is to say, given two circles on parallel planes with the same axis of symmetry, then any minimally immersed annulus with boundary being these two circles is actually a part of a translated catenoid defined implicitly by $z^2 + y^2 = a \cdot \cosh^2(x)$. However this catenoid need not be unique because these two circles will in general bound a part of a catenoid that is stable and a part of another catenoid which is unstable.

More generally, Shiffman considered the geometric properties of immersed minimal annuli whose boundary curves are circles in parallel planes. He proved in this case that the minimal annulus has the property that the intersection of an inbetween parallel plane with the surface is a circle. B. Riemann in [95] explicity expressed in terms of elliptic functions all minimal annuli with this property. The uniqueness of the catenoid then follows from Shiffman's theorem and the Riemann representations.

Shiffman also considered the case of minimal annuli whose boundary curves are convex curves on parallel planes. Shiffman's geometric results are given below (see [106]).

Theorem 13. *Let γ_1 and γ_2 be two Jordan curves on parallel planes P_1 and P_2, respectively, and let P be a parallel plane between P_1 and P_2. Let $f:M\to\mathbb{R}^3$ be a branched minimal annulus with boundary γ_1 and γ_2. Then:*

1. *If γ_1 and γ_2 are convex, then $f(M) \cap P$ is a convex Jordan curve.*
2. *If γ_1 and γ_2 are circles, then $f(M) \cap P$ is a circle.*

Section 9. Plateau's problems and the regularity of the solutions.

Let γ be a smooth Jordan curve in $\mathbb{R}^3$. Then there naturally arise three basic "Plateau problems" or least area problems which are listed below.

Plateau's Problems

(1) Does there exist a minimal disk $f:D\to\mathbb{R}^3$ of least area with $f(\partial D) = \gamma$?

(2) Does there exist a compact surface $f:M\to\mathbb{R}^3$ of least area with $f(\partial M) = \gamma$?

(3) Does there exist a compact orientable surface $f:M\to\mathbb{R}^3$ of least area with $f(\partial M) = \gamma$?

Note. The topological type of M is not fixed in problems (2) and (3) above.

The classical problem (1) was solved independently by Douglas [23] and Rado [92] in 1930. Their solution can be stated as follows (for general references see [49]).

Theorem 14. *If γ is a rectifiable Jordan curve in $\mathbb{R}^3$, then there exists a solution $f:D\to\mathbb{R}^3$ to problem (1) that is conformal, harmonic and such that $f|\partial D$ gives a monotonic parametrization of γ. Such a solution is called a Douglas solution to Plateau's Problem.*

For the purpose of fixing notation, let us recall the ideas of the proof. One observes initially that is suffices to minimize the Dirichlet integral also called the energy of f.

$$E = \frac{1}{2}\int_D (|f_x|^2 + f_y|^2)dxdy$$

For that, we first minimize E among all maps with finite energy that restricted to ∂D give a monotonic parametrization $\hat{g}: \partial D \to \gamma$ of γ. Each such parametrization gives rise to a solution $\tilde{g}: D \to \mathbb{R}^3$ called the *harmonic extension* of g which has least energy with respect to the parametrization g. The space of all such harmonic extensions is denoted by $\widetilde{H}(\gamma)$. By noticing that E is invariant under conformal transformations of the disk, we normalize all the elements in $\widetilde{H}(\gamma)$ by requiring that three fixed points of ∂D are taken to three fixed points of γ; the resulting space is denoted by $H(\gamma)$. The crucial point is now to prove that $H^N = \{\tilde{g} \in H(\gamma);\ E(\tilde{g}) \leq N\}$ is compact in the topology of uniform convergence. The theorem then follows from the lower semicontinuity of E in $H(\gamma)$. See [18] for further discussion.

Unfortunately the method of arriving at a Douglas solution to Plateau's problem tells us very little about the geometric properties of the solution. For a long time it was not known whether the resulting minimal disk could have branch points of not. In 1968 R. Osserman [84] proved the following interior regularity theorem.

Theorem 15. *A Douglas solution to Plateau's problem is an immersed surface in its interior.*

Osserman's proof was based on a cutting and glueing argument at an interior "geometric branch point". Later Gulliver [34] gave a proof that in fact the least energy solution of Douglas was actually an immersion and therefore there did not exist any "false interior branch point" which arise from bad parametrizations.

The problem of whether there exist boundary branch points for a Douglas solution to Plateau's problem is still unsolved. However, the following has been proved by Gulliver and Lesley [35].

Theorem 16. *If γ is an analytic Jordan curve and $f: D \to \mathbb{R}^3$ is a Douglas solution to Plateau's problem, then f is an immersion.*

While there exist continuous Jordan curves in $\mathbb{R}^3$ which do not bound compact surfaces of finite area, many Jordan curves such as all extremal Jordan curves do bound disks with finite area. (A Jordan curve is *extremal* if it lies on the boundary of its convex hull.) For such curves there exists a Douglas solution to Plateau's problem an while we can not expect boundary regularity, the following can be easily proved.

Theorem 17. *Let γ be a continuous Jordan curve. Suppose $f: D \to \mathbb{R}^3$ is a branched minimal immersion on $\overset{o}{D}$ whose restriction $f|\partial D$ gives a monotonic parametrization of a fixed Jordan curve γ. Then $f|\partial D$ gives rise to a homeomorphism with γ.*

The following simple proof of the above theorem was told to the author by Blaine Lawson: If $f: D \to \mathbb{R}^3$ is a minimal disk such that $f|\partial D$ gives a monotonic parametrization of γ which is not one-to-one, then some open interval I in ∂D is mapped to a point p which we may assume to be the origin in $\mathbb{R}^3$. Considering $\overset{o}{D}$ to be the upper half plane and the interval I as part of the real axis in $\mathbb{C}$, the harmonic coordinate functions of f can be extended analytically across I by applying the reflection principle [2], that is we extend f_i as a harmonic function by setting.

$$f_i(x,y) = f_i(x,-y)$$

for each i. Since the extension is analytic, the extension gives rise to a conformal harmonic map near I which is constant on I. Since the branch points on a minimal surface in $\mathbb{R}^3$ are easily seen to be isolated, using for example the Weierstrass representation in section 18, we arrive at a contradiction which proves the theorem.

We now return to the consideration of the other Plateau problems (2) and (3). With respect to the solutions of these problems we have the recent beautiful theorem of Hardt-Simon [38].

Theorem 18. *Let Γ be a finite collection of pairwise disjoint smooth Jordan curves in $\mathbb{R}^3$. Then*

a) There exists a branched minimal immersion $f: M \to \mathbb{R}^3$ of a compact surface M (possibly disconnected) *which bounds Γ and has least area with this property.*

b) There exists a branched minimal immersion $f: \tilde{M} \to \mathbb{R}^3$ of a compact orientable surface $\tilde{M}$ (possibly disconnected) *which has boundary Γ and has least area with this property.*

c) All such solutions f to (a) and (b) are smooth embeddings.

d) There is only a finite number of solutions to (b) and only a finite number of distinct topological types in part (a).

The proof of the theorem of Hardt-Simon depends on the existence of least area embedded "surfaces" with boundary Γ that arise in the theory of minimal currents. Their key result is a boundary regularity theorem for these surfaces. The finiteness of the number of solutions in part (b) uses the finiteness theory developed by F. Tomi which will be discussed in the next section.

A compact minimal surface is always contained in the convex hull of its boundary. Thus the next theorem [5] by F. Almgren and W. Thurston shows that a solution to Plateau's problems (2) and (3) may be forced to have large genus for purely topological reasons even for unknotted curves.

Theorem 19. *There exist unknotted analytic Jordan curves* γ_n *in* $\mathbb{R}^3$ *with total curvature* $4\pi + \frac{1}{n}$ *and such that if M is a compact orientable embedded surface with boundary* γ_n *and M is contained in the convex hull of* γ_n, *then the genus of M is greater than n.*

Section 10. Finiteness theorems for minimal disks.

One of the fundamental problems in minimal surface theory concerns the question of whether a smooth Jordan curve γ in $\mathbb{R}^3$ can be the boundary of an infinite number of compact minimal surfaces. One of the first and most basic results is a theorem of Tomi [110]. He proves for example that if every Douglas solution to Plateau's problem for γ is an immersion, then there are a finite number of Douglas solutions to Plateau's problem. Because of the unresolved question of boundary branch points for a Douglas solution to Plateau's problem his theorem does not imply that there are a finite number of solutions. However, the boundary regularity theorem 16 in section 9 and an application of the strong Hopf maximum principle show that for analytic or extremal γ, the Douglas solutions are immersed. Thus:

Theorem 20. *If* γ *is an analytic curve or if* γ *is smooth and extremal, then there is a finite number of Douglas solutions to Plateau's problem for* γ.

The following symmetric curve which is essentially the seam curve on a baseball shows that a fixed Jordan curve may be the boundary of two distinct solutions to Plateau's problem.

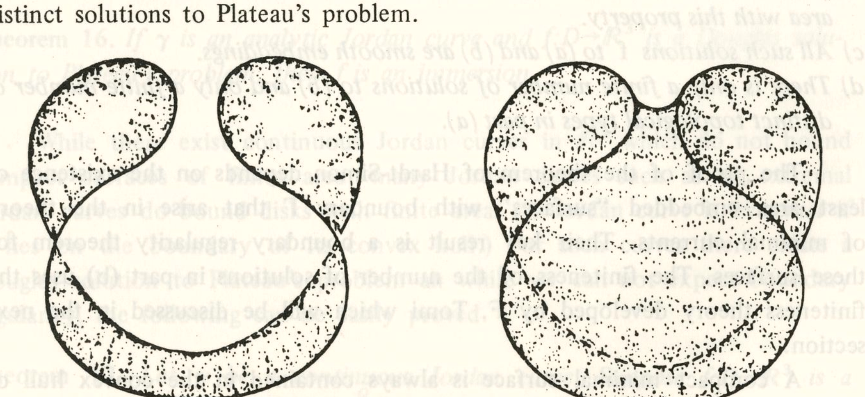


Fig. 2

Using the above example and the bridge principle to be discussed in section 14 it is possible to show that the following Jordan curve is the boundary curve of an infinite number of embedded stable minimal disks. This curve is formed from smaller and smaller replicas of the curve in figure 2 joined by thin "bridges".

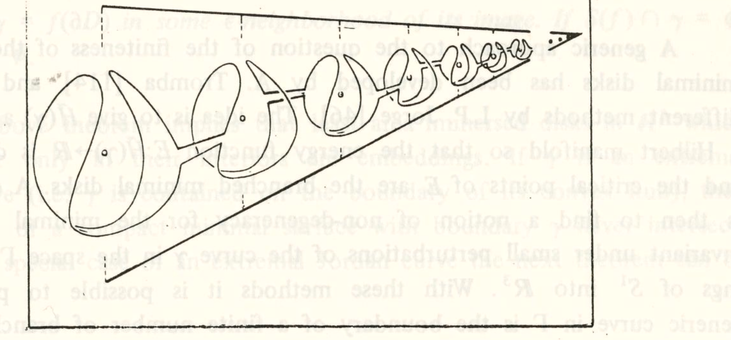


Fig. 3

After consideration of the above examples, it is natural to ask whether a smooth Jordan curve can bound an infinite number of compact minimal surface. While the answer to this question is still unresolved, F. Morgan has found a counter example if one allows more than one Jordan curve as the boundary. F. Morgan's examples [71] show that there exist four circles in three parallel planes which have the same axis of symmetry and which bound a connected embedded minimal surface of high genus. Since the boundary curves are invariant under a circle family of rotations, this minimal surface is part of continuous family of minimal surfaces. See the figure 4 below.

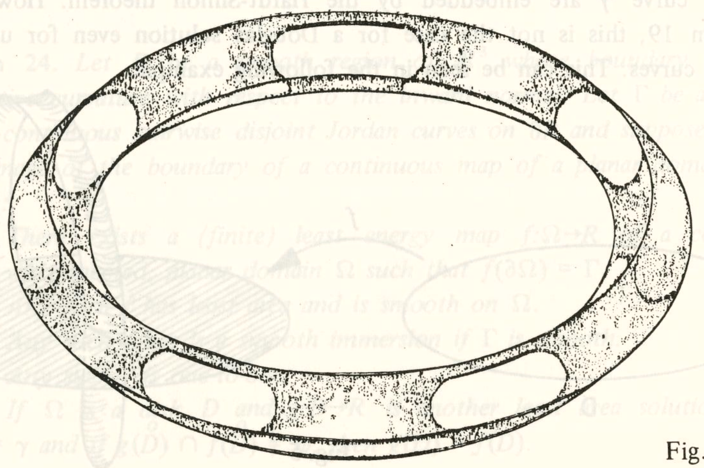


Fig. 4

Nitsche [83] proved the following interesting finiteness theorem which again makes an application of the stability theorem of Carmo and Barbosa in section 5.

Theorem 21. *If γ is a smooth Jordan curve with total curvature less than 6π, then γ is the boundary of a finite number of branched minimal disks.*

A generic approach to the question of the finiteness of the number of minimal disks has been developed by A. Tromba [114] and later, with different methods by L.P. Jorge [46]. The idea is to give $\tilde{H}(\gamma)$ a structure as a Hilbert manifold so that the energy function $E:\tilde{H}(\gamma)\to\mathbb{R}$ is differentiable and the critical points of E are the branched minimal disks. A crucial point is then to find a notion of non-degeneracy for the minimal disk that is invariant under small perturbations of the curve γ in the space Γ of embeddings of S^1 into $\mathbb{R}^3$. With these methods it is possible to prove that a generic curve in Γ is the boundary of a finite number of branched minimal disks. Recently, [11] Böhme and Tromba proved that there exists an open dense subset $\hat{\Gamma} \subset \Gamma$ in the C^∞ topology such that the curves in $\hat{\Gamma}$ bound a finite number of branched minimal disks. In [72] F. Morgan proved that a generic element of Γ is the boundary of a unique solution to the Plateau problems 2 and 3.

Section 11. The Meeks-Yau condition for the embedding of a Douglas solution to Plateau's problem.

Recall that solutions to Plateau's problems (2) and (3) for smooth Jordan curve γ are embedded by the Hardt-Simon theorem. However, by theorem 19, this is not the case for a Douglas solution even for unknotted Jordan curves. This can be seen in the following example.

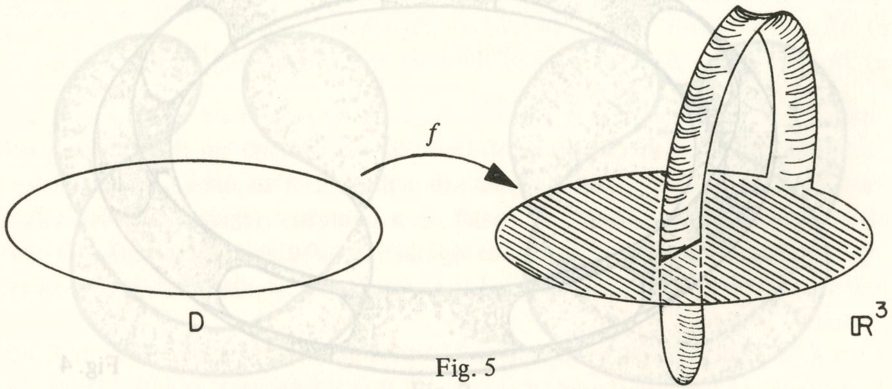


Fig. 5

In this case the self-intersection set

$$S(f) = \{p \in \mathbb{R}^3 \mid f^{-1}(p) \text{ has more than 1 point}\}$$

is nonempty. We now have the following theorem of Meeks-Yau [64] that gives a natural condition for S(f) to be empty.

Theorem 22. *Suppose $f:D\to\mathbb{R}^3$ is an immersion which has least area with respect to $\gamma = f(\partial D)$ in some ϵ-neighborhood of its image. If $S(f) \cap \gamma = \phi$, then $S(f) = \phi$.*

The above theorem implies that least area immersed disks in $\mathbb{R}^3$ which self-intersect only in their interiors are embeddings. If γ is an extremal Jordan curve (i.e. γ is contained on the boundary of its convex hull), then the interior of a compact minimal surface with boundary γ never intersects γ. For the special case of an extremal Jordan curve the next theorem can be proved.

Theorem 23. *A Douglas solution to Plateau's problem for a continuous extremal Jordan curve is one-to-one.*

The proof of the embedding of the Douglas solution to Plateau's problem in theorems 22 and 23 is based on the regularity theorems discussed in sections 3 and 9, a nontrivial approximation argument, the case when γ is analytic and a key topological construction called the tower construction which is discussed in section 24. The basic embedding argument is topological and hence is valid in three dimensional manifold. This generalization and the following theorem will be discussed further in section 16.

Theorem 24. *Let R be a smooth region of $\mathbb{R}^3$ whose boundary ∂R has nonnegative curvature with respect to the inward normal. Let Γ be a collection of continuous pairwise disjoint Jordan curves on ∂R and suppose that Γ is the image of the boundary of a continuous map of a planar domain into R. Then:*

1. *There exists a (finite) least energy map $f:\Omega\to R$ of a compact, possibly disconnected, planar domain Ω such that $f(\partial\Omega) = \Gamma$.*
2. *Any such f has least area and is smooth on Ω.*
3. *Any such f has is a smooth immersion if Γ is smooth.*
4. *Any such f is one-to-one.*
5. *If Ω is a disk D and $g:D\to R$ is another least area solution with $g(\partial D) = \gamma$ and if $g(\mathring{D}) \cap f(\mathring{D}) \neq \phi$, then $g(D) = f(D)$.*

In particular the above theorem shows that if the curves Γ bound an embedded planar domain in R, then they bound an embedded planar domain of least area in the space of embedded planar domains. This corollary to the theorem in the case γ is extremal was proved by Almgren-Simon using the theory of minimal currents. However, their methods do not show that their embedded minimal disk is actually a Douglas solution to Plateau's problem.

In [113] A. Tromba and F. Tomi showed that the space of smooth immersed minimal disks is a Banach manifold and the natural projection to the space of immersed circles in $\mathbb{R}^3$ is smooth. Using their manifold representation of the immersed minimal disks, they were also able to prove the existence of an embedded possibly unstable minimal disk in the case of an extremal Jordan curve (see also [112]).

The following theorem is essentially a converse to theorem 24 in the case Γ has one boundary curve (see [66]).

Theorem 25. *If a smooth Jordan curve γ in $\mathbb{R}^3$ is the boundary of an embedded stable minimal disk, then γ is contained on the boundary of a region which is a smooth ball having nonnegative mean curvature on its boundary.*

Section 12. A uniqueness theorem for extremal Jordan curves with total curvature less than or equal to 4π.

Theorem 24 in the previous section can be generalized to the case where the boundary of the region R is piecewise smooth with interior angles on the nonsmooth parts being less than or equal to π. One application for this generalization is the following.

Theorem 26. *Suppose γ_1 and γ_2 are two smooth Jordan curves on parallel planes. If γ_1 and γ_2 are the boundary of a connected compact branched minimal surface M, then γ_1 and γ_2 are the boundary of an embedded minimal annulus that has least area in a small regular neighborhood.*

The idea of the proof when M is embedded is as follows. Let T be the region between the parallel planes and let R be the closure of the unbounded component of T-M. Now the boundary of R is piecewise smooth with zero mean curvature on the boundary and the curves γ_1 and γ_2 which lie on the boundary of R are the boundary of a continuous map of an annulus into R. The piecewise smooth generalization to theorem 24 implies that there exists an embedded least area annulus in R with boundary γ_1 and γ_2.

Another nice application of theorem 24 is the following generalization of a special case of Nitsche's uniqueness theorem 12 (see [57], [61], [66]).

Theorem 27. *If γ is an extremal Jordan curve which is the boundary of a strictly unstable minimal disk or is the boundary of a compact non simply connected branched minimal surface, then γ is the boundary of at least two embedded minimal disks which have least area in some regular neighborhood. In particular, if the total curvature of γ in less than or equal to 4π, then γ is the boundary of a unique compact branched minimal surface which is an embedded minimal disk.*

We now give a sketch of the proof of the above theorem in the special case that γ is contained in the boundary sphere S of the unit ball B and γ is the boundary of a compact embedded minimal surface M which is a strictly unstable minimal disk or M is not a disk. In this case M disconnects the ball B into two compact regions R_1 and R_2. The boundaries of the regions R_1 and R_2 are piecewise smooth with interior angles less than π and with non-negative mean curvature on the smooth parts. Furthermore, by the Jordan curve theorem γ bounds two disks on the sphere S and hence γ bounds a disk in R_1 and in R_2. The piecewise smooth generalization of theorem 24 implies that γ is the boundary of an embedded least area disk D_1 in R_1 and D_2 in R_2. This completes the sketch of this special case of theorem.

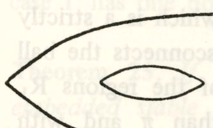
It is very possible that a smooth Jordan curve with total curvature less than or equal to 4π is the boundary of a unique compact branched minimal surface. If this were true, then theorem 18 would imply that the Douglas solution to Plateau's problem for γ is embedded. Thurston's examples in the next section show the uniqueness part of theorem 27 is sharp.

Section 13. Examples of W. Thurston.

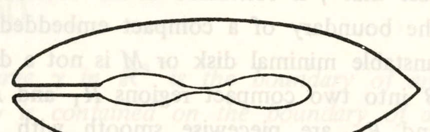
We now give a sequence of examples by W. Thurston which show that theorem 27 is sharp. W. Thurston constructs for every integer n an analytic curve γ_n on the unit sphere such that the length of γ_n is less than $\frac{1}{n}$ (thus γ_n is almost contained in a plane) and the total curvature of γ_n is less than $4\pi + \frac{1}{n}$. The curve γ_n has the property that for every k, $1<k<n$, γ_n is the boundary of n stable embedded minimal surfaces of genus k and the solution to Plateau's problem (2) has genus equal to n. The construction of γ_2 is as follows.

First consider a collection of three Jordan curves, as in figure 6A, made up of short arcs on large circles in the xy plane and such that the total curvature of these arcs is much smaller than $\frac{1}{n}$. Now connect these arcs by pairs α_1, α_2 and β_1, β_2 of very close arcs and perturb this example slightly in the plane so that the resulting curve $\tilde{\gamma}_2$ is analytic and has total curvature less than $4\pi + \frac{1}{2n}$ as in figure 6B. Let p be any point on $\tilde{\gamma}_2$. Consider for every r the sphere S_r of radius r with nonpositive z coordinates that is tangent to the xy plane at the point p.

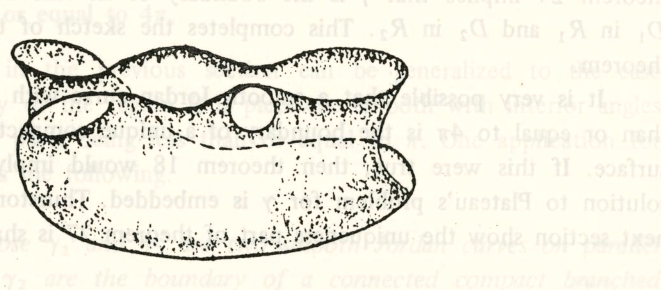
6-A



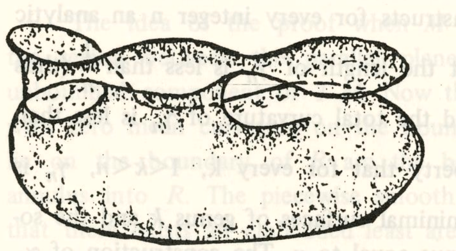
6-B



6-C



6-D



6-E

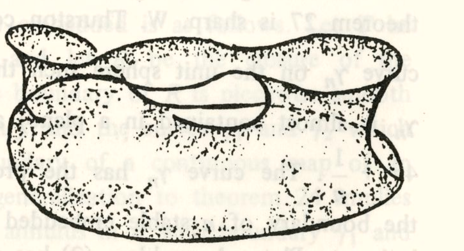


Fig. 6

Let $\gamma_2(r)$ be the intersection of the cylinder over $\tilde{\gamma}_2$ and the sphere S_r. If r is sufficiently large and the arcs α_1, α_2 and β_1, β_2 are sufficiently close, then

$$\gamma_2 = \left(\frac{1}{r}\,\gamma_2(r)\right) \subset S_1$$

is the required curve. We refer the reader to figure 6-CDE for the examples of the minimal surfaces produced. A rigorous proof of the existence of these surfaces would use the bridge principle in the next section. To create γ_n one uses more Jordan curves in the initial construction.

Section 14. The Bridge Principle.

The bridge principle is related to a physical property of soap films. The physical experiment concerning this principle is as follows: Consider two soap film surfaces with boundary curves being two bent steel wires. Change the wire configuration by joining these wires. Change the wire configuration by joining these wires by close parallel wires which we shall call a *bridge pair.* Mathematically we take the connected sum of two wire curves along what we will call a *bridge curve.* The physical property of the new wire configuration is that it is the border of a new soap film surface that is close to the old surfaces joined together with a soap film bridge joining the old surfaces (see figure 7).

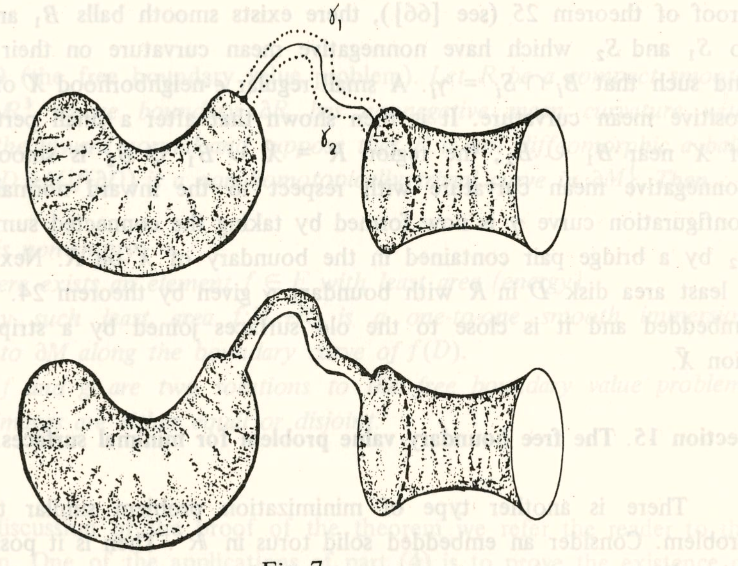


Fig. 7

Since soap films correspond to stable minimal surfaces, the bridge principle can be reformulated using the concept of stable minimal surfaces. Unfortunately the bridge principle is not true in general. This can be easily seen by considering S_1 and S_2 to be disks of radius 1 and radius 2, respectively, in the xy-plane. If the boundary curves of S_1 and S_2 are joined by a bridge pair in S_1–S_2, then the unique compact minimal surface with the new boundary does not contain a bridge. Thus, in order to prove the bridge principle some care must be taken with the choice of the bridge pair.

We now give a correct version of the bridge principle whose proof is based on the topological methods developed in [66].

Theorem 28 (the bridge principle). *Let* S_1 *and* S_2 *be two stable compact minimally immersed surfaces in* $\mathbb{R}^3$ *and* α *a bridge curve. Then given any* ϵ *there exists a bridge pair* ϵ *close to* α *and such that the new configuration is the boundary of a compact stable minimal surface* ϵ*-close to* S_1 *and* S_2 *joined by a strip* ϵ*-close to* α*. Furthermore if* S_1 *and* S_2 *are embedded and disjoint and* α *intersects* $S_1 \cup S_2$ *only at its boundary points, then the new surface is also embedded.*

The idea of the proof of the above bridge theorem in the geometric case where S_1 and S_2 are disjoint stable embedded minimal disks and the bridge curve α intersects S_1 and S_2 only on its boundary points is as follows. Let γ_1 and γ_2 be the boundary curves of S_1 and S_2, respectively. Using the proof of theorem 25 (see [66]), there exists smooth balls B_1 and B_2 close to S_1 and S_2 which have nonnegative mean curvature on their boundaries and such that $B_i \cap S_i = \gamma_i$. A small regular ϵ-neighborhood X of α also has positive mean curvature. It is then shown that after a small perturbation $\tilde{X}$ of X near $B_1 \cup B_2$, the region $R = \tilde{X} \cup B_1 \cup B_2$ is smooth and has nonnegative mean curvature with respect to the inward normal. The new configuration curve γ is now formed by taking the connected sum of γ_1 and γ_2 by a bridge pair contained in the boundary of $\tilde{X}$ on R. Next one takes a least area disk D in R with boundary γ given by theorem 24. This disk is embedded and it is close to the old surfaces joined by a strip in the region $\tilde{X}$.

Section 15. The free boundary value problem for minimal surfaces.

There is another type of minimization problem similar to Plateau's problem. Consider an embedded solid torus in $\mathbb{R}^3$. Then is it possible to cut this solid torus with a cross-section disk of least area?

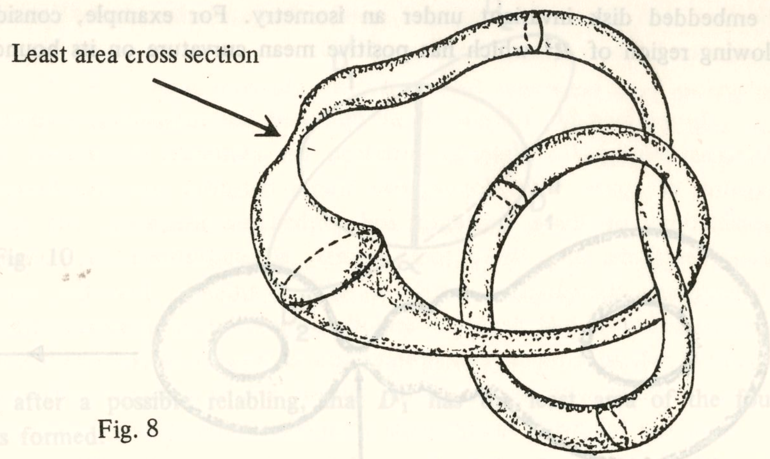


Fig. 8

In general, such an embedded least area cross-section does not exist. However, if the boundary of the solid torus has nonnegative mean curvature, then the following theorem in [65] shows that there is a cross-sectional disk of least area. Problems such as these in the calculus of variations where the minimizing map does not have a fixed boundary are called free boundary value problems. As stated, part 1 of this theorem also depends on results in [60].

Theorem 29 (the free boundary value problem). *Let* R *be a compact smooth region of* $\mathbb{R}^3$ *whose boundary* ∂R *has nonnegative mean curvature with respect to the inward normal and suppose that* R *is not diffeomorphic a ball. Let* $F = \{f: D \to M \mid f(\partial D)$ *is a non homotopically trivial curve in* $\partial M\}$*. Then*

1) *F is non-empty.*

2) *There exists an element $f \in F$ with least area (energy).*

3) *Any such least area $f: D \to R$ is a one-to-one smooth immersion orthogonal to ∂M along the boundary curve of $f(D)$.*

4) *If f and g are two solutions to the free boundary value problem, then their images are either equal or disjoint.*

For discussion of the proof of the theorem we refer the reader to the next section. One of the applications of part (4) is to prove the existence of an embedded disk invariant under an isometry. For example, consider the following region of $\mathbb{R}^3$ which has positive mean curvature on its boundary.

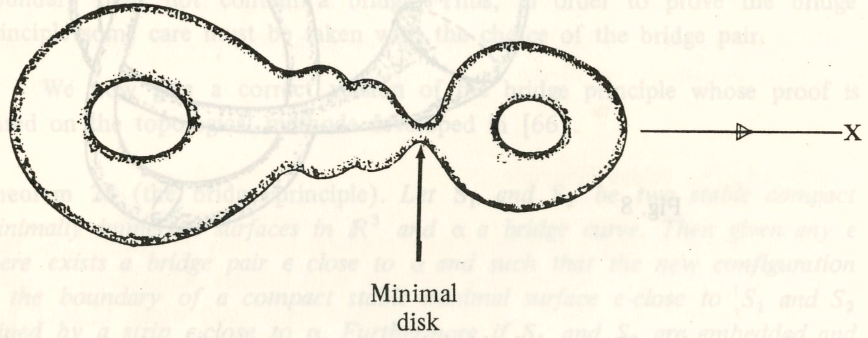


Fig. 9

The above region is invariant under rotation $g: R \to R$ around the x-axis. The solution of the free boundary value problem for R is invariant under g. To see this first note that the crossectional disk D of least area of this region must be near the center of the figure and disconnects R. Sing g is an isometry, $g(D)$ is another solution to the free boundary value problem and as such $g(D)$ equals D or is disjoint from D. Clearly D must intersect the x-axis and g has a fixed point on D. Therefore, $g(D)$ must equal D.

The idea of the proof of the disjointness of solutions of the free boundary value problem for a non simply connected compact region R with nonpositive mean curvature on ∂R is as follows. Let D_1 and D_2 be two such embedded least area solutions and suppose that D_1 and D_2 are in general position. In this case $D_1 \cap D_2$ is either empty or consists of a finite collection of Jordan arcs and curves.

Suppose that there is a Jordan arc α in $D_1 \cap D_2$. In this case α disconnects D_i into two subdisks D_i' and D_i'' for $i = 1,2$.

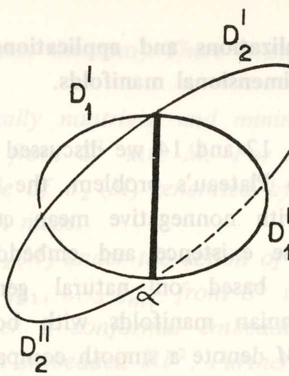


Fig. 10

Suppose, after a possible relabling, that D_1' has the least area of the four new disks formed.

Now consider the new piecewise smooth disks $D_3 = D_1' \cup D_2'$ and $D_4 = D_1' \cup D_2''$. Since these disks are not smooth along α, their areas can be decreased by a variation in the interior of M to get new disks $\tilde{D}_3$ and $\tilde{D}_4$. By the least area choice of D_1',

$$\text{Area}(\tilde{D}_3) < \text{Area}(D_2) \text{ and } \text{Area}(\tilde{D}_4) < \text{Area}(D_2).$$

Since D_2 is a solution to the free boundary value problem for R, the boundary curves of $\tilde{D}_3$ and $\tilde{D}_4$ must be homotopically trivial in ∂R. However, ∂D_2 can be expressed as a product of $\partial\tilde{D}_3$ and $\partial\tilde{D}_4$ in the fundamental group of ∂R. This shows that ∂D_2 is homotopically trivial contrary to the definition of a solution to the free boundary valued problem. This contradiction shows that $D_1 \cap D_2$ cannot contain a Jordan arc. A similar disk replacement argument implies that $D_1 \cap D_2$ does not contain a closed Jordan curve and hence $D_1 \cap D_2$ is empty. We refer the reader to [65] for a proof in the case that D_1 and D_2 are not in general position (see also the proof of theorem 69 in section 24).

Another problem considered in the calculus of variations is the partially free boundary value problem of which the following is a special case ([18] and [67]).

Theorem 30. *Let γ be a smooth Jordan curve in a plane P_1 and suppose that γ is one of the boundary curves of an immersed annulus Ω with the other boundary curve of Ω on a parallel plane P_2. Suppose that the area of Ω is less than the area of the disk in P_1 with boundary γ. Then*

1) There exists a map $f: \Omega \to \mathbb{R}^3$ with least area such that one of the boundary curves of f is γ and the other boundary curve is contained on the plane P_1.

2) Any such f is a one-to-one immersion which is orthogonal to P_2.

Section 16. Generalizations and applications of the Meeks-Yau embedding theorems in three dimensional manifolds.

In sections 11, 12 and 14 we discussed the embedding properties of the Douglas solution to Plateau's problem, the existence of minimal surfaces in regions of space with nonnegative mean curvature on the boundary, and applications of these existence and embedding theorems. The theorems in these sections are based on natural generalizations to compact three dimensional Riemannian manifolds with boundary. For the rest of this section we will let M denote a smooth compact three dimensional Riemannian manifold whose boundary is lipshitz and convex or smooth with non-negative mean curvature with respect to the inward normal vector. For these manifolds we have the following fundamental topological-geometric embedding theorems.

Theorem 31 (the geometric Dehn's lemma for planar domains). *Let* $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ *be a collection of pairwise disjoint continuous Jordan curves on* ∂M *and suppose that* Γ *is the boundary of a map of a compact planar domain in to* M. *Then*

1) *There exists a map* $f : \Omega \to M$ *of least area (energy) from a compact possibly disconnected planar domain* Ω *such that* $f(\partial\Omega) = \Gamma$.

2) *Any such* f *is one-to-one.*

3) *Any such* f *is a smooth immersion on* $\overset{\circ}{\Omega}$.

4) *If* Γ *and* ∂M *are smooth, then any such* f *is an immersion.*

5) *If* $n = 2$ *and* γ_1 *is homotopically nontrivial in* M *or if the curves in* Γ *generate a subgroup of rank* $n-1$ *in* $H_1(M,Z)$, *then* Ω *is path connected and if* $g : \Omega \to M$ *is another least area map, then either* $g(\overset{\circ}{\Omega}) \cap f(\overset{\circ}{\Omega}) = \Phi$ *or* $g(\Omega) = f(\Omega)$.

Theorem 32 (the geometric loop theorem). *Suppose that* $F = \{f : D \to M \mid f(\partial D)$ *is a homotopically non-trivial curve in* $\partial M\}$ *is non-empty. Then:*

1) *There exists an element* $f \in F$ *of least area (energy) called a solution to the free boundary value problem for* M.

2) *Any such* f *is one-to-one.*

3) *Any such* f *is a smooth immersion on* $\overset{\circ}{\Omega}$.

4) *If* ∂M *is smooth, then* f *is a smooth immersion orthogonal to* ∂M *along* $f(\partial D)$.

5) *If* $g : D \to M$ *is another least area solution to the free boundary value problem for* M, *then either* $g(D) = f(D)$ *or* $g(D) \cap f(D) = \phi$.

Theorem 33 (the geometric sphere theorem). *There exists maps* $f_1, f_2, \dots, f_k$ *from* S^2 *into* M *such that*

1) $f_1 : S^2 \to M$ *is homotopically nontrivial and minimizes area among all homotopically nontrivial maps from* S^2 *into* M. *For each* i, f_i *does not belong to the* $\pi_1(M)$ *submodule of* $\pi_2(M)$ *generated by* $\{f_1, \dots, f_{i-1}\}$ *and* f_i *minimizes area among all such maps.*

2) $\{f_1, \dots, f_k\}$ *generate* $\pi_2(M)$ *under the action of* $\pi_1(M)$.

3) *For any set of maps* $\{g_1, \dots, g_m\}$ *from* S^2 *into* M *which satisfy property* (1), *then* g_i *is either a conformal embedding or a two-to-one covering map whose image is an embedded* $\mathbb{R}P^2$. *Furthermore, if* $\{f_1, \dots, f_k\}$ *and* $\{g_1, \dots, g_m\}$ *are two sets of mappings satisfying property* (1), *then for all* i *and* j, *either the images of* f_i *and* g_j *are disjoint or equal.*

The above theorems (see [64], [65], [66], [67]) are called geometric because of analogous purely topological theorems in the theory of three dimensional manifolds (see [24], [39], [89], [103], [116]). We now give a quick summary of the elements in the proof of these theorems.

The existence of a solution to theorem 31 follows from a generalization of a theorem of Morrey to manifolds with boundary. Morrey first defines the concept of a "homogeneously regular" Riemannian manifold which includes as examples all compact manifolds and some noncompact manifolds such as $\mathbb{R}^n$. His theorem states that if $\tilde{M}$ is a complete homogeneously regular manifold and if Γ is a collection of disjoint rectifiable Jordan curves which are the boundary curves of a map of a planar domain, then there exists a map $f : \Omega \to \tilde{M}$ of least area (energy) and such an f is smooth on $\overset{\circ}{\Omega}$. In [64] and [66] it is shown that M can be extended to a complete homogeneously regular manifold $\tilde{M}$ in such a way that the Morrey solution in $\tilde{M}$ for Γ is always contained in M.

The existence of a solution to the free boundary value problem is a new result and its analytic proof appears in [65]. The existence of the minimal spheres in theorem 33 follows from a slight generalization of the beautiful existence theorem of Sachs-Uhlenbeck for minimal spheres in Riemannian manifolds [97].

The regularity results are based on the interior regularity theorem of Morrey [13] and on the regularity results for Plateau's problem discussed in section 9.

The proof of the embedding of the least area solutions is based on approximation arguments and on the tower construction in the theory of three dimensional manifolds. This type of construction was first used in the proof of some topological versions of the above theorems by Papakyriakopoulous. An application of a tower construction to the proof of theorem 31 is given in section 24.

Parts (5) in theorem 32 and (3) in theorem 33 have some very important applications to finite group actions in three dimensional manifolds. These applications arise from the following situation. Suppose that G is a finite group of diffeomorphisms acting on M. After first averaging a Riemannian metric on ∂M, there exists a metric on M which is invariant under G and which is a product metric on some ϵ-neighborhood of ∂M. Such metrics give M a convex boundary. Thus, we may assume that G acts as a group of isometries on M in a metric where the boundary of M is convex.

Now consider a disk $D \subset M$ which is a solution to the free boundary value problem. If $g \in G$, then $g(D)$ is another least area solution to the free boundary value problem and the disjointness property implies that $g(D) = D$ or $g(D) \cap D = \phi$. Thus theorem 32 yields the following new topological result [65].

Theorem 34 (equivariant loop theorem). *Suppose that M is a three dimensional manifold and suppose G is a finite group of diffeomorphisms of M. If the map $i_*: \pi_1(\partial M) \to \pi_1(M)$ induced by inclusion has a nontrivial kernel, then there exists an embedded disk D in M with ∂D being a homotopically nontrivial curve in ∂M and such that for all $g \in D, g(D) = D$ or $g(D) \cap D = \phi$.*

The above theorem has some deep applications to the theory of group actions on three dimensional manifolds. Combining an observation of Jordan and Litherland, the above equivariant loop theorem and a theorem of W. Thurston on the existence of an incompressible surface (which depends on a theorem of H. Bass), one can then settle in the affirmative the conjecture of P.A. Smith on the unknottedness of the fixed point set of a finite cyclic group action on S^3 (see [118]). The solution of the Smith conjecture implies that finite cyclic groups actions on S^3 with a fixed point must be conjugate to a linear action on S^3.

The least area spheres in theorem 33 enjoy the same disjointness equivariance properties that solutions to the free boundary valued problem satisfy. The associated equivariant sphere theorem implies that for a finite group acting smoothly on the connected sum of a nonsimply connected prime manifolds with fundamental group non-isomorphic to the integers, the action must split equivariantly up to the permutations of the factors. Hence, basically when we study finite group actions on a three dimensional manifold, we can assume that the manifold is prime.

The geometric shpere theorem also enables us to deal with finite group actions on noncompact manifolds. For example, combining the affirmative answer to the Smith conjecture with the geometric sphere theorem, one proves the following classification theorem [68].

Theorem 35. *If G is a compact group of orientation preserving diffeomorphisms of $\mathbb{R}^3$, then G is isomorphic to a compact subgroup of $SO(3)$. If G is not isomorphic to the icosahedral group A_5, then there is a diffeomorphism $f: \mathbb{R}^3 \to \mathbb{R}^3$ such that fGf^{-1} is a subgroup of $SO(3)$. In particular, any periodic diffeomorphism of $\mathbb{R}^3$ is conjugate to a rotation.*

Section 17. The existence of periodic minimal surfaces in $\mathbb{R}^3$ and a magical application of Abel's theorem.

There is a large class of known embedded proper complete minimal surfaces in $\mathbb{R}^3$. These minimal surfaces are periodic which means that they are invariant under a group of translations. For example, the helicoid and the superman surface S in figure 1 are part of global periodic minimal surfaces. Periodic minimal surfaces naturally arise when a complete minimal surface contains two parallel lines. Reflection in these lines leaves the surface invariant. Composed reflections in two parallel lines is a translation which leaves the surface invariant.

A specially interesting class of embedded periodic minimal surfaces are the surfaces M which are invariant under a lattice L_M of translations. After taking the quotient $\overline{M} = M/L_M$, we have a compact embedded minimal surface $\overline{M}$ in the flat torus $T^3 = \mathbb{R}^3/L_M$. It is straight forward to verify that a compact embedded minimal surface $\overline{M}$ of genus greater than one in a flat torus T^3 always yields a triply periodic minimal surface M in T^3 by taking the "lift" of $\overline{M}$ to the universal covering space $\mathbb{R}^3$ of T^3.

There is an interesting six dimensional family of triply periodic embedded minimal surfaces in $\mathbb{R}^3$ which arise from consideration of certain elementary facts about the canonical curve of a hyperelliptic Riemann surface M_3 of genus 3 ([33] and [56]). First, the canonical curve of a hyperelliptic Riemann surface M_3 can be choosen so that its image is the quadric $Q : X^2 + Y^2 + Z^2 = 0$ in CP^2. In this case $C: M_3 \to Q$ is a two sheeted cover of Q which is diffeomorphic to S^2 and conversely every two sheeted cover of Q branched over eight points corresponds to the canonical curve of a hyperelliptic Riemann surface of genus 3. This means that if $g: M_3 \to Q$ is a two sheeted branch cover of Q of genus 3, then g can be identified (up to a multiplication by a fixed complex number) with three holomorphic 1-forms on M_3

$$\{w_i(z) = f_i(z)dz \mid i = 1,2,3\}$$

such that $f_1^2(z) + f_2^2(z) + f_3^2(z) = 0$. From the discussion of the generalized Weierstrass representation for minimal surfaces in section 22 (see also section 18), it follows that such a $g:M_3 \to Q$ corresponds on the universal cover of M_3 to the Gauss map of a periodic minimal suface in $\mathbb{R}^3$. For these examples the local coordinate functions can be expressed in terms of elliptic integrals. The next theorem [56] shows that certain of these surfaces have embedded images.

For the construction of other interesting triply periodic minimal surfaces we refer the reader to [63], [76], [77] and [98].

Theorem 36. *Let M be a compact Riemann surface of genus three which can be represented conformally as a two sheeted covering of S^2 branched over eight antipodal points. Then there exist flat tori T_1 and T_2 and two conformal minimal embeddings $f_1:M \to T_1$ and $f_2:M \to T_2$ which give rise to distinct triply periodic minimal surfaces in $\mathbb{R}^3$.*

Further analysis of the examples in theorem 36 show that for every lattice L in $\mathbb{R}^3$ there exists an embedded triply periodic minimal surface invariant under translation by elements in L. Using a somewhat delicate analysis, the author believes he has a proof that every flat T^3 contains an infinite number of embedded minimal surfaces of genus three.

While compact minimal surfaces in the family described above are not always rigid in their tori, the following rigidity theorem [56] holds in $\mathbb{R}^3$.

Theorem 37. *Suppose $f,g:M \to \mathbb{R}^3$ are two proper triply periodic isometric minimal immersions of a Riemannian surface M. Then $f = R \circ g$ where R is a rigid motion of $\mathbb{R}^3$.*

The next beautiful theorem follows from a simple application of Abel's theorem [56].

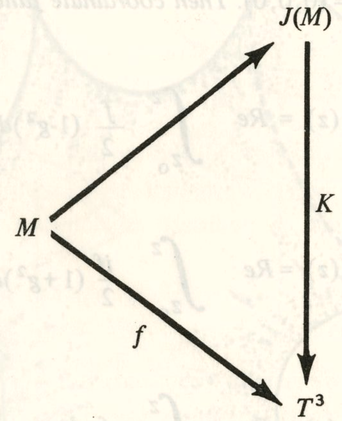
Theorem 38. *Suppose $f:M \to \mathbb{R}^3$ is a proper immersed triply periodic minimal surface with lattice L. Let R be any fixed fundamental region of the lattice and let $G:M \to S^2$ denote the Gauss map. For $v \in S^2$, define $\{p_1(v), \ldots, p_k(v)\} = G^{-1}(v) \cap R$ where $p_j(v)$ is listed with multiplicity. Then after a fixed translation of M, we have for all $v \in S^2$ that*

$$p_1(v) + p_2(v) + \ldots + p_k(v) \in L.$$

The idea of the proof is the following. First note that the above theorem will hold if for the associated compact immersed surface M in $T^3 = \mathbb{R}^3/L_M$ the sum of the points

$$\bar{p}_1(v) + \ldots + \bar{p}_k(v) = 0 \in T^3.$$

Since $G:\bar{M} \to S^2$ is conformal we can apply Abel's theorem to M. A simple application of Abel's theorem [33] implies that after a translation of $\bar{M}$ in its Jacobian variety $J(\bar{M})$ that the sum of points $\bar{p}_1(v) + \ldots + \bar{p}_k(v) = 0$ in $J(M)$. Since the minimal immersion $\bar{f}:\bar{M} \to T^3$ is given by the integration of harmonic oneforms on M, we have the diagram



where K is a "linear" homomorphism. Since K is linear, it follows that the sum

$$\bar{p}_1(v) + \ldots + \bar{p}_k(v) = 0 \text{ in } T^3.$$

The above theorem implies that the examples in theorem 36 are all invariant under an inversion through any zero of Gauss curvature. In particular, it follows that an embedded compact minimal surface of genus three in a flat torus has, after a translation, its zeroes of Gaussian curvature at precisely the points of T^3 which have order two in the group structure of T^3.

Section 18. The Gauss map, the Weierstrass representation, the associate surfaces for a minimal surface and the example of Jorge-Xavier.

The Gauss map in the theory of minimal surfaces plays a very important role and has special properties. First of all it is easy to check that the Gauss map of the surface is conformal with respect to the orientation on the sphere induced from the inward unit normal. Thus, $G:M \to S^2 = \mathbb{C} \cup \{\infty\}$ can be considered as a meromorphic function on M.

The Gauss map considered as meromorphic function on M enters into a formula due to Weierstrass for representing minimal surfaces is terms of holomorphic functions on M.

Theorem 39. (the Weierstrass representation). *Suppose that $\Psi: M \to \mathbb{R}^3$ is a minimal surface with $\Psi(z_0) = (0,0,0)$. Then coordinate functions of Ψ are given by*

$$\Psi_1(z) = Re \int_{z_0}^{z} \frac{f}{2}\,(1-g^2)dz$$

$$\Psi_2(z) = Re \int_{z_0}^{z} \frac{if}{2}\,(1+g^2)dz$$

$$\Psi_3(z) = Re \int_{z_0}^{z} fg dz$$

for some meromorphic 1-form $f(z)dz$. Here $g:M \to S^2$ is the Gauss map for M. Conversely, if the above integrals Ψ_i are well defined on a Riemann surface M for some meromorphic function g and meromorphic 1-form $f(z)dz$, then $\Psi = (\Psi_1, \Psi_2, \Psi_3): M \to \mathbb{R}^3$ is a conformal minimal immersion with branch points.

An interesting collection of complete examples are found by letting $M = S^2 - \{\theta_1, \ldots, \theta_n\}$ where θ_i is an n-th root of unity, $g(z) = z^{n-1}$, and $f(z) = \frac{1}{(z^n-1)^2}$. In the case $n = 2$, we have the catenoid. For all n, the examples are invariant under rotation by an angle of $2\pi/n$ around the z-axis. For $n = 3$ we have the following compact part of this surface, the existence of these examples appears in [47].

Compact part
of 3-end catenoid

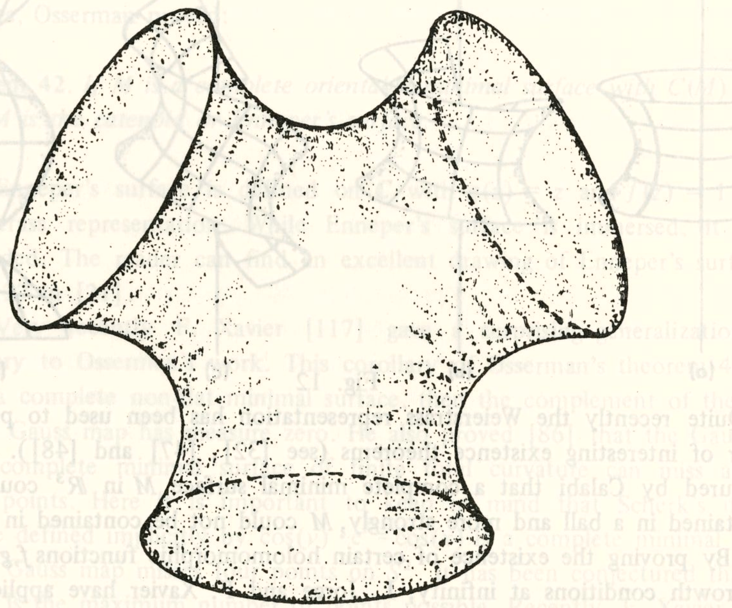


Fig. 11

If $\Psi: M \to \mathbb{R}^3$ is a minimal immersion of a simply connected minimal surface, then there exists an isometric family of minimal surfaces $\Psi_\theta: M \to \mathbb{R}^3$ where one uses the form $e^{i\theta} f(z)dz$ in place of $f(z)dz$ in the Weierstrass representation. These surfaces Ψ_θ, when they exist, are isometric to the original surface and are called associate surfaces to Ψ. The associate surface $\Psi_{\pi/2}$ is called the conjugate surface to Ψ.

If the minimal surface is simply connected, then the associate surfaces always exist. By taking the universal covering space of the minimal surface, we may always suppose that the associate surface exists. The helicoid and the catenoid are conjugate surfaces in this sense. The following sequence of figures shows how to deform a connected part of the catenoid to a part of the helicoid through associate surfaces.

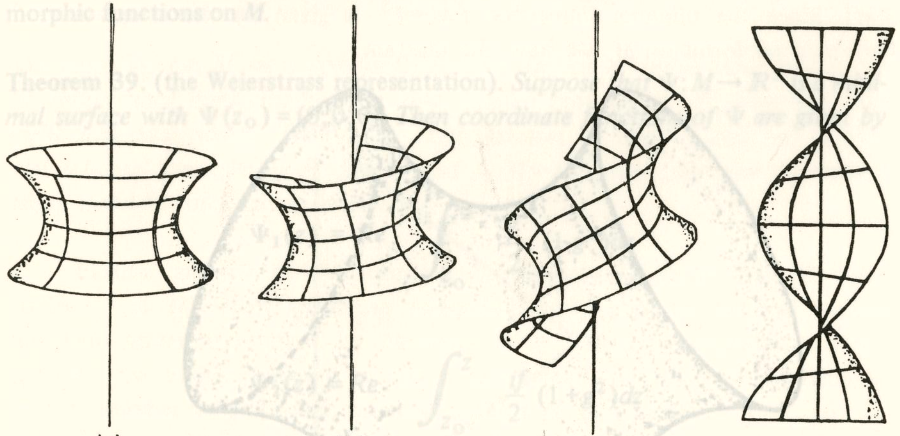


Fig. 12

Quite recently the Weierstrass representation has been used to prove a number of interesting existence theorems (see [32], [47] and [48]). It was conjectured by Calabi that a complete minimal surface M in $\mathbb{R}^3$ could not be contained in a ball and more strongly, M could not be contained in a half space. By proving the existence of certain holomomorphic functions $f,g: D \to C$ with growth conditions at infinity, L. Jorge and F. Xavier have applied the Weierstrass representation to show that the strong version of Calabi's conjecture is not true. They prove the following.

Theorem 40. *There exists a complete immersed minimal surface M in $\mathbb{R}^3$ which is contained between two parallel planes.*

Section 19. The theorems of R. Osserman and of F. Xavier concerning the image of the Gauss map for a complete minimal surface.

R. Osserman proved some deep properties concerning the Gauss map G for a complete minimal surface. For example [88]:

Theorem 41. *Let M be a complete orientable minimal surface in $\mathbb{R}^3$. If the total Gaussian curvature $C(M) = \int_M KdA$ is infinite, then for almost every $v \in S^2$, $G^{-1}(v)$ is an infinite set. If $C(M)$ is finite, then M is conformally equivalent to a compact Riemann surface $\overline{M}$ punctured in a finite number of points and the Gauss map extends conformally to $\overline{M}$. In particular, $C(M)$ is either infinite or an integer multiple of -4π.*

Applying theorem 41 and the Weierstrass representation for minimal surfaces, Osserman proved:

Theorem 42. *If M is a complete orientable minimal surface with $C(M) = -4\pi$, then M is the catenoid or Enneper's surface.*

Enneper's surface is defined on $\mathbb{C}$ with $g(z) = z$ and $f(z) = 1$ in the Weierstrass representation. While Enneper's surface is immersed, it is not embedded. The reader can find an excellent drawing of Enneper's surface on page 127 in [21].

Very recently F. Xavier [117] gave a surprising generalization to a corollary to Osserman's work. This corollary of Osserman's theorem 41 is: if M is a complete non-flat minimal surface, then the complement of the image of the Gauss map has measure zero. He also proved [86] that the Gauss map of a complete minimal surface of finite total curvature can miss at most three points. Here it is important to keep in mind that Scherk's minimal surface defined implicitly by $\cos(y) \cdot e^t = \cos(x)$ is a complete minimal surface whose Gauss map misses four points on S^2. It has been conjectured that four points is the maximum number of points possible. Recently, F. Xavier proved the following.

Theorem 43. *The Gauss map of a non-flat complete minimal surface in $\mathbb{R}^3$ can omit at most 6 points.*

Section 20. The topology of complete minimal surfaces of finite total curvature.

Let M be a complete minimally immersed surface in $\mathbb{R}^3$ with finite total curvature $C(M)$. By theorem 41 M is conformally equivalent to $\overline{M} - \{p_1, \ldots, p_k\}$ where $\overline{M}$ is a compact Riemann surface and the Gauss map extends conformally to $\overline{M}$ to give $G:\overline{M} \to S^2$.

The following theorems arise from a topological-geometric study of these surfaces. The intersection of a small disk centered at a point p_j on $\overline{M}$ with M will be called an end E_j of M (see [47]).

Theorem 44. *Let $M = \overline{M} - \{p_1, \ldots, p_k\}$ be as above. Then*

1) *The immersion of M is proper.*

2) *For each end E_j of M the curves $(\frac{1}{r} E_j) \cap S^2$ converge smothly to a geodesic on S^2 with multiplicity I_j as r goes to infinity.*

3) $C(M) = 2\pi(\chi(M) - \sum_{j=1}^{k} I_j)$.

4) $C(M) = 2\pi(\chi(M) - k)$ *if and only if for each j there is an embedded end E_j.*

5) *For every $k > 1$, there exists an example M which is diffeomorphic to $S^2 - \{p_1, \ldots, p_k\}$ and such that $C(M) = 2\pi(\chi(M)) - k)$.*

The above theorem describes the important topological properties of M. The proofs of part (1) and (2) are carried out by analyzing the intersection of the ends of M with planes. Part (2) shows that all such examples viewed from infinity look like a finite collection of planes with multiplicity that pass through the origin. For example, the catenoid viewed from infinity looks like a single plane with multiplicity two. The formulas in parts (3) and (4) give a geometric interpretation for $C(M)$ in terms of the behavior of M at infinity in $\mathbb{R}^3$. Part (5) follows by letting $g(z) = z^{k-1}$ and $f(z) = \frac{1}{(z^k - 1)^2}$ in the Weierstrass representation.

Theorem 45. *Let M be as in theorem 44 and suppose that M is embedded. Then*

1) *If M has one topological end, then M is a flat plane.*

2) *After a rotation of M in $\mathbb{R}^3$, the Gauss map has zeroes and poles at the ends of M.*

3) *If M is diffeomorphic to an annulus, then M is a catenoid.*

4) *M cannot be diffeomorphic to the complement of 3, 4 or 5 points on S^2.*

5) *Let R_1 and R_2 be the two components of $\mathbb{R}^3 - M$. Then for $B_r = \{x \in \mathbb{R}^3 \,\|x\| \leq r\}$, $L = \lim \frac{Vol\,(R_1 \cap B_r)}{Vol\,(R_2 \cap B_r)} = 0, 1, \infty$ where 1 occurs if and only if M has an odd number of ends.*

6) *Let R_1 and R_2 be the two components of $\mathbb{R}^3 - M$ such that L in (5) equals 0 or 1. Suppose that M has k ends and that the genus of $\overline{M}$ is g. If $L = 1$, then R_1 and R_2 are diffeomorphic to the interior of a solid $(g + \frac{1}{2}(k-1))$-holed torus. If $L = 0$, then R_1 is diffeomorphic to the interior of a solid $(g + \frac{k}{2})$-holed torus and R_2 is diffeomorphic to the interior of a solid $(g - 1 + \frac{k}{2})$-holed torus.*

7) *If M' is another embedded example which is diffeomorphic to M and if M has two ends, then M and M' are isotopic in $\mathbb{R}^3$.*

The proof of theorem 45 is largely based on interpreting theorem 44. In reference to (1), the reader should keep in mind that there exist immersed examples with one end such as Enneper's surface. Part (2) of theorem 45 and part (4) of theorem 43 combine to show that the holomorphic functions in the Weierstrass representation of the minimal surface are very special. Parts (3) and (4) of the theorem 45 follow easily from an analysis of this representation. Further calculations with the Weierstrass representation will probably show that the plane and the catenoid are the unique embedded minimal planar domains which have finite total Gaussian curvature. The proof of parts (6) and (7) are based on the uniqueness theorems in [57] and [60].

Section 21. Geometric examples of minimal surfaces of finite total curvature.

In 1975 Gackstatter and Kunert [31] proved the following very nice existence theorem for complete minimal surfaces of finite total curvature.

Theorem 46. *Let M be a compact Riemann surface. Then there exists a finite number of points $B = \{p_1, p_2, \ldots, p_k\}$ on M and a conformal minimal immersion $f: M - B \to \mathbb{R}^3$ such that in the induced metric the surface is complete and the total curvature is finite.*

The proof of the above theorem is by way of the Weierstrass representation. Gackstatter and Kunert apply the Riemann-Roch theorem to prove the existence of the required functions. Unfortunately their examples have relatively large total curvature. Recently new examples of low genus and low total curvature were given ([32] and [47]) by using the Weierstrass representations.

The difficulty in finding these and other new examples by using the Weierstrass is two fold. First one must guess the holomorphic function $g(z)$ and the holomorphic 1-form $f(z)dz$ in the Weierstrass on a fixed compact Riemann surface. Second one must prove that the associated 1-forms in the Weierstrass representation have imaginary periods and real residues. This method requires a lot of good guessing and calculation even in the simplest explicit examples.

In [63] the author has developed a geometric method of finding new examples of minimal surfaces of low finite total curvature. This new method uses a conjugate minimal surface construction. More precisely, consider a geometrically appealing possibility such as one of the examples in the figure given below. These "topological examples" X have such a large group of plane symmetries that there exists a simply connected fundamental region R that generates X after a finite number of plane reflections. Here the boundary of R consists of plane curves.

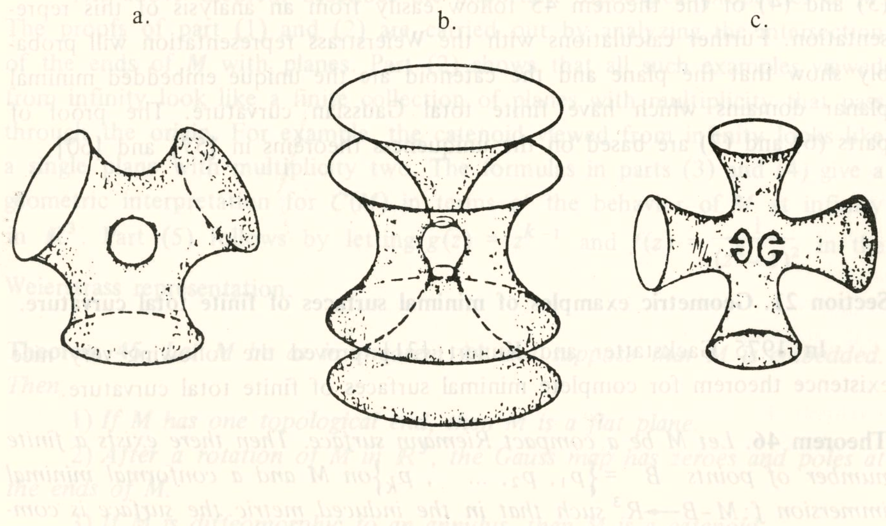


Fig. 13

On the would-be conjugate surface R^* of R, the boundary ∂R^* would consist of straight lines perpendicular to the planes containing the corresponding boundary curves of R. Our method produces an appropriate R^* which then proves the existence of R.

All examples X given so for by this conjugate minimal surface construction have catenoid type ends. This implies by theorem 45 that these surfaces have total curvature $C(X) = 2\pi(\chi(X)-k)$ where k is the number of ends of M. To create an approximate finite portion of ∂R^* in Figure 11, we use the following Jordan curve γ_L where α_2 is part of a helix.

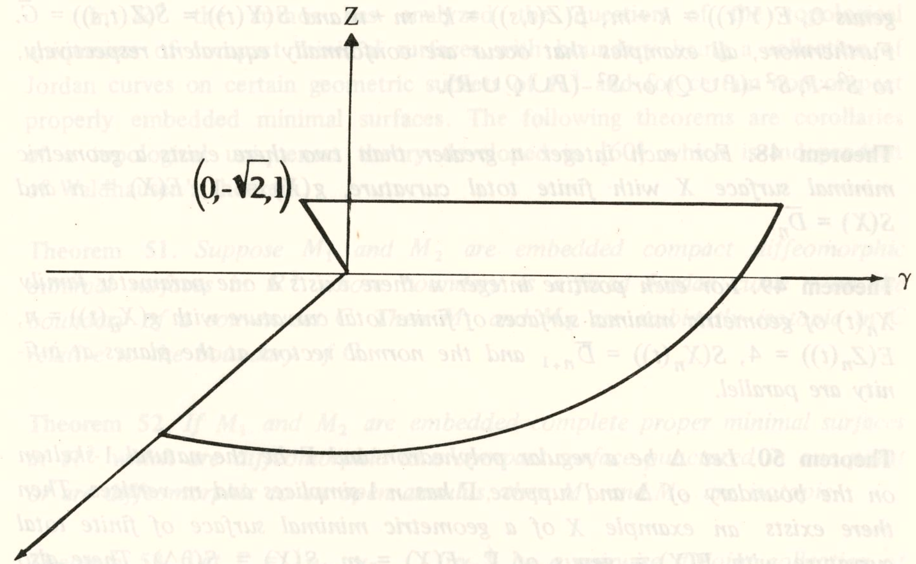


Fig. 14

The curve γ_L bounds a unique minimal disk D_L whose interior is a graph over the plane orthogonal to the vector $(0,\sqrt{2},1)$. Letting L go to infinity the disks D_L converge to the required surface R^*. Since $R = -R^{**}$, we have a method for producing R and hence by plane reflections the example in Figure 11. By changing the angle θ between α_2 and the vector $(0,-1,0)$ to $\theta_n = \pi/n$ one acquires an n-ended catenoid with symmetry group isomorphic to the Z_2 extension $\overline{D}_n \subset O(3)$ of the dyhedral group D_n.

The method outlined above using plastic "soap film" models of a compact part of R^* can be used to produce plastic models of surfaces by the conjugate surface construction without ever explicitly knowing the coordinate functions for the surfaces. This method yields the next existence theorems which appear in [63].

Definition. If M is a surface in $\mathbb{R}^3$, then the space group $S(M)$ is the group of isometries of $\mathbb{R}^3$ which leave M invariant. Let $g(M)$ denote the genus of M and let $E(M)$ denote the number of ends of M. If G is a subgroup of $SO(3)$, then let $\bar{G}$ denote the Z_2-extension in $O(3)$.

Theorem 47. *Let G be a finite subgroup of $SO(3)$ and $P = \{p_1, \ldots, p_k\}$, $Q = \{q_1, \ldots, q_m\}$, $R = \{r_1, \ldots, r_n\}$ be distinct singular orbits of G on S^2. Then there exists a unique geometric minimal surface X with finite total curvature, $g(X) = 0$, $E(X) = k$ and $S(X) = \{f \in O(3) | f(P) = P\}$. There exist one and two parameter families of examples $Y(t)$, $Z(t,s)$ respectively, with genus 0, $E(Y(t)) = k+m$, $E(Z(t,s)) = k+m+n$ and $S(Y(t)) = S(Z(t,s)) = \bar{G}$. Furthermore, all examples that occur are conformally equivalent respectively, to S^2-P, S^2-$(P \cup Q)$ or S^2-$(P \cup Q \cup R)$.*

Theorem 48. *For each integer n greater than two there exists a geometric minimal surface X with finite total curvature, $g(X) = 1$, $E(X) = n$ and $S(X) = \bar{D}_n$.*

Theorem 49. *For each positive integer n there exists a one parameter family $X_n(t)$ of geometric minimal surfaces of finite total curvature with $g(X_n(t)) = n$, $E(Z_n(t)) = 4$, $S(X_n(t)) = \bar{D}_{n+1}$ and the normal vectors at the planes at infinity are parallel.*

Theorem 50. *Let Δ be a regular polyhedron and Γ be the natural 1-skelton on the boundary of Δ and suppose Γ has n 1-simplices and m vertices. Then there exists an example X of a geometric minimal surface of finite total curvature with $E(X)$ = genus of Γ, $E(X) = m$, $S(X) = S(\partial\Delta)$. There also exists a one parameter family $Y(t)$ of examples with $g(Y(t))$ = genus of Γ, $E(Y(t)) = m+n$ and $S(Y(t)) = S(\Delta)$.*

Remark. It is clear from the construction of X in theorems 48 and 50 that X must be unique. The examples in theorem 50 may be embedded. In the case of genus equal to one in theorem 50 the examples correspond to compact surfaces which are rectangular elliptic curves. The coordinate functions of these minimal surface examples can be explicitly written down in terms of classical functions. The author plans to check by computer whether or not these examples of genus one are embedded.

Section 22. The topological uniqueness theorems for minimal surfaces in $\mathbb{R}^3$.

The first topological uniqueness theorem for minimal surfaces was given by B. Lawson [51] who proved that two embedded diffeomorphic minimal surfaces M in the three dimensional sphere S^3 in $\mathbb{R}^4$ are isotopic in S^3. The first part of his proof is to show by a variational argument due to Synge that the fundamental group of the closures of the complements S^3-M are generated by curves on ∂M. Then a theorem of Papakyriakopoulous shows that the complements S^3-M are diffeomorphic to solid g-holed tori. Then Lawson applies the deep result of Waldhausen concerning the topological uniqueness up to isotopy of such decompositions of S^3.

In $\mathbb{R}^3$ the author has analyzed the question of the topological uniqueness of compact minimal surfaces with boundary being a collection of Jordan curves on certain geometric subsets of $\mathbb{R}^3$ and for certain noncompact properly embedded minimal surfaces. The following theorems are corollaries of a topological uniqueness theory developed in [60] which is independent of Waldhausen's theorem.

Theorem 51. *Suppose M_1 and M_2 are embedded compact diffeomorphic minimal surfaces in $\mathbb{R}^3$ whose boundary is a fixed Jordan curve γ on the boundary of a convex set C. Then M_1 and M_2 are ambiently isotopic in C relative to the boundary of C.*

Theorem 52. *If M_1 and M_2 are embedded complete proper minimal surfaces in $\mathbb{R}^3$ which are diffeomorphic to a compact surface punctured in one point or are diffeomorphic to an open annulus, then M_1 and M_2 are isotopic.*

Theorem 53. *Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ be a pairwise disjoint collection of Jordan curves with γ_1 in a plane P and $\gamma_2, \ldots, \gamma_k$ in a parallel plane. If M_1 and M_2 are connected embedded diffeomorphic compact minimal surfaces in $\mathbb{R}^3$ with boundary Γ, then M_1 and M_2 are ambiently isotopic relative to the parallel planes.*

Theorem 54. *If M is a proper embedded complete minimal surface in $\mathbb{R}^3$, then the complements $\mathbb{R}^3$-M are diffeomorphic to one-dimensional CW-subcomplex of $\mathbb{R}^3$.*

Below are two simple examples of stable diffeomorphic minimal surfaces which bound the same Jordan curve on an ellipsoid. These examples arise from joining parts of stable "catenoids" by thin bridges. We challenge the reader to prove directly that the two minimal surfaces are isotopic.

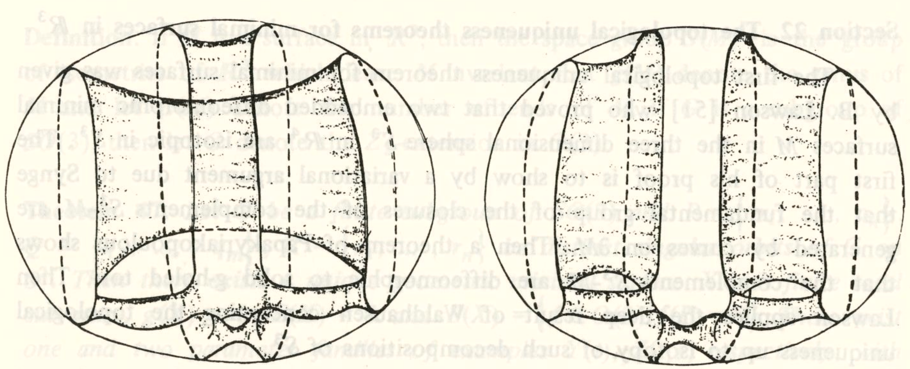


Fig. 15

The following two examples are isotopic by theorem 51. They are made of stable catenoids and planar minimal disks connected by thin bridges (see the bridge principle in section 15). For fun we also challenge the reader to construct the required isotopy in this case.

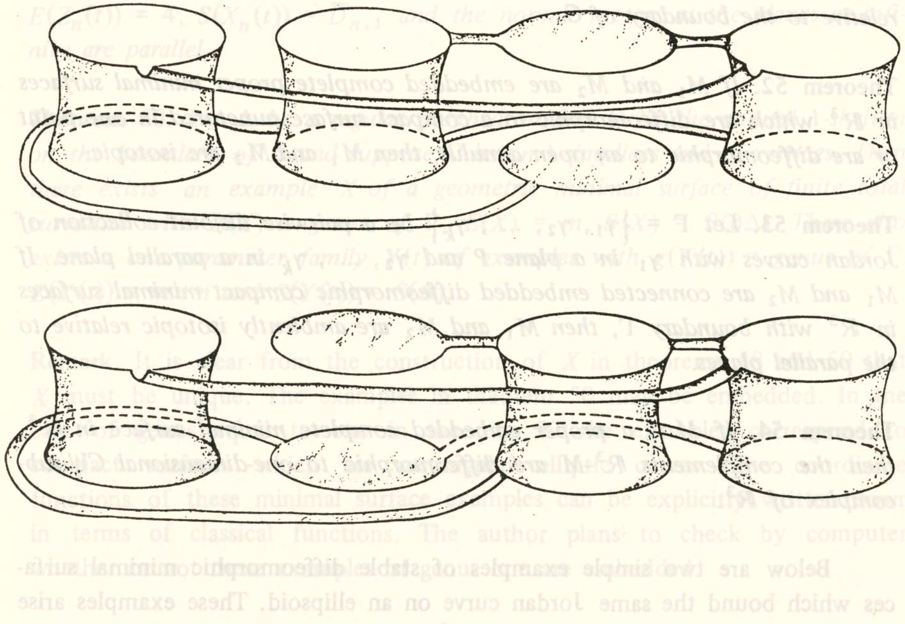


Fig. 16

Section 23. Generalizations to Riemannian manifolds and applications of minimal surface theory to geometry, topology and the theory of relativity.

The theory of minimal surfaces in $\mathbb{R}^3$ extends naturally to the study of minimal surfaces in an n-dimensional manifold. For example, the Douglas solution to Plateau's problem discussed in section 9 continues to exist for a Jordan curve in $\mathbb{R}^n$. More generally, as discussed in section 16, Morrey, solved Plateau's problem in the larger class of homogeneously regular manifolds. The regularity theorems for minimal surfaces given in section 3 continue to hold in the case the ambient space is a manifold. However, the least area surfaces that arise in variational problems in higher dimensions may be forced to have branch point singularities. The existence of a branch point for a least area surface occurs in the following example. Define $f{:}C\to C^2$ by $f(z) = (z^2, z^3)$. Then the restriction $f|D$ to the unit disk is the unique Douglas solution to Plateau's problem for $f(\partial D)$ up to conformal reparametrization. The proof that $f|D$ has least area follows from the area minimizing properties of complex subvarieties of a Kähler manifold which will be discussed shortly. In general, F. Morgan [70] proved the following related property for singular points of a two dimensional area minimizing surface.

Theorem 55. *The tangent cone to an oriented two dimensional area minimizing surface in $\mathbb{R}^n$ consists entirely of complex planes for some orthogonal linear complex structure on its span.*

Examples of minimal surfaces in $\mathbb{R}^n$ depend to a large extent on the Weierstrass representation of these surfaces. Every simply connected minimal surface $f{:}M\to \mathbb{R}^n$ arises from the real projection Re $(\sqrt{2}\tilde{f})$ where $\tilde{f}{:}M\to C^n$ is a holomorphic curve isometric to f and $\tilde{f}$ is defined by the integration of n-holomorphic one-forms $\{w_1, w_2, \ldots, w_n\}$ such that if $w_i = g_i(z)dz$ in local coordinates, then $\sum_{i=1}^{n} g_i^2(z) = 0$. The generalized Gauss map $G{:}M\to CP^{n-1}$ is then defined by $G(p)$ = the complex line passing throught the point $(g_1(p), \ldots, g_n(p))$. The image of the Gauss map is contained in the quadric $Q_{n-1}{:}Z_1^2 + \ldots + Z_n^2 = 0$. In the case $n = 3$, Q is naturally isometric to S^2 and by composing with this isometry, the generalized Gauss map can be identified with the usual Gauss map for a minimal surface in $\mathbb{R}^3$ (see [49]).

The rigidity theorems for minimal surfaces in $\mathbb{R}^3$ (see [49] and [56]) depend on the following rigidity theorem of Calabi for holomorphic curves.

Theorem 56. *Two isometric holomorphic curves in C^n differ by a rigid motion.*

By studing the generalized Gauss map for a complete minimal surfaces M in $\mathbb{R}^n$ with finite total curvature R. Ossermann and S.S. Chern proved that the Gauss map on M extends conformally to a compact Riemann surface $\overline{M}$ where M is $\overline{M}$ punctured in a finite number of points. This easily implies that the total curvature of M is an integer multiple of -2π. Recently a number of new results have been obtained concerning these surfaces and we refer the interested reader to [16] and [43] for a thorough discussion.

The existence of a bounded complete minimal surfaces has recently been found by P. Jones [45]. He proves the stronger theorem that there exists a complete bounded immersed holomorphic embedding of the disk in C^3.

Calabi earlier noted the existence of a holomorphic curve properly contained in a ball in C^3 by considering the ball to be the universal covering space of a compact algebraic surface and then lifting an algebraic curve into the universal cover.

In the classical geometric setting of compact constant curvature three manifolds there are some special deep results. In [50] B. Lawson proved a number of beautiful theorems on the geometry of minimal surfaces in the three sphere. In particular he proved.

Theorem 57. *Every compact minimal surface except the projective plane can be minimally immersed in S^3.*

2. *Every compact orientable surface embeds minimally in S^3 and if the genus is not prime there are at least two non-isometric examples.*

3. *Two diffeomorphic embedded compact minimal surfaces in S^3 are isotopic.*

The deeper results on the minimal surfaces in flat tori are usually involved with the conformal structure. For example Abel's theorem in section 17 gives an important instance of this relationship between the conformal structure and the geometry of these surfaces. In higher dimensions every Riemann surfaces embeds conformally and minimally in a flat torus. In fact it embeds holomorphically in its Jacobian variety. Minimal surfaces in flat tori are interesting because their lifts to $\mathbb{R}^n$ give rise to n-periodic minimal surfaces. Applying the generalized Weierstrass representation for minimal surfaces in $\mathbb{R}^n$, the following can be proved [56].

Theorem 58. (1) *A compact Riemann surface of genus three admits a full conformal minimal immersion into a flat torus T^5 if and only if the surface is hyperelliptic.*

(2) *If the genus g of a compact Riemann surface is greater than three, then the surface immerses conformally, minimally and fully in a flat torus T^{2g-1}.*

One of the most beautiful existence theorems for minimal surfaces in compact manifolds is the following recent theorem of Sachs-Uhlenbeck [97].

Theorem 59. *If M is a compact Riemannian manifold whose universal covering space is not contractible, then there exists a branched minimal sphere in M. Furthermore, there is a collection of branched minimal spheres of least area in their homotopy classes which generate $\pi_2(M)$ under the action of $\pi_1(M)$.*

The method of proof of the above theorem is to first consider the following perturbed energy functions for an immersion $f: S^2 \to M$

$$E_\alpha(f) = \int_{S^2} (1 + E(f))^{1+\alpha} dA$$

where $E(f)$ is the usual energy of f and α is a nonnegative number. The E_α energy satisfies condition (C) of Palais-Smale for α positive. Their existence theorem follows from a delicate analysis of what happens to the critical points of E_α as α goes to zero.

According to the result of Sachs-Uhlenbeck and its slightly improved version by Meeks-Yau, the infimum of the energies of a map of S^2 to M representing an element of $\pi_2(M)$ can be achieved by a sum of stable harmonic maps f_i from S^2 into M. For Kähler manifolds of positive holomorphic curvature such stable harmonic maps can be shown to be holomorphic or antiholomorphic (see [108]). Y.T. Siu and S.T. Yau recently proved this result and combined it together with some geometrical arguments and theorem in algebraic geometry to prove an old conjecture of Frankel. Their theorem is:

Theorem 60. *Every compact Kähler manifold of positive bisectional curvature is biholomorphic to a complex projective space.*

To prove the existence of minimal surfaces of other topological types it is usually necessary to assume something about the fundamental group of the surface. A map $f:X \to M$ of a surface is called incompressible if the induced map

$$f_*: \pi_1(X) \to \pi_1(M)$$

is injective on the fundamental groups.

Theorem 61. *Suppose M is a compact manifold and $f:X \to M$ is a closed incompressible surface in M. Then there exists a branched minimal immersion $g:X \to M$ of least area which has the same image on the fundamental groups (up to conjugacy) as f.*

A proof of the above theorem was given independently by Schoen-Yau and Sachs-Uhlenbeck. The idea in the proof is to minimize the energy in the homotopy class of f as map for all conformal structures on X. The injectivity on the fundamental group is used to show that the energy gets large near the boundary of the appropriate Teichmüller space (after composing f with all diffeomorphisms on the surface). Thus the least energy occurs for some conformal structure on X. For this conformal structure the least energy map also has least area.

Freedman, Hass and Scot [119] have recently been able to show that in dimension three certain of the least maps given in the above theorem are actually embeddings. Their technique is similar to that developed by Meeks-Yau as discussed in Sections 16 and 24.

Theorem 62. *Suppose M is a compact irreducible three dimensional manifold and $g:X \to M$ is a closed embedded incompressible surface. Then there is a least area immersion $f:X \to M$ in the homotopy class of g and any such f is one-to-one or two-to-one. In particular f(X) is an embedded surface.*

The above theorem has some important applications to finite group actions in three manifolds. These applications and some related embedding theorems were found independently by Meeks, Simon and Yau [69]. One of the results in [69] shows that the universal covering space of an irreducible three manifold is irreducible.

R. Schoen and S.T. Yau [102] later applied their existence of least area incompressible surfaces to get some interesting results for three dimensional manifolds of nonnegative scalar curvature. For example, they prove that a nonnegative scalar curvature metric on T^3 is actually flat. With some

deep generalizations of this technique they were able to prove that an asymptotically flat scalar curvature metric on $\mathbb{R}^3$ is actually flat. The basic construction in the proof of this theorem is to show that there exists a stable complete minimal surface in this metric which is a plane. They then use the Gauss-Bonnet theorem to get a contradiction to the stability of this plane unless the metric on $\mathbb{R}^3$ is actually flat. This type of theorem by Schoen-Yau [101] led them to a proof of the famous positive mass conjecture in the theory of general relativity which can be simply stated as follows.

Theorem 63. *Let M be a space-time whose local mass density is nonnegative everywhere. Then the total mass of M as viewed from spacial infinity must be positive unless M is the flat Minkowski space-time.*

Later R. Schoen and D. Fischer-Colbrie proved a related theorem whose important corollary is that every complete stable orientable minimal surface in $\mathbb{R}^3$ is a plane. This corollary is not known in the case M is nonorientable [27].

Theorem 64. *Let N be a complete oriented three manifold of nonnegative scalar curvature. Let M be an oriented complete stable minimal surface in N. Then there are two possibilities:*

1. *If M is compact, then M is a sphere or a totally geodesic flat torus. If the scalar curvature is positive, then M is a sphere.*
2. *If M is not compact, then M is conformally equivalent to the complex plane C or $C-\{0\}$.*

Recently Yau and Schoen [120] have made another beautyful application of minimal surface theory to three dimensional geometry. Their theorem given below makes use of the existence of a complete stable minimal surface and some estimates in [27]

Theorem 65. *A complete non-compact three dimensional manifold with positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$.*

An important class of minimal submanifolds are complex submanifolds of Kähler manifolds. An easy application of Wirtinger's inequality and the fact that the Kähler form is closed implies the following (see [49]).

Theorem 66. *Let $\overline{M}$ be any Kähler manifold and let $f:M \to \overline{M}$ be a complex submanifold where M is compact with boundary which is possibly empty.*

Then the volume of M in the induced metric is less than or equal to the volume of any other compact submanifold which is homologous to M in $\overline{M}$.

Another way of proving the existence of minimal surfaces in a three dimensional manifold M is to apply the regularity and existence theorems for minimal currents in geometric measure theory [25]. The theorem of Hardt-Simon in section 9 is a beautiful application of this theory to the classical Plateau problem of minimal surfaces in $\mathbb{R}^3$. The earlier interior regularity theorem (see [52] for references) shows that every two dimensional homology class in an orientable three dimensional manifold can be realized by an embedded least area closed surface with multiplicity on its components. In particular, there exists a stable embedded minimal surface when the second homology group is nonzero.

Recently J. Pitts showed, using geometric measure theory, that every closed three manifold contains an embedded closed minimal surface. He has generalized his theorem to prove [90].

Theorem 67. *A closed manifold of dimension less than or equal to six contains a closed embedded codimension-one minimal submanifold.*

R. Schoen and L. Simon have improved this theorem up to dimension of the ambient manifold equalling eight.

Much of the theory of minimal currents was developed to solve the generalized Bernstein problem. In 1920 Bernstein [9] proved that a graph of a smooth function $f: \mathbb{R}^2 \to \mathbb{R}$ is a minimal surface if and only if the function f is linear. The stability theorem 8, Osserman's theorem 41 and Xavier's theorem 43 give strong generalizations of this Bernstein theorem in $\mathbb{R}^3$. For example, it follows from the Carmo-Barbosa stability theorem that a minimal graph is stable and hence by theorem 8 it must be a plane. Fleming gave a proof of the Bernstein theorem using his theorems concerning minimal currents. The pioneering work of Fleming eventually led to the following theorem of J. Simons (see [52] for a discussion and history).

Theorem 68. *Let $f: \mathbb{R}^n \to \mathbb{R}$ be a smooth function whose graph is a minimal submanifold in $\mathbb{R}^{n+1}$. If $n \leqslant 7$, then f is linear.*

This theorem is closely related to the interior regularity of minimal currents [107]. The work in [12] shows that the theorem of Simons is not true for $n > 7$.

Section 24. Appendix: The proof of the embedding of the analytic case of the geometric Dehn's lemma.

In this section we give a somewhat simplified proof of the basic topological construction in the proof of the geometric Dehn's lemma in section 16. For further details see [64].

Theorem 69. *Suppose M is a compact analytic three dimensional Riemannian manifold. Suppose that γ is an analytic curve on ∂M and that $f:D \to M$ is a least area (energy) map with $f(\partial D) = f(D) \cap \partial N = \gamma$. Then f is one-to-one.*

Proof of the theorem. The proof of the theorem will depend on the following sequence of lemmas.

Lemma 1. *$f:D \to M$ is an analytic immersion.*

Proof of lemma 1. By the regularity theorems of Gulliver and Osserman in section 9, f is an immersion on $\overset{\circ}{D}$. f is analytic on $\overset{\circ}{D}$ by Morrey's interior regularity theorem [73]. f is analytic on D by the boundary regularity theorems discussed in section 3. By the theorem 15 of Gulliver-Lesley f is an immersion on D.

Lemma 2. *$f:D \to M$ is simplicial with respect to fixed triangulations of D and M.*

Proof of lemma 2. By lemma 1 f is analytic and it follows that $f(D)$ is a semi-analytic subset of M. Also it follows from the triangulation theorems in [55] that the semi-analytic subset $f(D)$ of M is a two dimensional subcomplex of some triangulation of M. Since f is an immersion, the triangulation of $f(D)$ induces a triangulation of D such that $f:D \to M$ is simplicial.

Lemma 3. *Suppose D_1 and D_2 are distinct analytic embedded disks in an open three dimensional Riemannian manifold N and that D_1 and D_2 have least area with respect to their boundary curves. Then if $D_1 \cap D_2 \subset \overset{\circ}{D}_1 \cap \overset{\circ}{D}_2$, then $D_1 \cap D_2 = \phi$.*

Proof of lemma 3. Suppose first that D_1 and D_2 are in general position which is the generic case. If $D_1 \cap D_2$ is non-empty, then $D_1 \cap D_2$ is one-dimensional submanifold of $\overset{\circ}{D}_1$ and of $\overset{\circ}{D}_2$. By the classification of one-dimensional submanifolds $D_1 \cap D_2$ is a finite collection of Jordan curves. Let γ be a Jordan curve in $D_1 \cap D_2$. Then the Jordan curve theorem implies that γ is the boundary of a subdisk D_1' of D_1 and a subdisk D_2' of D_2.

Suppose that the area of D_1' is less than or equal to the area of D_2'.

Then consider the new piecewise smooth disk.

$$D_3 = (D_2 - D_2') \cup D_1'$$

The area of D_3 is less than or equal to the area D_2. The area of D_3 can now be decreased along γ which contradicts the hypothesis that D_2 has least area with respect to its boundary curve.

If D_1 and D_2 are not in general position, then there are two ways to reduce to the general position case. The first way is by approximation. The second is by way of the following:

Assertion. If $D_1 \cap D_2 \subset \overset{o}{D}_1 \cap \overset{o}{D}_2$ is non-empty, then $D_1 \cap D_2$ contains a closed Jordan curve.

Proof. Since D_1 and D_2 are analytic, $\Gamma = D_1 \cap D_2$ is a compact triangulable analytic subset analytic subset of $\overset{o}{D}_1$. We first note that Γ has no isolated vertices. If Γ has an isolated vertex p, then p corresponds to a point on D_1 where D_1 is locally on one side of D_2. By the maximum principal for minimal surfaces, D_1 and D_2 intersect in an open set near p so the vertex p is not isolated. Also Γ cannot contain a 2-simplex, for by the uniqueness of analytic continuation, D_1 and D_2 must agree on an open set that goes to the boundary of D_1 or D_2. However, this is impossible since the intersection of D_1 and D_2 does not by hypothesis include points on the boundaries.

The argument used above shows that Γ is a one-dimensional subcomplex of some triangulation of D_1 and Γ contains no isolated vertices. Analytic one-dimensional subsets of a disk have an even number of edges at every vertex. This implies that Γ represents a one-cycle in the simplicial one-chains of D_1 using Z_2-coefficients. Since the first homology group with Z_2 coefficients of D_1 is zero, geometric intersection theory implies that Γ must disconnect D_1. A boundary curve of a component of $D_1 - \Gamma$ which is different from ∂D_1 is the required Jordan curve in the assertion.

The existence of the Jordan curve in $D_1 \cap D_2$ together with the disk replacement argument used in the general position case gives a contradiction. Hence, $D_1 \cap D_2$ must be empty which proves the lemma.

Lemma 4. *Suppose N is a triangulated three dimensional manifold and $f{:}D \to N$ is a simplicial immersion of a disk with respect to some triangulation T of D. Then there exists a subdivision of the triangulation of N so that $f{:}D \to N$ is still simplicial with respect to T and such that the simplicial neighborhood of $f(D)$ is a simplicial regular neighborhood of $f(D)$.*

Proof of lemma 4. This elementary result follows after subdividing two times the triangulation of N. Each time the subdivision includes the baricenters of

the simplices which are not contained in $f(D)$. This proves lemma 4.

We now carry out the construction of a tower for $f{:}D \to M$ in order to simplify the self-intersection or singular set for $f{:}D \to M$ which by lemma 2 is simplicial. First, let N_1 be a regular neighborhood of $f(D)$. After restricting the range space of f to N_1, there is a new map $f_1{:}D \to N_1$. If N_1 is not simply connected, then let $P_1{:}\tilde{N}_1 \to N_1$ be the universal covering space of N_1 and let $\tilde{f}_1{:}D \to \tilde{N}_1$ be a lift of f_1 to this covering space. Then restricting the range space of $\tilde{f}_1$ to a regular neighborhood N_2 of $\tilde{f}_1(D)$, we get another map $f_2{:}D \to N_2$.

If N_2 is not simply connected, then we can repeat the construction in the previous paragraph to get a lift $\tilde{f}_2{:}D \to \tilde{N}_2$ to the universal covering space $P_2{:}\tilde{N}_2 \to N_2$ of N_2. After restricting the lift $\tilde{f}_2$ to a regular neighborhood N_3 of $f(D)$, we get $f_3{:}D \to N_3$.

Repeating k-times, the construction outlined above yields a tower

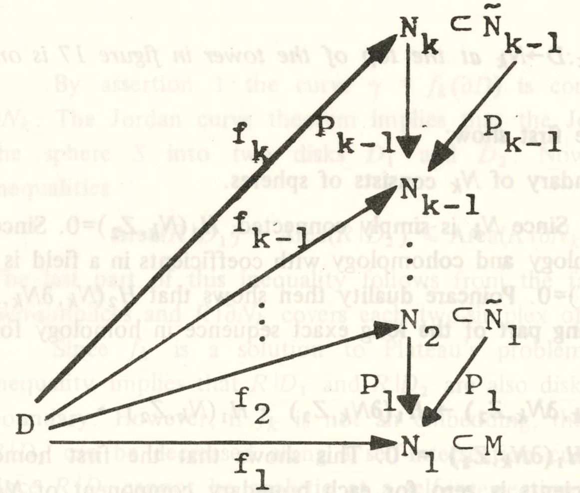


Fig. 17.

where $P_i{:}N_{i+1} \to N_i$ is the restriction of $P_i{:}\tilde{N}_i \to N_i$ to N_{i+1}.

Each N_i in the above tower is a Riemannian manifold with respect to the pulled back metric. Each of the lifts $f_i{:}D \to N_i$ is a solution to Plateau's problem for the Jordan curve $f_i(\partial D)$ with respect to this metric. Otherwise there is an immersion $g{:}D \to N_i$ with $g(\partial D) = f_i(\partial D)$ and with respect to the pulled back metric on D, Area$(g) \leqslant$ Area (f_i). We would then have Area $(P_1 o P_2 o \ldots o P_{i-1} o g)$ = Area (g) < Area (f_i) = Area (f) which is impossible.

By lemmas 2 and 4 we may assume that each map $f_i{:}D \to N_i$ in the tower is simplicial with respect to a fixed triangulation T for which $f_1{:}D \to N_1$ is simplicial. Note that the triangulation on N_i is induced from the triangulation on N_{i-1} pulled back to $\tilde{N}_i$ by $P_i{:}\tilde{N}_i \to N_i$. We now use this fact to prove that the tower construction terminates after a finite number k of steps with N_k being simply connected.

Lemma 5. *If* $S(f_i) = \{(\sigma,\tau) \in T \times T \mid \sigma \neq \tau$ *and* $f(\sigma) = f(\tau)\}$, *then* $S(f_{i+1}) < S(f_i)$. *Hence, the tower construction terminates at some* k *with* N_k *simply connected.*

Proof of lemma 5. Since $f_i = P_i \circ f_{i+1}$ where P_{i+1} is simplicial, $S(f_{i+1}) \leqslant S(f_i)$. If $S(f_{i+1}) = S(f_i)$, then $h = P_i|f_{i+1}(D)$ induces a homeomorphism between $f_{i+1}(D)$ and $f_i(D)$. Using h we can define a lift of the inclusion map $i: f_i(D) \to N_i$ to $\tilde{N}_i$ by $\tilde{\imath}: f_i(D) \to \tilde{N}_i$ where $\tilde{\imath} = h^{-1} o i$. Since N_1 is a regular neighborhood of $f_i(D)$, $i_*: \pi_1(f_i(D)) \to \pi_1(N_i)$ is an isomorphism. Since $\tilde{N}_i$ is simply connected, the lifting criterion for maps in covering space theory implies that N_i is simply connected. Thus, we may assume that $S(f_{i+1}) < S(f_i)$ which proves the lemma.

Lemma 6. *The lift* $f_k: D \to N_k$ *at the top of the tower in figure 17 is one-to-one.*

Proof of lemma 6. We first show:

Assertion 1. The boundary of N_k consists of spheres.

Proof of Assertion 1. Since N_k is simply connected, $H_1(N_k, Z_2) = 0$. Since the pairing between homology and cohomology with coefficients in a field is non-degenerate, $H^1(N_k, Z_2) = 0$. Poincaré duality then shows that $H_2(N_k, \partial N_k, Z_2) = 0$. From the following part of the long exact sequence in homology for the pair $(N_k, \partial N_k)$.

$$\to H_2(N_k, \partial N_k, Z_2) \to H_1(\partial N_k, Z_2) \to H_1(N_k, Z_2) \to$$

one computes that $H_1(\partial N_k, Z_2) = 0$. This shows that the first homology group with Z_2 coefficients is zero for each boundary component of N_k. By the classification theorem for compact surfaces, each component of the boundary of N_k is a sphere which proves the assertion.

We shall now use the fact that the boundary of N_k consists entirely of spheres to show that $f_k: D \to N_k$ is an embedding. First note that since N_k is a simplicial regular neighborhood, there is, after a subdivision, a simplicial retraction $S: N_k \to f_k(D)$ whose restriction $R = S|\partial N_k \to f_k(D)$ has the property: R covers each open two-simplex of $f_k(D)$ exactly two times and $R|(\partial N_k - f_k(\partial D))$ is locally one-to-one. The existence of such a retraction follows directly from the definition of a simplicial regular neighborhood and the collapsing properties of such a neighborhood onto an immersed codimension-one simplicial submanifold whose boundary is the intersection of the

submanifold with the boundary of the ambient manifold. For a proof of the existence we refer the reader to [65]. The existence of such a retraction is easy to see in the one lower dimensional picture below.

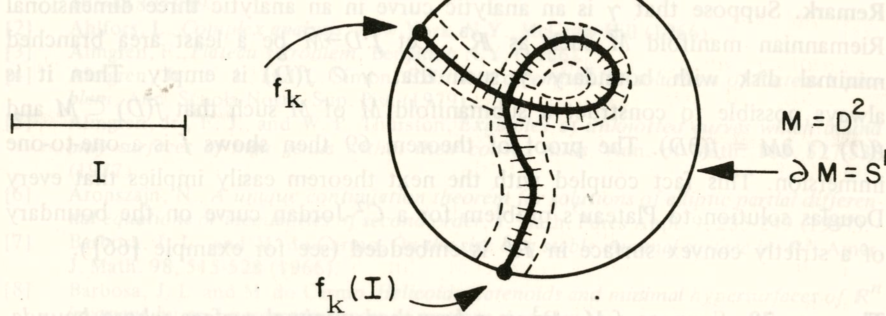


Fig. 18

By assertion 1 the curve $\gamma = f_k(\partial D)$ is contained in a sphere S in ∂N_k. The Jordan curve theorem implies that the Jordan curve γ disconnects the sphere S into two disks D_1 and D_2. Now consider the following inequalities

$$\text{Area}(R|D_1) + \text{Area}(R|D_2) \leqslant \text{Area}(R|\partial N_k) \leqslant 2\ \text{Area}(f_k)$$

The last part of this inequality follows from the fact that area is carried by two-simplices and $R|\partial N_k$ covers each two-simplex of $f_k(D)$ two times.

Since f_k is a solution to Plateau's problem for γ, the above area inequality implies that $R|D_1$ and $R|D_2$ are also disks of least area with γ for boundary. However, if f_k is not an embedding, then the area of $R|D_1$ and $R|D_2$ can be decreased along a self-intersection curve of $f_k(D)$. This is true since $R|D_i$ cannot be analytic at a self-intersection point of $f_k(D)$ as is clear in the one lower dimensional figure 18. Since this contradicts the least area property of f_k, the map f_k must be an embedding which proves the lemma.

We now complete the proof of the theorem. If $f: D \to M$ is not an embedding, then we may assume by the previous lemma that k is greater than one with $f_{k-1}: D \to N_{k-1}$ not one-to-one. Let E be the embedded disk $iof_k(D) \subset \tilde{N}_{k-1}$ where $i: N_k \to \tilde{N}_{k-1}$ is the inclusion map. Since f_{k-1} is not one-to-one and $N_{k-1} = \tilde{N}_{k-1}/G$ where G is the group of covering transformations, there exists a nontrivial covering transformation $\tau: \tilde{N}_{k-1} \to \tilde{N}_{k-1}$ such that $\tau(E) \cap E$ is nonempty. Since the covering transformation τ is an isometry on $\tilde{N}_{k-1}$, the disk $\tau(E)$ has least area with respect to its boundary curve. The hypothesis in the theorem that $f(\partial D) = f(D) \cap \partial M = \gamma$ implies that $E \cap \cap \tau(E) \subset \overset{o}{E} \cap \tau(\overset{o}{E})$. Lemma 3 shows this containment is impossible which implies that $f{:}D \to M$ must in fact be an embedding. This completes the proof of the theorem.

Remark. Suppose that γ is an analytic curve in an analytic three dimensional Riemannian manifold $\tilde{M}$ such as $\mathbb{R}^3$. Let $f{:}D \to \tilde{M}$ be a least area branched minimal disk with boundary γ such that $\gamma \cap f(\overset{o}{D})$ is empty. Then it is always possible to construct a submanifold M of $\tilde{M}$ such that $f(D) \subset M$ and $f(D) \cap \partial M = f(\partial D)$. The proof of theorem 69 then shows f is a one-to-one immersion. This fact coupled with the next theorem easily implies that every Douglas solution to Plateau's problem for a C^2-Jordan curve on the boundary of a strictly convex surface in $\mathbb{R}^3$ is embedded (see for example [66]).

Theorem 70. *Suppose $f{:}M \to \mathbb{R}^3$ is a branched minimal surface whose boundary curves $f(\partial M)$ are C^2. Suppose that $f(M)$ is contained in a region R of $\mathbb{R}^3$ with nonnegative mean curvature with respect to the inward normal and $f(M) \cap \partial R = f(\partial M)$. Then $f|\partial M$ is an immersion.*

A continuous version of the following very important compact smooth convergence theorem enters in an essential way in the proof that every Douglas solution to Plateau's problem for a continuous extremal Jordan curve is embedded. An easy application of a continuous version of the next theorem, the embedding of the Douglas solution for an extremal analytic curve and theorem 69 imply that an extremal Jordan curve bounds at least one embedded Douglas solution.

Theorem 71. *Let γ_k be a sequence of smooth Jordan curves that converge smoothly in the smooth topology to a smooth Jordan curve γ. Let $f_k{:}D \to \mathbb{R}^3$ be branched minimal disks with boundary curves γ_k which are normalized so that for three distinct points $p_1, p_2, p_3 \in \partial D$ the triples of points $(f_k(p_1), f_k(p_2), f_k(p_3))$ converges to three distinct points on γ. Then a subsequence of the f_k converge smoothly on D to a branched minimal immersion with boundary γ.*

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Note. This bibliography is not intended to be complete.

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Instituto de Matemática Pura e Aplicada (IMPA)
Rua Luis de Camões, 68
20.060 – Rio de Janeiro, Brasil