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Linearity and residues for foliations

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0. Introduction.

In [H] we constructed residues for a vector field preserving a foliation. These residues depend a priori on the germs of the vector field and the foliation on the singular set of the vector field. In this note we show that for certain vector fields, the residues actually depend only on the one jets of the vector field and the foliation. For definitions and notation, we refer to [H].

1. Continuity of the residues.

In this section we show that if τ_1 and τ_2 are foliations which are C^k close ($k \geqslant 2$), and if X_1 and X_2 are vector fields preserving τ_1 and τ_2 respectively, with the same singular set, and if X_1 is C^k close to X_2, then the residues determined by τ_1, X_1 are close to those determined by τ_2, X_2.

Let M be a smooth, n dimensional, oriented manifold with tangent bundle TM. For any bundle E over M we denote the space of smooth sections of E by $C^\infty(E)$. Each element of $C_0^\infty(\Lambda^p TM)$, the subspace of $C^\infty(\Lambda^p TM)$ consisting of non-zero sections, determines a smooth p dimensional subbundle of TM, and every such subbundle is so obtained. Denote by $F^q(M)$ the subset of $C_0^\infty(\Lambda^{n-q} TM)$ consisting of those sections which determine oriented foliations. The space $C_0^\infty(\Lambda^{n-q} TM)$ has a natural C^k topology on it (the topology of uniform C^k convergence on compact sets) and so it induces a topology on $F^q(M)$, the C^k topology.

Let $\tau \in F^q(M)$ and X a Γ vector field for τ. We assume that

$$\text{Sing } X = \{x \in M | X(x) \in \tau_x\}$$

is a connected leaf N of τ, and that M is an R^q bundle over N.

Theorem 1. *Let τ_t, $t \in R$ be a family of foliations on M, so that $\lim_{t\to 0} \tau_t = \tau_0$ in the C^k ($k \geqslant 2$) topology on $F^q(M)$. For each $t \in R$, let X_t be a Γ vector field for τ_t. Assume that Sing $X_t = N$ for all t, and that $\lim_{t\to 0} X_t =$*

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$= X_0$ *in the* C^k *topology. Then for each* $\varphi \in I_q(WO_q)$, $\lim_{t \to 0} Res_\varphi (\tau_t, X_t, N) =$
$= Res_\varphi (\tau_0, X_0, N)$.

By $\lim_{t\to 0} \text{Res}_\varphi\ (\tau_t,X_t,N) = Res_\varphi\ (\tau_0,X_0,N)$ we mean that it is possible to choose a family of differential forms $\beta_t \in \text{Res}_\varphi\ (\tau_t,X_t,N)$ so that on any compact set $B \subset N$, $\lim_{t\to 0} \beta_t = \beta_0$ in the C^0 topology.

To prove the theorem, it is only necessary to exhibit a family of basic X_t connections θ^t on ν_t so that for t close to 0, θ^t is C^1 close to θ^0.

Proof of Theorem 1. Since the residues are local we may assume that each τ_t is transverse to the fibers of M and that the X_t are tangent to the fibers of M. We will identify the normal bundle of τ_t with TR^q the tangent bundle along the fiber of M. We assume without loss of generality that N is compact.

In addition, in order to avoid a plethora of ϵ's and δ's we will be somewhat imprecise and will speak of C^k closeness without writing down the requisite ϵ's and δ's.

Choose a smooth metric g on TM. Let D be an open disc sub-bundle of M, with closure $\bar{D} \subset M$. Let $\mathfrak{U}'$ be an open neighborhood of M-D whose closure is disjoint from N. For $t \in R$ let $\bar{\omega}_t$ be the one form on $\mathfrak{U}'$ satisfying

i) $\bar{\omega}_t|_{\tau_t} \equiv 0$
ii) $\bar{\omega}_t(X_t) \equiv 1$
iii) if $Y \in TR^q$ and $g(Y,X_t) = 0$ then $\bar{\omega}_t(Y) = 0$.

Let $\mathfrak{U}$ be an open set with

$$\mathfrak{U}' \supset \mathfrak{U} \supset \bar{\mathfrak{U}} \supset M\text{-}D$$

and let φ be a smooth function on M with

$$\varphi|_{M-\mathfrak{U}'} \equiv 0, \varphi|_{\mathfrak{U}} \equiv 1$$

and set

$$\omega_t = \varphi \cdot \bar{\omega}_t.$$

Now if X_t is C^k close to X_0 and τ_t is C^k close to τ_0 then

(*) $\qquad \omega_t$ is C^k close to ω_0.

Let $\rho_t: TM \to TR^q$ be the orthogonal projection associated to τ_t. For τ_t C^k close to τ_0

(**) $\qquad \rho_t$ is C^k close to ρ_0.

The statements (*) and (**) are essentially lemmas in linear algebra and are left to the reader.

Let $\{W_a\}$ be a finite open cover of M where each W_a is a trivializing neighborhood for M, i.e., W_a is diffeomorphic to $U_a \times R^q$ where U_a is a coordinate chart on N with coordinates $x_1, \ldots, x_{n-q}$, and R^q has coordinates $s_1, \ldots, s_q$. Define the connection θ_a^t on TR^q over W_a by requiring its covariant derivative to satisfy

i) $\nabla_y^t\ \partial/\partial s_r = \rho_t\ ([Y, \partial/\partial s_r])$ for all $Y \in C^\infty(\tau_t|W_a)$
ii) $\nabla_{\partial/\partial s_p}^t\ \partial/\partial s_r = \omega_t\ (\partial/\partial s_p)\ [X_t, \partial/\partial s_r]$.

Then θ_a^t is a basic X connection for τ_t, X_t on W_a supported off $W_a \cap \mathfrak{U}$.

One now calculates directly that θ_a^t is C^{k-1} close to θ_a^0. Gluing the θ_a^t together with a partition of unity we obtain θ^t, a basic X connection for τ_t, X_t supported off $\mathfrak{U}$. For τ_t, X_t C^k close to τ_0, X_0, θ^t is C^{k-1} close to θ^0. The theorem follows.

2. Linearity of the residues.

In this section we show that for certain Γ vector fields the residues are determined by the one jets of the foliation and Γ vector field on the singular set. That this is not true in general follows from [H], example 2. Let M be an R^q bundle over a smooth manifold N. Let τ be a codimension q foliation on M transverse to its fibers and let X be a Γ vector field for τ tangent to the fibers of M with singular set N.

This situation may be alternately described as follows. Let Diff$(R^q,0)$ be the group of smooth diffeomorphisms of R^q fixing 0. Let $h{:}\Pi_1(N) \to$ Diff$(R^q,0)$ be a homomorphism. Let $X \in C^\infty(TR^q)$ be a vector field on R^q invariant under $h(\Pi_1(N))$ with $X = 0$ only at the origin. Let $\tilde{N}$ be the universal cover of N. Form the bundle $\tilde{N} \times R^q$. It has a natural foliation τ on it with leaves of the form $\{x\} \times R^q$. The vector field X on R^q induces a vector field on $\tilde{N} \times R^q$, also denoted X, which is a Γ vector field for τ. Let $M = \tilde{N} \times_h R^q$ be the associated foliated bundle over N with associated Γ vector field X. This is the point of view we shall adopt in this section. We assume for convenience that N is compact.

Definition. Let $X \in C^\infty(TR^q)$. We say that X is a Γ *vector field for h* if X is invariant under h and is 0 only at the origin. We say that X is a *linear* Γ *vector field for h* if the linear map $\mathcal{L}_{X_0}$, induced by the Lie derivative with respect to X on TR^q_0, is an isomorphism.

Let $X \in C^\infty(TR^q)$ be a linear Γ vector field for h. Define the vector field X_0 on R^q by

$$X_0(x) = \mathcal{L}_{X_0} \cdot x$$

(i.e., if $\mathcal{L}_{X_0} = [a_j^i]$ then $X_0(x) = \Sigma\, a_j^i x_j\ \partial/\partial x_i$).

Let $h_0:\Pi_1(N) \to GL_q$ be the action induced on TR_0^q by $h, h_0(\gamma) = (h(\gamma))_*|TR_0^q$. A direct computation shows

Proposition. *If X is a linear Γ vector field for h, then X_0 is a Γ vector field for h_0.*

Let M_0 be the bundle $\tilde{N} \times_{h_0} R^q$ over N and denote by τ_0 the natural foliation on M_0 and by X_0 the Γ vector field for τ_0 induced by $X_0 \in C^\infty(TR^q)$.

Theorem 2. *For all $\varphi \in I_q(WO_q)$*

$$Res_\varphi(\tau, X, N) = Res_\varphi(\tau_0, X_0, N).$$

Thus for linear Γ vector fields the residues are determined by the linear parts of τ and X on the singular set of X.

Proof. We will construct a family τ_t, X_t $t > 0$, of foliations and Γ vector fields on M_0, all of which are diffeomorphic to τ, X and which converge C^k to τ_0, X_0. As the residues are invariant under diffeomorphism we will have

$$\mathrm{Res}_\varphi(\tau, X, N) = \mathrm{Res}_\varphi(\tau_t, X_t, N)$$

but

$$\lim_{t\to 0} \mathrm{Res}_\varphi(\tau_t, X_t, N) = \mathrm{Res}_\varphi(\tau_0, X_0, N)$$

by Theorem 1.

For $t > 0$, let $\varphi_t \in \mathrm{Diff}(R^q, 0)$ be

$$\varphi_t(x) = x/t$$

and define $h_t:\Pi_1(N) \to \mathrm{Diff}(R^q, 0)$ to be

$$h_t(\gamma) = \varphi_t h(\gamma) \varphi_t^{-1}.$$

This is a homomorphism and it is easy to calculate that for any k, h_t converges uniformly in C^k on compact sets to h_0.

For $t > 0$, define $X_t \in C^\infty(TR^q)$ to be

$$X_t = \varphi_{t*} X.$$

It is clear that X_t is a Γ vector field for h_t. An obvious calculation establishes

Lemma. *For any k, $\lim_{t\to 0} X_t = X_0$ in C^k.*

Denote by M_t the bundle $\tilde{N} \times_{h_t} R^q$. On this bundle we have the foliation τ_t and Γ vector field X_t.

Lemma. *For all $t > 0$ there is a diffeomorphism $\Phi_t:M \to M_t$ taking τ to τ_t and X to X_t.*

Proof. Define $\tilde{\Phi}_t:\tilde{N} \times R^q \to \tilde{N} \times R^q$ by

$$\tilde{\Phi}_t(x, s) = (x, s/t).$$

Then

$$\tilde{\Phi}_t \circ h = h_t \circ \tilde{\Phi}_t$$

and so $\tilde{\Phi}_t$ induces

$$\Phi_t:\tilde{N} \times_h R^q \to \tilde{N} \times_{h_t} R^q.$$

Since $\tilde{\Phi}_t$ takes τ to τ_t and X to X_t, so does Φ_t.

The problem now is to transfer this family of foliations and Γ vector fields on the family of bundles M_t to a family of foliations and Γ vector fields on M_0 converging in C^k $(k \geq 2)$ to τ_0, X_0.

Choose a smooth metric on the tangent bundle to M and let g be the associated h invariant metric on $\tilde{N} \times R^q$. For each $t > 0$, and $(x, s) \in \tilde{N} \times R^q$ set

$$g^t_{(x,s)} = g_{(x,st)}.$$

Another direct calculation gives

Lemma. *g^t is invariant under h_t.*

Note that as $t \to 0$, $g^t_{(x,s)}$ converges in C^k to $g_{(x,0)}$ uniformly on compact sets and that the metric g_0, given by $(g_0)_{(x,s)} = g_{(x,0)}$, is invariant under h_0.

For each $t \geq 0$, let ν_t be the tangent bundle along the fiber of M_t restricted to N. Note that

$$\nu_t = \tilde{N} \times_{h_0} R^q$$

for all t. Let $\exp_t:\nu_t \to M_t$ be the exponential map associated to g_t, and consider the family of maps

$$\psi_t:M_t \to M_0$$

given by

$$\psi_t = \exp_0 \circ \exp_t^{-1}.$$

These maps are only well defined in a neighborhood of N, and are diffeomorphisms in a neighborhood of N. This is sufficient for our purposes so we assume that each ψ_t is a globally well defined diffeomorphism. On M_0 we have the foliation $\psi_{t*}(\tau_t)$ and the Γ vector field $\psi_{t*}(X_t)$. We denote them by τ_t and X_t respectively.

Lemma. *For the family τ_t, X_t on M_0 we have*

$$\lim_{t\to 0} \tau_t = \tau_0$$

$$\lim_{t\to 0} X_t = X_0$$

in the C^k topology.

Proof. Pull everything up to the universal cover $\tilde{N}\times R^q$. Consider the diagram

$$\begin{array}{ccccc} \tilde{N}\times R^q & \xrightarrow{\tilde{\exp}_t^{-1}} & \tilde{N}\times R^q & \xrightarrow{\tilde{\exp}_0} & \tilde{N}\times R^q \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{N}\times_{h_t} R^q & \xrightarrow[\exp_t]{} & \tilde{N}\times_{h_0} R^q & \xrightarrow[\exp_0]{} & \tilde{N}\times_{h_0} R^q \end{array}$$

On $\tilde{N}\times R^q$ the foliations induced by τ_t and τ_0 are identical. For t close to 0, X_t is C^k close to X_0. The diffeomorphism $\tilde{\psi}_t = \tilde{\exp}_0 \circ \tilde{\exp}_t^{-1}$ is smooth and as $t \to 0$ it converges in the C^k topology uniformly on compact sets to identify. Thus for t close to 0, $(\tilde{\psi}_t)_*\tau_t$ and $(\tilde{\psi}_t)_*X_t$ are C^k close to τ_0 and X_0 respectively. Therefore $(\psi_t)_*\tau_t$ and $(\psi_t)_*X_t$ are C^k close to τ_0 and X_0 on M_0.

3. The residue.

We close with some remarks about the actual calculation of the residues in the linear case. Let $h{:}\Pi_1(N) \to GL_q$ be a homomorphism and let $X = Ax$, $A \in GL_q$, be a vector field on R^q invariant under h. This information determines the cohomology classes $\mathrm{Res}_\varphi(\tau,X,N) \in H^*(N;R)$, where $\varphi \in I_q(WO_q)$. The bundle $\tilde{N}\times_h R^q$ is a flat bundle and there is a canonical homomorphism

$$\alpha{:}H^*(gl_q,SO_q) \to H^*(N;R)$$

which measures the incompatibility of the flat structure with an SO_q structure. Recall that

$$H^*(gl_{2k+1},SO_{2k+1}) = \Lambda(h_1,h_3,\ldots,h_{2k+1})$$

and

$$H^*(gl_{2k},SO_{2k}) = \Lambda(h_1,h_3,\ldots,h_{2k-1},X).$$

For $\varphi \in I_q(WO_q)$ of the form $\varphi = \hat{c}_{i_1}\ldots\hat{c}_{i_k}c_{j_1}\ldots c_{j_l} = \hat{c}_{i_1}\ldots\hat{c}_{i_k}c_J$ we have from [H], 5.11 (see also [L])

$$\mathrm{Res}_\varphi(\tau,X,N) = \alpha(h_{i_1}\ldots h_{i_k})\,\mathrm{Res}_{c_J}(\tau,X,N).$$

In order to compute Res_{c_J} proceed as follows. Let ω be a one form on $\tilde{N}\times_h R^q$ so that off a disc sub-bundle $\omega(X) = 1$. Lift this form to $\tilde{N}\times R^q$. On the tangent bundle along the fiber of $\tilde{N}\times R^q$ define a connection θ by requiring its covariant derivative to satisfy

$$\nabla_Y\, \partial/\partial s_i = 0$$

for all Y tangent to N and

$$\nabla_Z\, \partial/\partial s_i = \omega(Z)[X, \partial/\partial s_i]$$

for all Z tangent to R^q. The connection θ is then a basic X connection on TR^q and it is not difficult to show that it defines a basic X connection of TR^q over $\tilde{N}\times_h R^q$. (See [H] 5.6). To evaluate Res_{c_J} we must compute the curvature Ω of θ and integrate $c_J(\Omega)$ over the fiber of $\tilde{N}\times_h R^q$. It is immediate that with respect to the basis $\partial/\partial s_1,\ldots,\partial/\partial s_q$ of TR^q, the local connection form is

$$-\omega\cdot A.$$

The local curvature is then

$$-d\omega\cdot A$$

and

$$c_J(\Omega) = c_J(A)\,(-d\omega)^q.$$

Therefore

$$\mathrm{Res}_{\hat{c}_{i_1}\ldots\hat{c}_{i_k}c_J}(\tau,X,N) = (-1)^q\, c_J(A)\,\alpha(h_{i_1}\ldots h_{i_k})\,\gamma_R(d\omega^q)$$

where γ_R is integration over the fiber of $\tilde{N}\times_h R^q$. Thus the computation of the residue in the case of a linear Γ vector field is reduced to two problems:

i) compute the map $\alpha{:}H^*(gl_q,SO_q) \to H^*(N;R)$,

ii) determine the class in $H^*(N;R)$ represented by $\gamma_R(d\omega^q)$.

For several examples where these computations are worked out in detail see [H], examples 3 and 4. We conjecture that $\gamma_R(d\omega^q)$ is always a multiple (possibly zero) of the Euler class of the bundle $\tilde{N}\times_h R^q$.

References

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