BOL. SOC. BRAS. MAT., VOL 12 Nº 1 (1981), 101-106

On moduli of stability of two-dimensional vector fields

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Abstract.

In this paper the author proves that, on a compact connected and orientable two-dimensional C^∞ manifold M, the gradient of a C^3-Morse Function has finite modulus of stability under conjugacy and modulus zero under topological equivalence. It is also proved that generically the modulus of stability under conjugacy of a graph of a C^2 vector field on the plane is at least twice the number of its saddles. Some new conjugacy invariants arise in the proofs of these results.

Let $\chi^r(M)$ be the Banach space of C^r vector fields on M with the C^r topology, $r \geqslant 2$. We will consider the subspace $\mathrm{Grad}^r(M)$ of the elements of $\chi^r(M)$ which are gradients of C^{r+1} real functions on M. Given $X \in \chi^r(M)$ let X_t be its flow.

We say that $X, Y \in \chi^r(M)$ are conjugate if there is a homemorphism $h: M \to M$ such that $h\ X_t = Y_t h, t \in \mathbb{R}$. If there is a homeomorphism $h: M \to M$ seding orbits of X onto orbits of Y, preserving the orientation of the orbits, then we say that X and Y are topologically equivalent.

We say that a vector field is stable with respect to one of these relations if its equivalence class is open.

These equivalence relations can also be considered in $\mathrm{Grad}^r(M)$.

Consider now an element unstable with respect to one of the relations above. If it is possible to describe all the equivalence classes in some neighbourhood of such an element by a finite number of real parameters, then we say that it has *finite modulus of stability* with respect to that relation. In this case the minimum number of such parameters is called the modulus of stability of the element given.

If in some neighbourhood of that element there are at most a denumerable number of equivalence classes, then we say that its *modulus of stability is zero* with respect to the relation considered.

If for that element none of the above conditions are satisfied then we say that its *modulus of stability is infinite* with respect to the relation considered.

(*) During the preparation of this paper the author was a visiting Professor at IMPA; and was partially suported by Financiadora de Estudo e Projetos (FINEP).

Recebido em 31/10/79.

Let $GM^r(M) \subset \mathrm{Grad}^r(M)$ be the (open and dense) set of the gradients of the C^{r+1} real Morse Functions on M.

Then we have:

Theorem A. *If $\nabla f \in GM^r(M)$, then ∇f has modulus of stability finite under conjugacy and modulus zero under topological equivalence.*

This result remains true for $GM^r(M)$ considered as a subset of $\chi^r(M)$ instead of as a subset of $\mathrm{Grad}^r(M)$, and the proof is the same.

Let M be a connected orientable tow-dimensional manifold, possibly non-compact. Consider in $\chi^r(M)$ the C^r Whitney Topology, $r \geqslant 2$.

Let $\Sigma \subset \chi^r(M)$ be the set of the structurally stable vector fields (i.e., stable under topological equivalence). With the usual concept of dimensional graph, we have the following.

Remark. There is an open and dense set $\beta \subset \chi^r(M)$ such that $\beta - \Sigma$ is dense in $\chi^r(M) - \Sigma$, and if G is a graph of an element $X \in \beta$, then the modulus of stability of X near G is at least twice the number of the saddles of G.

The proofs of these results are based on ([3]) and ([4]), where a linearization theorem for hyperbolic two-dimensional saddles ([1]) is used. This is the reason of the assumption $r \geqslant 2$ in $\chi^r(M)$ and $\mathrm{Grad}^r(M)$. Theorem A and the Remark are steps in the way to solve the following problems.

Problem 1. Describe as completely as possible the two-dimensional vector fields which have finite modulus of stability under conjugacy.

Problem 2. Describe as completely as possible the two-dimensional vector fields which have zero modulus of stability under topological equivalence.

Problem 3. Are Problems 1 and 2 equivalent?

This work is divided into three parts. In the first we state notations, definitions and some useful theorems.

The second and the third parts are dedicated to the proofs of the results.

Aknowledgements are due to W. Melo and J. Palis for helpful conversations.

1. Preliminares.

Let $X \in \chi^r(M)$ and $p \in M$ be a hyperbolic saddle of X. The connected components of $W^s_{\mathrm{loc}}(p) - \{p\}$ and $W^s(p) - \{p\}$ are indicated by ξ^{s+}_p, ξ^{s-}_p and γ^{s+}_p, γ^{s-}_p, respectively. The pairs (p, ξ^{s+}_p), (p, ξ^{s-}_p) are called *local separatrices* and the pairs $(p, \gamma^{s+}_p), (p, \gamma^{s-}_p)$ are called *separatrices*. Similar notations and definitions are used for the unstable manifold of p.

If $p, q \in M$ are hyperbolic saddles of X such that $\gamma^{u+}_p = \gamma^{s+}_q$ ($\gamma^{u+}_p = \gamma^{s-}_q$, $\gamma^{u-}_p = \gamma^{s+}_q$, $\gamma^{u-}_p = \gamma^{s-}_q$), then the ordered triple $(p, q, \gamma^{u+}_q)((p, q, \gamma^{u+}_p), (p, q, \gamma^{u-}_q), (p, q, \gamma^{u-}_p))$ is called a *saddle connection.*

Let $(p, q, \gamma = \gamma^{u+}_p)$ be a saddle connection of $X \in \chi^r(M)$, $r \geqslant 2$. Consider C^1 linearizations ϕ and ψ in small neighbourhoods of p and q ([1]), and take points $A \in W^s_{loc}(p) - \{p\}$, $B \in W^u_{loc}(p) \cap \gamma$, $C \in W^u_{loc}(q) \cap \gamma$, $D \in W^u_{loc}(q) - \{q\}$, such that A and D are in the same side of γ. Set $\phi(A) = (0,a)$, $\phi(B) = (b,0)$, $\psi(C) = (c,0)$ and $\psi(D) = (0,d)$. Let Σ_p and Σ_q be segments parallel to $\phi(W^s_{loc}(p))$ and $\psi(W^u_{loc}(q))$, through the points (b,o) and (c,o) respectively. Now we define the *Poincare' Transformations of X, from Σ_p to Σ_q* as the orientation preserving diffeomorphism $g: \Sigma_p \to \Sigma_q$ sending (b,y) to the first intersection of its X-orbit with Σ_q after (b,y).

The number $k = \lim\limits_{y \to 0} \dfrac{z}{y} = g'(0) > 0$ is called *vertical distortion.* This number depends on X, on the linearizations and on the choices of B and C.

It is easy to prove the following

Lemma 1.1. a) *If $\phi = (\phi^u, \phi^s)$, $\psi = (\psi^s, \psi^u)$ and $C = X_T(B)$, $T > 0$, then* $k = \dfrac{\partial}{\partial s}(\psi^u \circ X_T \circ \phi^{-1})|_{(b,\mathrm{o})}$

b) *By suitable choices of B and $C = X_T(B)$, $T > 0$, k can be made any given positive real number.*

c) *If $\bar{\phi} = (\bar{\phi}^u, \bar{\phi}^s)$, $\bar{\psi} = (\bar{\psi}^s; \bar{\psi}^u)$ are also linearizations as above, and β^s_p and β^u_q are the derivatives at 0 of the local diffeomorphisms*

$$y \to \bar{\phi}^s \circ \phi^{-1}(b,y) \text{ and } z \to \bar{\psi}^u \circ \psi^{-1}(c,z),$$

then $\bar{k} = \beta^u_q \cdot k \cdot \dfrac{1}{\beta^s_p}$

With the same notations, the number $K = \dfrac{c}{b}$ is called *horizontal distortion.* It depends on X, on the linearizations and on the choices of B and $C = X_T(B)$, $T > 0$.

Analogously we have the

Lemma 1.2. a) *If* $\phi = (\phi^u, \phi^s)$, $\psi = (\psi^s, \psi^u)$ *and* $C = X_T(B)$, $T > 0$, *then* $K = \frac{\partial}{\partial u}(\psi^s \circ X_T \circ \phi^{-1})|_{(b,o)}$

b) *By suitable choices of* B *and* $C = X_T(B)$, $T > 0$, K *can be made any given positive real number.*

c) *If* $\bar\phi = (\bar\phi^u, \bar\phi^s)$, $\bar\psi = (\bar\psi^s, \bar\psi^u)$ *are also linearizations as above, and* β^s_p *and* β^u_q *are the derivatives at b and c of the local diffeomorphisms:* $x \to \bar\phi^u \circ \phi^{-1}(x,0)$ *and* $x \to \bar\psi^s \circ \psi^{-1}(x,0)$ *respectively, then* $\bar K = \frac{1}{\beta^s_p} \cdot K \cdot \beta^u_q$.

We notice that in (1.2-b) we have $\bar K = \frac{b\,\bar c}{\bar b\, c}.K$, by definition.

Remark 1.3. The number k can be defined using g^{-1} instead of g, and K as $\frac{b}{c}$ instead of $\frac{c}{b}$. The analogous of (1.1) and (1.2) are also true. In each situation we will use the more convenient of these definitions, which will be clear from the context. Now we will recall a result from ([3]) to be used latter on.

Let $(p,q,\gamma = \gamma^{u+}_p)$ be a saddle connection of $X \in \chi^r(M)$, $r \geqslant 2$, such that (p, ξ^{s+}_p) and (q, ξ^{u+}_q) are in the same side of γ. Let $A \in \xi^{s+}_p$, $B \in W^u_{loc}(p) \cap \gamma$, $C \in W^s_{loc}(q) \cap \gamma$, $D \in \xi^{u+}_q$ be such that $\phi(A) = (0,a)$ $\phi(B) = (b,0)$, $\psi(C) = (c,0)$, $\psi(D) = (0,d)$, where ϕ and ψ are C^1 linearizations in neighbourhoods of p and q, respectively. Let λ_p be the negative eigenvalue of $DX(p)$ and μ_q be the positive engenvalue of $DX(q)$. Let $(p',q', \gamma^{u+}_{q'})$ be a saddle connection of $X' \in \chi^r(M)$, $r \geqslant 2$. Using similar notations for X', we get ([3]).

Theorem 1.4. *X and X' are conjugate in neighbourhoods of* $\bar\gamma$ *and* $\bar\gamma'$ *iff* $\frac{\mu_q}{\lambda_{p'}} = \frac{\mu'_{q'}}{\lambda'_{p'}}$. *(This relation will be called the eigenvalue condition).*

Remark 1.5. It follows from the proof of (1.4) that if (p,q,γ) and (p',q',γ') satisfy the eigenvalue condition and if $h: \xi^{s+}_p \cup \bar\gamma \to \xi'^{s+}_{p'} \cup \bar\gamma'$ conjugates X and X', then h can be extended as conjugacy to neighbourhoods of $\bar\gamma$ and $\bar\gamma'$. Moreover, if $\phi' \circ h \circ \phi^{-1}(0,a) = (0,a')$ and $\psi' \circ h \circ \psi^{-1}(0,d) = d'$, then $\frac{d'}{a'} = \frac{k'}{k^{\mu'_{q'}/\mu_q}} \cdot (\frac{d}{a})^{\mu'_{q'}/\mu_q}$. The number $\frac{k'}{k^{\mu'_{q'}/\mu_q}}$ depends only on the linearizations and on the definition of $h: \gamma \to \gamma'$, but not on the choices of B and C, with $h(B) = B'$ and $h(C) = C'$.

Let $S_1, \dots, S_l$ be hyperbolic saddles, and $\gamma_1, \dots, \gamma_{l-1}$ be regular orbits of $X \in \chi^r(M)$, $r \geqslant 2$, such that

a) $\alpha(\gamma_j) = S_j$ and $\omega(\gamma_j) = S_{j+1}$

or b) $\omega(\gamma_j) = S_j$ and $\alpha(\gamma_j) = S_{j+1}$

for $j = 1, \dots, l-1$.

Let $\Gamma = \bigcup_{j=1}^{l-1} \bar\gamma_j$.

If for each $j = 2, \dots, l-1$, S_j is a source or a sink of $X_{|\Gamma}$, then Γ is called a *type I generalized separatrix* of X.

Suppose now that Γ is such that either a) holds for all $j = 1, \dots, l-1$ or b) holds for all $j = 1, \dots, l-1$. We consider each $\bar\gamma_j$ as an oriented manifold with boundary, with the orientation induced by X. Let u_j be the positive unit tangent vector to $\bar\gamma_{j-1}$ at S_j and v_j be the positive unity tangent vector to $\bar\gamma_j$ at S_j, $j = 2, \dots, l-1$. Fix an orientation on M. We say that Γ is a *type II generalized separatrix* of X, if the basis $\{u_j, v_j\}$ belongs to the orientation of M, for each $j = 2, \dots, l-1$. If this does not occurs for each $j = 2, \dots, l-1$, then we say that Γ is a *type III generalized separatrix* of X.

Let $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$, where each Γ_i is a generalized separatrix of type I, II or III, $i = 1, \dots, m$. If the last saddle of Γ_i is the first saddle of Γ_{i+1}, $i = 1, \dots, m-1$, then Γ is called a *generalized separatrix* of X. Moreover, if Γ is a generalized separatrix such that the last saddle of Γ_m is the first saddle of Γ_1, Γ is called a *cycle of separatrices.* A cycle of separatrices such that each Γ_i is of type II (type III) is called a *graph.*

Let $G = \Gamma$ be a graph of $X \in \chi^r(M)$, $r \geqslant 2$, whose saddles are $S_1, \dots, S_l$. Let $\lambda_i < 0 < \mu_i$ be the eigenvalues of $DX(S_i)$, $i = 1, \dots, l$. Denote $r_i = -\frac{\lambda_i}{\mu_i} > 0$, and $\alpha = r_1 \dots r_l$. Suppose $\alpha \neq 1$. Then

Proposition 1.6. ([4]). *The graph G is an atractor iff* $\alpha > 1$, *and it is a repellor iff* $\alpha < 1$.

From (1.6) it follows easily that

Corollary 1.7. *There is an open and dense set $\beta \subset \chi^r(M)$, $r \geqslant 2$, such that:*

i) if $X \in \beta$, then the critical elements of X are all hiperbolic;

ii) $\beta - \Sigma$ is dense in $\chi^r(M) - \Sigma$;

iii) if $X \in \beta$ and G is a graph of X, then G is either an attractor or a repellor.

This result remains true if M is a two-dimensional manifold without boundary and the topology of $\chi^r(M)$, $r \geqslant 2$, is the Whitney C^r topology.

2. Proof of Theorem A. We start with some notations, definitions, lemmas and propositions.

In this section we are concerned with the case of the gradient vector field $\nabla f \in GM^r(M)$, $r \geqslant 2$, of a C^{r+1} Morse function $f{:}M \to \mathbb{R}$. Let its sources be $F_1, \dots, F_{l'}$, its saddles be $S_1, \dots, S_l$, and its sinks be $P_1, \dots, P_{l''}$. Fix the signs + and − for the saddle separatrices.

We say that the saddle separatrix γ_i^{u+} (γ_i^{u-}), of S_i, is a *stabilized separatrix*, if $\omega(\gamma_i^{u+})$ $(\omega(\gamma_i^{u-}))$ is a sink, and that the saddle separatrix γ_i^{s+} (γ_i^{s-}), of S_i is a *stabilized separatrix* if $\alpha(\gamma_i^{s+})$ $(\alpha(\gamma_i^{s-}))$ is a source.

It is well known that there is a sufficiently small neighbourhood $\mathcal{N}(\nabla f) \subset GM^r(M)$, such that each $\nabla g \in \mathcal{N}(\nabla f)$ has $F_1(g), \dots, F_{l'}(g)$ as its sources $S_1(g), \dots, S_l(g)$ as its saddles and $P_1(g), \dots, P_{l''}(g)$ as its sinks, and the following properties hold:

1) $F_i(g)$ is arbitrarily near F_i, $i = 1, \dots, l'$;

2) $S_i(g)$ is arbitrarily near S_i, $i = 1, \dots, l$;

3) $P_i(g)$ is arbitrarily near P_i, $i = 1, \dots, l''$.

Moreover, if we choose the saddles separatrices signs for $S_i(g)$ compatibly with the choices made for S_i, and if γ_i is a stabilized separatrix of S_i, then $\gamma_i(g)$ is a stabilized separatrix of $S_i(g)$. In particular, the number of saddle connections of ∇g is less or equal than the number of saddle connections of ∇f. ([2])

We say that ∇g, $\nabla g' \in \mathcal{N}(\nabla f)$ have *isomorphic phase diagrams* when the following conditions are satisfied:

i) $(S_i(g), S_j(g), \gamma_{ij}(g))$ is a saddle connection of ∇g iff $(S_i(g'), S_j(g'), \gamma_{ij}(g'))$ is a saddle connection of $\nabla g'$;

ii) $\gamma_i(g)$ is a stabilized separatrix of ∇g iff $\gamma_i(g')$ is a stabilized separatrix of $\nabla g'$, and moreover $\alpha(\gamma_i(g)) \cup \omega(\gamma_i(g)) = \{S_i(g), F_j(g)\}$ iff $\alpha(\gamma_i'(g')) \cup \cup\, \omega(\gamma_i(g')) = \{S_i(g'), F_j(g')\}$ or $\alpha(\gamma_i(g)) \cup \omega(\gamma_i(g)) = \{S_i(g), P_j(g)\}$ iff $\alpha(\gamma_i(g')) \cup \omega(\gamma_i(g')) = S_i(g'), P_j(g')$.

Given $\nabla g \in \mathcal{N}(\nabla f)$, we define $I(\nabla g) = \{\nabla g' \in \mathcal{N}(\nabla f)$ such that g and g' have isomorphic phase diagrams$\}$.

It is clear that the collection $\{I(\nabla g), \nabla g \in \mathcal{N}(\nabla f)\}$ is a partition of $\mathcal{N}(\nabla f)$ into a finite number of subsets.

From now on, we allways will consider $\nabla g \in \mathcal{N}(\nabla f)$ and $\nabla g' \in I(\nabla g)$.

Using that g is decreasing along the orbits of ∇g, one can verify that any type I generalized separatrix of ∇g has only one point, and that any cycle of separatrices of ∇g has at least a positive even number of type I generalized separatrices.

Let $\Gamma = \bigcup_{i=1}^{2m} \Gamma_i$ be a cycle of separatrices of ∇g, such that $\Gamma_i = S_i$, $i = 1, 3, \dots, 2m-1$ are type I generalized separatrices and Γ_j, $j = 2, 4, \dots, 2m$ are type II (type III) generalized separatrices, each of them having an even number of saddles. In this case Γ is called a *distinguished cycle* of ∇g.

Indexing the saddles of Γ, according by to the positive orientation of Γ, let I be the set of odd indexes, starting at S_1, and J be the set of even indexes, starting at the first saddle of Γ_2. Let $I \cup J = \{1, 2, \dots, 2N\}$.

We can assume that $S_{r_1} = S_1 = \Gamma_1$ is a source of $\nabla g_{|\Gamma}$. Thus, $S_{r_{2i-1}} = \Gamma_{2i-1}$ is a source of $\nabla g_{|\Gamma}$, when $i = 1, 3, \dots, m-1$, and is a sink of $\nabla g_{|\Gamma}$, when $i = 2, 4, \dots, m$. (Note that m is even). It is easy to see that the set of indexes of Γ_j is $\{r \in I \cup J : r_{2i-1} \leqslant r \leqslant r_{2i+1}\}$, for $j = 2i$, $i = 1, \dots, m$, $r_{2m+1} = r_1$.

The number $D(\Gamma) = (\prod_{i \in I} k_i)(\prod_{j \in J} K_j)$, where k_i and K_j are the vertical and horizontal distortions of saddle connections of Γ, related to its positive orientation, is called the *distortion of* Γ.

Proposition 2.1. *Let $\nabla g' \in I(\nabla g)$, and Γ and Γ' be corresponding distinguished cycles of ∇g and $\nabla g'$, respectively. If $D(\nabla g)$ (S_r) and $D(\nabla g')$ (S_r') have the same engenvalues for $r = 1, \dots, 2N$, and $\dfrac{D(\Gamma')}{D(\Gamma)} = 1$, then ∇g and $\nabla g'$ are conjugated in neighbourhoods of Γ and Γ'.*

Proof. First we choose linearizations in neighbourhoods of S_r and S_r', $r = 1, \dots, 2N$.

We will consider only the case where Γ_{2i} and Γ_{2i}' are type II generalized separatrices, $i = 1, \dots, m$.

We can assume that $S_1 = S_1(g)$ and $S_1' = S_1(g')$ are sources of $\nabla g_{|\Gamma}$ and $\nabla g'_{|\Gamma}$, respectively.

It is obvious that $\Gamma = \bigcup_{i=1}^{m} \Gamma_{2i}$ and $\Gamma' = \bigcup_{i=1}^{m} \Gamma_{2i}$.

We define the conjugacy $h: W_1^s \cup (W_1^u \cap \Gamma) \to W_1'^s \cup (W_1'^u \cap \Gamma')$, between ∇g and $\nabla g'$, arbitrarily. Then, by the hypothesis, (1.4) induces a unique definition of $h: W_2^u \to W_2'^u$. We complete the definition of h from W_2^s to $W_2'^s$, and repeat the argument. Using finite induction and the hypothesis that the number of saddles of Γ_2 and Γ_2' are even and the same, we get $h: W_{r_3}^u \to W_{r_3}'^u$ univocally defined.

The reiteration of this reasoning for $\Gamma_4, \Gamma_6, \ldots, \Gamma_{2m}$ could give us a new definition for $h: W_1'^s \to W_1'^s$.

We claim that this is not the case and, moreover, h can be extended to neighbourhoods of Γ and Γ'.

The first part of our claim obviously imply the second. Therefore, it is enough to prove that the above procedure gives us the same $h: W_1'^s \to W_1'^s$ defined before.

Let ξ_1^{u+}, Γ_2 and ξ_1^{s+} be a such that one can go from ξ_1^{u+} to ξ_1^{s+}, in the positive sense, without crossing any other local separatrix of S_1. Similarly, let $\xi_{r_3}^{s+}$, Γ_2 and $\xi_{r_3}^{u+}$ be such that one can go from $\xi_{r_3}^{u+}$ to $\xi_{r_3}^{s+}$, in the positive sense, without corssing any other local separatrix of S_{r_3}. Let $A_1 \in \xi_1^{s+}$, $B_2 \in \xi_{r_3}^{u+}$ and $A_1' = h(A_1)$, $B_2' = h(B_2)$.

Thus, using the hypothesis, (1.4), the definition of horizontal and vertical distortions, and a finite induction, we get

$$\frac{b_2'}{a_1'} = \left(\frac{k_1' \cdot K_2' \cdot \ldots \cdot k'_{r_3-1}}{k_1 \cdot K_2 \cdot \ldots \cdot k_{r_3-1}}\right) \cdot \frac{b_2}{a_1}$$

This argument applyied to $\Gamma_2, \Gamma_4, \ldots, \Gamma_{2m}$, allows us to get the equality

$\dfrac{D(\Gamma')}{D(\Gamma)} = \dfrac{a'_{m+1}}{a_1'} = 1$, which ends the proof of (2.1).

Now we are going to prove that the number $\dfrac{D(\Gamma')}{D(\Gamma)}$, in (2.1), is an invariant of $\nabla g'$.

Lemma 2.2. *With the notations of* (2.1) *if* $D(\nabla g)$ (S_r) *and* $D(\nabla g')$ (S_r') *have the same eigenvalues,* $r = 1, \ldots, 2N$, *then* $\dfrac{D(\Gamma')}{D(\Gamma)}$ *is a continuous positive real function of* $\nabla g'$, *whose image contains* 1.

Proof. Assuming that $\dfrac{D(\Gamma')}{D(\Gamma)}$ is a function of $\nabla g'$, then (2.2) follows from (1.1-a)), (1.2-a)), and the continuity of the linearizations and of the flow, with respect to the vector field $\nabla g'$.

By (1.1-c)) and (1.2-c)), $D(\Gamma')$ and $D(\Gamma)$ does not depend on the linearizations. Thus, $\dfrac{D(\Gamma')}{D(\Gamma)}$ does not depend either. Then, from the hypothesis and the calculations made on the proof of (2.1), we have $\dfrac{D(\Gamma')}{D(\Gamma)} = \dfrac{a'_{m+1}}{a_1'}$. The right hand side of this equality depends on $\nabla g'$, and the numbers a_1' and a'_{m+1} are not involved in the definition of its left hand side. This ends the proof of (2.2).

Remark 2.3. The continuity of $\dfrac{D(\Gamma')}{D(\Gamma)}$ with respect to $\nabla g'$ is with respect to the C^1-topology of $\mathrm{Grad}^r(M)$, $r \geqslant 2$.

Proof of Theorem A. The first part of Theorem A follows from:

Proposition 2.4. *Let* $\nabla g' \in I(\nabla g)$. *If for any two corresponding saddles* S *and* S', $D(\nabla g)$ (S) *and* $D(\nabla g')$ (S') *have the same eigenvalues, and any two corresponding distinguished cycles have the same distortions, then* ∇g *and* $\nabla g'$ *are conjugated.*

Proof. Let $F_i(g) = F_i$, $i = 1, \ldots, l'$, be the sources, $S_j(g) = S_j$, $j = 1, \ldots, l$, be the saddles and $P_k(g) = P_k'$, $k = 1, \ldots, l''$ be the sinks of ∇g.

Let γ_{j1}, γ_{j4} be the stable separatrices and γ_{j2}, γ_{j3} be the unstable separatrices of S_j, $j = 1, \ldots, l$.

Similar and compatible notations will be used for $\nabla g'$.

We can suppose that $i < j \Longrightarrow g(S_i) \geqslant g(S_j)$.

Now we are going to define the conjugacy h, between ∇g and ∇g.

Firstly we put $h(F_i) = F_i'$, $i = 1, \ldots, l'$; $h(S_j) = S_j$, $j = 1, \ldots, l$; $h(P_k) = P_k'$ $k = 1, \ldots, l''$. We will define

$h: \bigcup_{j=1}^{l} \left(\bigcup_{r=1}^{4} \gamma_{jr}\right) \to \bigcup_{j=1}^{l} \left(\bigcup_{r=1}^{4} \gamma'_{jr}\right)$ by induction on j.

Using that g is decreasing along the orbits of ∇g, we see that $\alpha(\gamma_{11}) = F_{i_1}$ and $\alpha(\gamma_{14}) = F_{i_2}$, $i_1, i_2 \in \{1, \ldots, l'\}$. Thus, $\alpha(\gamma_{11}') = F_{i_1}'$ and $\alpha(\gamma_{14}') = F_{i_2}'$. Then, we can define

$h: \bigcup_{r=1}^{4} \gamma_{1r} \to \bigcup_{r=1}^{4} \gamma_{1r}$ arbitrarily.

Suppose that $h: \bigcup_{r=1}^{4} \gamma_{jr} \to \bigcup_{r=1}^{4} \gamma'_{jr}$ is defined for j = 1, … , n-1, and let us define $h: \bigcup_{r=1}^{4} \gamma_{nr} \to \bigcup_{r=1}^{4} \gamma'_{nr}$. If $\alpha(\gamma_{n1})$ and $\alpha(\gamma'_{n1})$ are sources we define $h: \bigcup_{r=1}^{3} \gamma_{nr} \to \bigcup_{r=1}^{3} \gamma'_{nr}$ arbitrarily. If $\alpha(\gamma_{n1})$ is a saddle, by using again that g is decreasing along the orbits of ∇g, we have $\alpha(\gamma_{n1}) = S_j$, for some $j < n$. Thus, $\alpha(\gamma'_{n1}) = S'_j$. Then, by the induction hypothesis, (1.4) induces the definition of h from $\bigcup_{r=1}^{3} \gamma_{nr}$ to $\bigcup_{r=1}^{3} \gamma'_{nr}$. If $\alpha(\gamma_{n4})$ and $\alpha(\gamma'_{n4})$ are sources we define $h: \gamma_{n4} \to \gamma'_{n4}$ arbitrarily. If $\alpha(\gamma_{n4})$ and $\alpha(\gamma'_{n4})$ are corresponding saddles, then, by the above argument and by the induction hypothesis, $h: \gamma_{n4} \to \gamma'_{n4}$ is alreaddy defined.

We notice that, taking care in the definition of h where it was arbitrarily defined, by (2.1) and by the hypothesis on the corresponding distinguished cycles, we get a well defined conjugacy

$$h: \bigcup_{j=1}^{l} \left(\bigcup_{r=1}^{r} \gamma_{jr}\right) \to \bigcup_{j=1}^{l} \left(\bigcup_{r=1}^{4} \gamma'_{jr}\right), \text{ between } \nabla g \text{ and } \nabla g'.$$

To extend this conjugacy to the whole manifold M one can now use the classical Tubular Family Theory ([2]). This ends the proof of (2.4).

To finish the proof of Theorem A, we will use the proof of (2.4) and some lemmas.

Let (p,q,γ) be a saddle connection of a vector field $X \in \mathfrak{X}^r(M)$, $r \geq 2$, such that $\xi_p^{u+} \cup \xi_q^{s+} \subset \gamma$. Choose $B \in \xi_p^{u+}$, $C \in \xi_q^{s+}$ and $T > 0$ such that $X_T(B) = C$, and fix linearizations in neighbourhoods of p and q. Let $K > 0$ and $k > 0$ be the corresponding horizontal and vertical distortions of X.

Lemma 2.5. *For each* $\overline{K} \in (e^{T\lambda_q} K, e^{T\mu_p} K)$ *there exist a positive* C^∞ *function* $\overline{\eta}: M \to \mathbb{R}$ *and points* $\overline{B} \in \xi_p^{u+}$, $C \in \xi_q^{s+}$, *such that the vector field* $\overline{X} = \overline{\eta} \cdot X$ *has the following properties:*

1) $X_T(\overline{B}) = \overline{C}$;

2) *the horizontal distortion of* $\overline{X}$, *relative to the fixed linearizations and to* $\overline{B}$, $\overline{C}$, *is exactly* $\overline{K}$;

3) $\overline{X} = X$ *outside an arbitrarily thiner long flow box of* X *along* γ, *from* B *to* C;

4) *the identity of* M *is a topological equivalence between* X *and* $\overline{X}$.

Proof. It is enough to consider the case $\overline{K} \in (e^{T\lambda_q} K, K)$. There is a unique $t_0 \in (0,T)$, such that $X_{T+t_0}(B) = \overline{C}$ and that the horizontal distortion of X, relatively to $\overline{B} = B$, $\overline{C}$ and the fixed linearizations, is $e^{t_0\lambda_q} K = \overline{K}$.

Let $\phi: (-\epsilon, T+\epsilon) \times (-3\delta, 3\delta) \to U$ be a long flow box of X, such that $\phi(0,0) = B$, $\phi(T,0) = C$, $D\phi(t,s)\cdot Z(t,s) = X(\phi(t,s))$, where $Z(t,s) = (1,0)$.

There is a C^∞ positive function $\beta: (-\epsilon, T+\epsilon) \times (-3\delta, 3\delta) \to \mathbb{R}$, such that

$$\begin{cases} \beta(t,s) = 1 \text{ if } |s| \geq 2\delta, \text{ for all } t \in (-\epsilon, T+\epsilon); \\ \beta(t,s) = 1 \text{ if } t \leq 0 \text{ or if } t \geq T, \text{ for all } s \in (-3\delta, 3\delta); \\ \int_0^T \beta(t,s)\,dt = T - t_0 \text{ for all } s \in (-\delta, \delta). \end{cases}$$

Let $\overline{Z}(t,s) = \beta(t,s)\cdot Z(t,s) = (\beta(t,s), 0)$. We define

$$\begin{cases} \overline{X}(x) = D\phi(\phi^{-1}(x))\cdot\overline{Z}(\phi^{-1}(x)), \text{ if } x \in U, \text{ and} \\ \overline{X}(x) = X(x), \text{ if } x \notin U. \end{cases}$$

It is obvious that $\overline{X}$ has the desired properties.

Remark 2.6. Consider a saddle connection (p',q',γ') of a vector field $X' \in \mathfrak{X}^r(M)$, $r \geq 2$. With similar notations to the ones used for (p,q,γ) above, if $\lambda_p = \lambda_{p'}$, $\mu_p = \mu_{p'}$, $\lambda_q = \lambda_{q'}$, $\mu_q = \mu_{q'}$, then the number $\frac{K'}{K}$ does not depend on the choices of the point B, and of the time $T > 0$, $(h(B) = B', X_T(B) = C, h(C) = C', X'_T(B') = C')$. By (1.4) the same is true for the number $\frac{k'}{k}$.

Lemma 2.7. *Suppose that in (2.6) we have* $\lambda_p = \lambda'_{p'} = \lambda_q = \lambda'_{q'} = -1$ *and* $\mu_p = \mu'_{p'} = \mu_q = \mu'_{q'} = 2$. *If* $K' < K$, *then, there exist a conjugacy* $h: \gamma \to \gamma'$, *points* $\overline{B}$, $\overline{C} \in \gamma$, *and a positive* C^∞ *function* $\eta: M \to \mathbb{R}$, *with the following properties:*

1) $\eta = 1$ *outside an arbitrarily thiner long flow box of* X, *along* γ, *from* B *to* C;

2) *if* $\overline{k}$ *and* $\overline{K}$ *are the distortions of* $\overline{X} = \eta\cdot X$, *relatively to* $\overline{B}$ *and* $\overline{C} = \overline{X}_T(\overline{B})$, *and* k' *and* K' *are the ones of* X', *relatively to* $\overline{B}' = h(\overline{B})$ *and* $\overline{C}' = h(\overline{C})$, *then* $\frac{k'}{\overline{k}} = \frac{K'}{\overline{K}} = 1$. *(h will be a conjugacy between* $\overline{X}$ *and* X', *from* γ *to* γ').

Proof. Let $\tilde{h}: \gamma \to \gamma'$ be a conjugacy between X and X' such that $\tilde{h}(\tilde{B}) = B'$, $\tilde{h}(\tilde{C}) = C'$, $X_T(\tilde{B}) = \tilde{C}$ and $X'_T(B') = C'$.

We will choose a new point $B \in \gamma$ and will define a new conjugacy $h: \gamma \to \gamma'$, between X and X', puting $h(B) = B'$, such that $\frac{k'}{k} = 1$. We have $e^{-\tilde{s}}\tilde{y} = y$, $e^{2\tilde{s}}\tilde{z} = z$. Thus, $k = e^{3\tilde{s}}\,\tilde{k}$. That is $\frac{k'}{k} = e^{-3\tilde{s}} \cdot \frac{k'}{\tilde{k}}$. Let $\tilde{s} = \frac{1}{3}\log\frac{k'}{\tilde{k}}$.

By (2.6) we can choose $\tilde{B} \in \gamma$ and $T > 0$, such that $B = X_{\tilde{s}}(\tilde{B})$ and $C = X_T(B)$ are in the linearized neighbourhoods of p and q. Then, we get $\frac{k'}{k} = 1$.

Consider $s > 0$, $t > 0$, and define $X_s(\overline{B}) = B$ and $X_t(\overline{C}) = C$. If $\overline{k}$ and $\overline{K}$ are the distortions of X with respect to $\overline{B}$ and $\overline{C}$, then

$$\overline{k} = e^{-s+2t}k \text{ and } \overline{K} = e^{2s-t}K.$$

We want that $\frac{k'}{\overline{k}} = \frac{K'}{\overline{K}} = 1$. Thus, to get this equalities it is enough to solve the following system:

$$\begin{cases} -s + 2t = 0 \\ 2s - t = \log\left(\frac{K'}{K}\right) < 0 \\ s > 0,\ t > 0,\ s + t < T \end{cases}$$

By (2.6), one can choose T very large, and the left hand sides of the two first equalities does not change. Then, the system has unique solution (s_0, t_0).

To end the proof of (2.7) we modify the vector field X as in (2.5), using a C^∞ function β such that $\int_0^T \beta(t,s)dt = T-(s_0+t_0)$, for all $s \in (-\delta,\delta)$.

End of the proof of Theorem A.

Let $\eta_1, \eta'_1: M \to \mathbb{R}$ be positive C^∞ functions, such that $\eta_1 = 1$ and $\eta'_1 = 1$ outside small linearized neighbourhoods of S_j and S'_j, and the eigenvalues of $DX_1(S_j)$ and $DX'_1(S'_j)$ are -1 and 2, for $j = 1, \ldots, l$, where $X_1 = \eta_1 \cdot \nabla g$ and $X'_1 = \eta'_1 \cdot \nabla g'$.

Following the proof of (2.4) and applying (2.7) to each pair of corresponding saddle connections. We get from X_1 and X'_1 two conjugated vector fields $X = \eta \cdot \nabla g$ and $X' = \eta' \nabla g'$, where $\eta, \eta': M \to \mathbb{R}$ are positive C^∞ functions. This, of course implies that ∇g and $\nabla g'$ are topologically equivalent, as desired.

3. Proof of the remark. Let G be a graph of $X \in \mathfrak{X}^r(M)$, $r \geq 2$. We can assume that G is a type II generalized separatrix, whose saddles are $S_1, \ldots, S_l$, and whose regular orbits are $\gamma_1, \ldots, \gamma_l$ with $\alpha(\gamma_i) = S_i$, $\omega(\gamma_i) = S_{i+1}$, $i = 1, \ldots, l$, $S_{l+1} = S_1$.

We can also assume that $\xi_i^{u+} \subset \gamma_i$ and $\xi_i^{s+} \subset \gamma_{i-i}$, $i = 1, \ldots, l$, $\gamma_0 = \gamma_l$.

Let ϕ_i be a linearization of X in a neighbourhood of S_i, and $\lambda_i < 0 < \mu_i$ be the eigenvalues of $DX(S_i)$, $i = 1, \ldots, l$.

Take $A_i \in \xi_i^{s+}$, $B_i \in \xi_i^{u+}$ such that $\phi_i(A_i) = (0,1)$, $\phi_i(B_i) = (1,0)$, an horizontal segment $\Sigma_i \supset (0,1)$ and a vertical segment $\tilde{\Sigma}_i \supset (1,0)$. Consider the Poincaré transformation $g_i: \tilde{\Sigma}_i \to \Sigma_{i+1}$, and define the function $\overline{g}_i(l, y_i) \equiv \overline{g}_i(y_i) = \frac{g_i(y_i)}{y_i}$ if $y_i > 0$, and $\overline{g}_i(0) = Dg_i(0)$, $i = 1, \ldots, l$, $\Sigma_{l+1} = \Sigma_1$. It is clear that $\overline{g}_i$ is a continuous function, whose image is contained in closed interval $[a_i, b_i] \subset (0, +\infty)$.

We define the Poincaré Transformation of G, $f: (0,\delta] \subset \Sigma_1 \to (0,\epsilon] \subset \Sigma_1$, relative to Σ_1, in the usual way.

Consider the numbers $r_i = -\frac{\lambda_i}{\mu_i} > 0$ and $\alpha = r_1 \ldots r_l$.

The proof of (1.5) is done showing that $f(x) = \phi(x)x^\alpha$, where $\phi: (0,\delta] \to \mathbb{R}$ is a continuous function such that $0 < m < \phi(x) \leq M$, for all $x \in (0,\delta]$ and for some $m, M \in \mathbb{R}$.

We can assume that $\alpha > 1$, (that is, f is a contraction).

Now we will make estimatives for the time $T_n(x)$, spent by the orbit of x from $f^n(x)$ to $f^{n+1}(x)$, $n = 0, 1, 2, \ldots$.

Let $(x_1)_n \in \Sigma_i$ and $(y_i)_n \in \tilde{\Sigma}_i$, be the points where the orbit of x crosses Σ_i and $\tilde{\Sigma}_i$ for the first time after $f^n(x)$.

Let $(t_i)_n$ be the time spent by this orbit from $(x_i)_n$ to $(y_i)_n$, and $(s_i)_n$ be the time from $(y_i)_n$ to $(x_{i+1})_n$, $i = 1, \ldots, l$, $(x_{l+1})_n = f^{n+1}(x)$.

Then we have $t_n(x) = \sum_{i=1}^{l} (t_i)_n + \sum_{i=1}^{l} (s_i)_n = \sum_{i=1}^{l} \frac{1}{\mu_i} \cdot \log\frac{1}{(x_i)_n} + \sum_{i=1}^{l} (s_i)_n$.

One can easily verify that

$$(x_{i+1})_n = \overline{g}_{i-1}((y_{i-1})_n) \cdot (y_i)_n = \overline{g}_i((y_i)_n \cdot (x_i)_n^{r_i}.$$

Thus, by induction, we get

$$(x_i)_n = \overline{g}_{i-1}(y_{i-1})_n) \cdot [\overline{g}_{i-2}((y_{i-2})_n)]^{r_{i-1}} \cdot \ldots \cdot [\overline{g}_1((y_i)_n)]^{r_{i-1} \cdot \ldots \cdot r_2} \cdot (x_1)_n$$

But there are numbers $0 < a < b$ such that $a \leq \overline{g}_i(y) \leq b$, for all $i = 1, \ldots, l$, and for all small $y \geq 0$.

Then, $a^{1+r_{i-1}+\dots+(r_{i-1}\cdot\dots\cdot r_2)}\,[(x_1)_n]^{r_1\cdot\dots\cdot r_{i-1}} \leqslant (x_i)_n$, which implies that $T_n(x) \leqslant \sum_{i=1}^{l} \frac{1}{\mu_i}\,[1+r_{i-1}+\dots+(r_{i-1}\cdot\dots\cdot r_2)]\,\log\frac{1}{a} + \sum_{i=1}^{l}\left[\frac{r_1\cdot\dots\cdot r_{i-1}}{\mu_i}\right]$ $\log\frac{1}{(x_1)_n} + \sum_{i=1}^{l}(s_i)_n$.

Using that $f(x)=\phi(x)\cdot x^{\alpha}$, $0<m\leqslant\phi(x)\leqslant M$, and that $(x_1)_n = f^n(x)$, we obtain $m^{1+\alpha+\dots+\alpha^{n-1}}\cdot x^{\alpha^n} \leqslant (x_1)_n$. The number $\sum_{i=1}^{l}(s_i)_n$ is bounded as a real function of $(n,x)\in N\times(0,\delta]$. Then, for any $(n,x)\in N\times(0,\delta]$, we have $T_n(x)\leqslant A-K\cdot\alpha^n\log(m^{\frac{1}{\alpha-1}}x)$ where $A\in\mathbb{R}$ and $K=\sum_{i=1}^{l}\frac{r_1\cdot\dots\cdot r_{i-1}}{\mu_i}$.

Analogoulsy, we get

$T_n(x)\geqslant \bar{A}-K\cdot\alpha^n\log(M^{\frac{1}{\alpha-1}}\cdot x)$, where $\bar{A}\in\mathbb{R}$, for all $(n,x)\in N\times(0,\delta]$.

Now, let G' be a graph of $X'\in\chi^r(M)$, $r\geqslant 2$, such that X and X' are conjugated by a homeomorphism h, in neighbourhoods of G and G'.

We consider for G' similar notations to the ones stated for G.

We can assume that h is an orientation preserving homeomorphism such that $h(S_i)=S'_i$, i = 1, ... , l. Let $A'_i = h(A_i)$ and $B'_i = h(B_i)$, i = 1, ... , l. Changing the linearization ϕ'_i, if necessary, we can suppose that $\phi'_i(A'_i) = (0,1)$ and $\phi'_i(B'_i) = (1,0)$, i = 1, ... , l.

Given $x\in(0,\delta]\subset\Sigma_1$, we consider the point $x'\in\Sigma'_1$, nearest to $h(x)$ in the orbit of $h(x)$ by X'. Then, using that h is a conjugacy, we get $\lim_{n\to+\infty}(T_n(x)-T'_n(x'))=0$, independent of the choice of x and, a fortiori, of x'.

By (1.4), we have $\frac{\mu_{i+1}}{\lambda_i}=\frac{\mu'_{i+1}}{\lambda'_i}$, i = 1, ... , l, $\mu_{l+1}=\mu_1$, $\mu_{l+1}=\mu'_1$.

Then, $\alpha=\alpha'$ and, by the same calculations we made before, we get:

$T'_n(x')\leqslant A'-K'\alpha^n\log(m'^{\frac{1}{\alpha-1}}x')$, $A'\in\mathbb{R}$, and $T'_n(x')\geqslant\bar{A}'-K'\alpha^n\log(M'^{\frac{1}{\alpha-1}}x')$, $\bar{A}'\in\mathbb{R}$, where $K'=\sum_{i=1}^{l}\frac{r'_1\cdot\dots\cdot r'_{i-1}}{\mu_i}$.

We claim that $K=K'$.

In fact, from our estimatives on $T_n(x)$ and $T'_n(x')$ we obtain

$$\text{(i)}\quad\begin{cases} T_n(x)-T'_n(x')\leqslant B+[K'\log(M'^{\frac{1}{\alpha-1}}x')-K\log(m^{\frac{1}{\alpha-1}}x)]\,\alpha^n \\ T_n(x)-T'_n(x')\geqslant \bar{B}+[K'\log(m'^{\frac{1}{\alpha-1}}x')-K\log(M^{\frac{1}{\alpha-1}}x)]\,\alpha^n, \\ B,\ \bar{B}\in\mathbb{R}. \end{cases}$$

Suppose, by contradiction, that $K'\neq K$.

It is enough to consider only the case $K'-K>0$.

It is obvious that (i) is equivalent to

$$\text{(ii)}\quad\begin{cases} T_n(x)-T'_n(x')\leqslant B+(K'-K)\left[\log\left(\frac{M'}{m}\right)^{\frac{1}{\alpha-1}}+\log\left(\frac{x'}{x}\right)\right]\ \alpha^n \\ T_n(x)-T'_n(x')\geqslant \bar{B}+(K'-K)\left[\log\left(\frac{m'}{M}\right)^{\frac{1}{\alpha-1}}+\log\left(\frac{x'}{x}\right)\right]\ \alpha^n \end{cases}$$

Then, $0<\left(\frac{m}{M'}\right)^{\frac{1}{\alpha-1}}\leqslant\frac{x'}{x}\leqslant\left(\frac{m'}{M}\right)^{\frac{1}{\alpha-1}}$, because $\alpha>1$, $K'-K>0$, and $(T_n(x)-T'_n(x'))\to 0$.

We can rewrite the first of the inequalities (i) in the following form:

$$T_n(x)-T'_n(x')\leqslant B+\ [K'\log M'^{\frac{1}{\alpha-1}}-K\log m^{\frac{1}{\alpha-1}}+K'\log\left(\frac{x'}{x}\right)+(K'-K)\log x]\ \alpha^n.$$

Using that $K'-K>0$ and $\frac{x'}{x}$ is bounded, we see that the left hand side of this inequality tends to $-\infty$, when $x\to 0$. This contradicts the fact that, $T_n(x)-T'_n(x')\to 0$, which proves our claim.

Recalling that $\frac{\lambda_i}{\mu_{i+1}}=\frac{\lambda'_i}{\mu'_{i+1}}$

$K=\frac{1}{\mu_1}\cdot\left[\sum_{i=1}^{l}\left(\frac{-\lambda_1}{\mu_i}\right)\cdot\dots\cdot\left(\frac{-\lambda_{i-1}}{\mu_i}\right)\right]$ and $K'=\frac{1}{\mu'_1}\left[\sum_{i=1}^{l}\left(\frac{-\lambda'_1}{\mu'_2}\right)\cdot\dots\cdot\left(\frac{-\lambda'_{i-1}}{\mu'_i}\right)\right]$

from $K=K'$ we get $\mu_1=\mu'_1$.

Starting the above argument at each i = 1, ... , l, we obtain $\mu_i=\mu'_i$ and then $\lambda_i=\lambda'_i$, for each i = 1, ... , l. This and (1.7) finish the proof of the Remark.

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