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Successive coefficients of univalent functions

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1. Introduction.

Let S be the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \quad (|z| < 1)$$

analytic and univalent in the unit disc.

Let Σ denote the class of functions

$$g(z) = z + b_0 + b_1 z^{-1} + \dots \quad (|z| > 1)$$

that are univalent in $\Delta = \{z \in \mathbb{C}_\infty : |z| > 1\}$. A problem which has attracted considerable attention is to estimate $d_n = \| a_{n+1} | - | a_n \|$, $n = 2, 3, \dots$, the difference of the moduli of successive coefficients of functions in S. Goluzin [1] showed $d_n = 0(n^{1/4} log n)$. Then in 1963 Hayman [2] settled the order of growth problem by showing that $d_n \leq A$, where A is an absolute constant. Milin [3] found an alternate proof, simpler than Hayman's which led to the best bound known: $d_n < 4.18$. In 1966 M. S. Robertson [5] has proved that the inequalities

$$\frac{8}{n+1} - 6 \leq |a_{n+1}| - |a_{n-1}| \leq 4 - 4/_{n+1} \quad (n = 2, 3, 4, \dots)$$

hold for all close-to-convex functions in $|z| < 1$. It is well known that such a function is necessarily univalent in $|z| < 1$.

In this paper we get an improvement of the Milin and M. S. Robertson results.

2. Preliminary results.

We require the following inequality of Milin-Lebedev [3] and Grunsky ([4], p. 60).

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Lemma A. (Grunsky inequality). *If* $g \in \Sigma$ *and* $\lambda_k \in \mathbb{C}(k = 1, 2, \ldots)$ *then*

$$\sum_{k=1}^{\infty} k \left| \sum_{\ell=1}^{\infty} b_{k\ell}\, \lambda_\ell \right|^2 \le \sum_{k=1}^{\infty} \frac{|\lambda_k|^2}{k}.$$

For an arbitrary sequence of complex numbers $\{\alpha_k\}_1^\infty$, which generates a sequence $\{\beta_n\}_0^\infty$ by means of the expansion

$$\text{(2.1)} \qquad \sum_{n=0}^{\infty} \beta_n z^n = \exp\left[\sum_{k=1}^{\infty} \alpha_k z^k\right]$$

we may obtain, for $n = 1, 2, \ldots$, the following three inequalities.

Lemma B. (Milin-Lebedev). *If we define*

$$\text{(1)} \quad \sigma_n = \frac{1}{n+1}\, exp\left[\sum_{k=1}^{n} \frac{1}{k+1} - \sum_{k=1}^{n} \frac{k(n-k+1)}{n+1} |\alpha_k|^2\right] \sum_{j=0}^{n} |\beta_j|^2$$

for $n = 0, 1, \ldots$ *then*

$$\sigma_n \le \sigma_{n-1} \le 1 \quad (n = 1, 2, \ldots).$$

In particular,

$$\sum_{j=0}^{n} |\beta_j|^2 \le (n+1)\, exp\left[\sum_{k=1}^{n+1} k|\alpha_k|^2 - \sum_{k=1}^{n+1} \frac{1}{k} + \right.$$

$$\left. + 1 - \frac{1}{n+1} \sum_{k=1}^{n+1} k^2 |\alpha_k|^2\right].$$

Lemma C. *If* $\sigma_n(n = 1, 2, \ldots)$ is defined by (1) then

$$|\beta_n| \le exp\left[\frac{1}{2} \sum_{k=1}^{n} (k|\alpha_k|^2 - 1/k)\right].$$

Lemma D. *If the right-hand side converges then*

$$\sum_{n=0}^{\infty} |\beta_n|^2 \le exp\left[\sum_{k=1}^{\infty} k|\alpha_k|^2\right].$$

3. Main result.

In this section we will prove the following theorem:

Theorem 1. *Let* $f \in S$. *Then*

$$-5.50 + \frac{12 + 3\sqrt{2e}}{8(n-1)} < |a_{n+1}| - |a_{n-1}| < 6.60 + $$

$$+ \frac{5.06 - 3\sqrt{2e}}{8(n-1)} \quad (n = 2, 3, 4, \ldots)$$

In particular,

$$-5.50 < |a_{n+1}| - |a_{n-1}| < 6.60 \quad (n = 2, 3, 4, \ldots).$$

Proof.

The proof of theorem 1 will be given in four steps (a) – (d). In Step (a) we apply Milin's Method (cf. [3]) to obtain the power series about ∞ for the functions

$$\frac{d}{dz} \frac{z^2 - t^2}{g(z) - g(t)} \quad \text{and} \quad \frac{2z}{z-t} - \frac{(z+t)\, g'(z)}{g(z) - g(t)},$$

where

$$g(z) = \frac{1}{f(z^{-1})},$$

($|z| > 1$) belongs to Σ and $f(\xi)$ ($|\xi| < 1$) is an arbitrary function in S. Step (b) derives a complicated representation for

$$(n-1)\,(a_{n+1} - t^2\, a_{n-1})$$

which leads to very economical estimates. Step (c) is devoted to determining an upper bound for the right-hand side of expression obtained in Step (b). For this, we use the Grunsky inequality, Schwarz's inequality, Parseval's formula and Milin's Method. Finally, we obtain in Step (d) the desired inequality using the well-known inequality $|a_n| \le n$ for $n = 2, 3$ and the results obtained in Steps (a) – (c).

(a) Let us take an arbitrary function $f(\xi) \in S$, and let us form from it the function $g(z) \in \Sigma$:

$$\text{(3.1)} \qquad g(z) = [f(z^{-1})]^{-1}, \quad |z| > 1 \quad (\text{cf. [4], p. 12}).$$

Let $b_{k\ell}$ $(k, \ell = 1, 2, \ldots)$ be the Grunsky coefficients of $g(z)$, (cf. [4], p. 58). If we define

$$\text{(3.2)} \qquad \alpha_k(t) = \sum_{\ell=1}^{\infty} b_{k\ell} t^{-\ell} \quad (|\, t\,| > 1,\ k = 1, 2, \ldots)$$

$$\text{(3.3)} \qquad \frac{z-t}{g(z)-g(t)} = \sum_{r=0}^{\infty} \beta_r(t) z^{-r} \quad (|\, z\,| > 1,\ |\, t\,| > 1)$$

we obtain from ([4], (8), p. 58) that

$$\text{(3.4)} \qquad \log \frac{z-t}{g(z)-g(t)} = \sum_{k=1}^{\infty} \alpha_k(t) z^{-k}.$$

Thus $\alpha_k = \alpha_k(t)$ and $\beta_n = \beta_n(t)$ are related by the exponential relation (2.1). Furthermore, by (3.4) and (3.3),

$$\text{(3.5)} \qquad \frac{d}{dz} \log \frac{z^2-t^2}{g(z)-t} = \frac{2z}{z^2-t^2} - \frac{g'(z)}{g(z)-g(t)}$$

and

$$\text{(3.6)} \qquad \frac{d}{dz}\, \frac{z^2-t^2}{g(z)-g(t)} = \frac{z-t}{g(z)-g(t)} + (z+t)\, \frac{d}{dz}\, \frac{z-t}{g(z)-g(t)}$$

$$= \sum_{r=0}^{\infty} \beta_r(t) z^{-r} + z\left(\sum_{r=1}^{\infty} (-r)\, \beta_r(t) z^{-r-1}\right)$$

$$+\, t\left(\sum_{r=1}^{\infty} (-r)\, \beta_r(t) z^{-r-1}\right)$$

$$= \sum_{r=0}^{\infty} \beta_r(t) z^{-r} - \sum_{r=1}^{\infty} r\, \beta_r(t) z^{-r} - \sum_{r=1}^{\infty} rt\, \beta_r(t) z^{-r-1}.$$

Also,

$$\text{(3.7)} \qquad \frac{d}{dz} \log \frac{z^2-t^2}{g(z)-g(t)} = \frac{d}{dz} \log(z+t) + \frac{d}{dz} \log \frac{z-t}{g(z)-g(t)}$$

$$= \frac{1}{z+t} - \sum_{k=1}^{\infty} k\, \alpha_k(t) z^{-k-1}$$

Thus we obtain from (3.5) and (3.7)

$$\text{(3.8)} \qquad \frac{2z}{z-t} - \frac{(z+t)\, g'(z)}{g(z)-g(t)} = 1 - (z+t) \sum_{k=1}^{\infty} k\, \alpha_k(t) z^{-k-1}$$

(b) Let us now select a value of t on the circle $|\, z\,| = r > 1$, such that

$$\text{(3.9)} \qquad |\, g(t)\,| = \min_{|z|=r} |\, g(z)\,| = \frac{1}{M(r)}, \quad |\, t\,| = r$$

where

$$\text{(3.10)} \qquad M(r) = \max_{|s| = \frac{1}{r}} |\, f(s)\,|,$$

and let us form the function $(1 - t^2/z^2)\, f'(1/z)$. According to the Cauchy formula for the Taylor coefficients of this function, we shall have

$$\text{(3.11)} \qquad (n+1)a_{n+1} - (n-1)a_{n-1}t^2 = \frac{1}{2\pi i} \int_{|z|=r} (1 - t^2/z^2)\, f'(1/z) z^n \frac{dz}{z}$$

Noting that

$$f'(1/z) = z^2\, g'(z)/(g(z))^2$$

we may rewrite Equation (3.11) thus:

$$\text{(3.12)} \qquad (n+1)a_{n+1} - (n-1)a_{n-1}t^2 = \frac{1}{2\pi i} \int_{|z|=r} \left[\frac{g'(z)}{g(z)-g(t)}\, \frac{z^2-t^2}{g(z)-g(t)} \left(1 - \frac{g(t)}{g(z)}\right) - \frac{1}{z^2}\, f'(1/z)\, \frac{z^2-t^2}{g(z)-g(t)}\, g(t) \right] z^n \frac{dz}{z}.$$

Noting still that

$$\frac{d}{dz} \log \frac{z^2-t^2}{g(z)-g(t)} = \frac{2z}{z^2-t^2} - \frac{g'(z)}{g(z)-g(t)}$$

and

$$\frac{d}{dz}\, \frac{z^2-t^2}{g(z)-g(t)} = \left[\frac{d}{dz} \log \frac{z^2-t^2}{g(z)-g(t)}\right] \frac{z^2-t^2}{g(z)-g(t)},$$

we may transform the first terms of the integrand in Equation (3.12) in the following way:

$$\frac{g'(z)}{g(z)-g(t)}\, \frac{z^2-t^2}{g(z)-g(t)} \left(1 - \frac{g(t)}{g(z)}\right) = -\frac{d}{dz}\, \frac{z^2-t^2}{g(z)-g(t)}$$

$$- \left[\frac{g'(z)}{g(z)-g(t)} - \frac{2z}{z^2-t^2}\right] \frac{z^2-t^2}{g(z)-g(t)}\, \frac{g(t)}{g(z)} + \frac{2z}{g(z)}.$$

If we take this into account, and if we also use (3.6) and the Taylor series expansions for $|\, z\,| > 1$ of the function $2z/g(z)$, namely

$$\frac{2z}{g(z)} = \sum_{k=0}^{\infty} 2a_{k+1} z^{-k}, \quad a_1 = 1$$

we may rewrite formula (3.12) as follows:

$$(n-1)a_{n+1} - (n-1)a_{n-1}t^2 = (n-1)\beta_n + t(n-1)\beta_{n-1}$$

$$-\frac{1}{2\pi i}\int_{|z|=r}\left\{\left[\frac{g'(z)}{g(z)-g(t)} - \frac{2z}{z^2-t^2}\right]\frac{z^2-t^2}{g(z)-g(t)}\,\frac{g(t)}{g(z)}\right.$$

$$\left.+\frac{1}{z^2}f'(1/z)\,\frac{z^2-t^2}{g(z)-g(t)}\,g(t)\right\} z^n \frac{dz}{z}$$

We shall write α_k and β_r instead $\alpha_k(t)$ and $\beta_r(t)$. We obtain by (3.3) and (3.8) that

$$-(n-1)(a_{n+1}-t^2a_{n-1}) = -(n-1)\beta_n - t(n-1)\beta_{n-1}$$

$$+\frac{1}{2\pi i}\int_{|z|=r}\left[(z+t)\sum_{k=1}^{n-1}\frac{k\alpha_k}{k+1} - 1\right]\sum_{j=0}^{\infty}\frac{\beta_j}{z^j}\,\frac{g(t)}{g(z)}\, z^n\frac{dz}{z}$$

$$+\frac{g(t)}{|2\pi i|}\int_{|z|=r}(z+t)\sum_{k=1}^{n-1}\frac{ka_k}{z^{k+1}}\sum_{j=0}^{\infty}\frac{\beta_j}{z^j}\,z^n\frac{dz}{z}.$$

Remark. We are allowed to replace the functions by suitable partial sums of their power series expansions because the remainders do not give any contribution to the values of the integrals.

And so we get

$$-(n-1)(a_{n+1}-t^2a_{n-1}) = -(n-1)\beta_n - t(n-1)\beta_{n-1} +$$

$$+\frac{1}{2\pi i}\int_{|z|=r}(z+t)\sum_{k=1}^{n-1}\frac{k\alpha_k}{z^{k+1}}\sum_{j=0}^{n-2}\frac{\beta_j}{z^j}\,\frac{g(t)}{g(z)}\,z^n\frac{dz}{z}$$

$$-\frac{g(t)}{2\pi i}\int_{|z|=r}\sum_{j=0}^{\infty}\frac{\beta_j}{z^j}\sum_{k=0}^{n-1}\frac{a_{k+1}}{z^{k+1}}\,z^n\frac{dz}{z}$$

$$+\frac{g(t)}{|2\pi i|}\int_{|z|=r}(z+t)\sum_{k=1}^{n-1}\frac{k^a k}{z^{k+1}}\sum_{j=0}^{\infty}\frac{\beta_j}{z^j}\,z^n\frac{dz}{z}$$

$$= -(n-1)\beta_n - t(n-1)\beta_{n-1} +$$

$$+\frac{1}{2\pi i}\int_{|z|=r}(z+t)\sum_{k=1}^{n-1}\frac{k\alpha_k}{z^{k+1}}\sum_{j=0}^{n-2}\frac{\beta_j}{z^j}\,\frac{g(t)}{g(z)}\,z^n\frac{dz}{z}$$

$$+\frac{g(t)}{2\pi i}\int_{|z|=r}\left(t\sum_{k=1}^{n-1}\frac{ka_k}{z^{k+1}} + \sum_{k=1}^{n-2}\frac{ka_{k+1}}{z^{k+1}} - \frac{a_n}{z^n}\right)\sum_{j=0}^{\infty}\frac{\beta_j}{z^j}\,z^n\frac{dz}{z}.$$

Hence we obtain

$$-(n-1)(a_{n+1}-t^2a_{n-1}) = -(n-1)\beta_n - t(n-1)\beta_{n-1}$$

(3.13)
$$+\frac{1}{2\pi i}\int_{|z|=r}(z+t)\sum_{k=1}^{n-1}\frac{k\alpha_k}{z^{k+1}}\sum_{j=0}^{n-2}\frac{\beta_j}{z^j}\,\frac{g(t)}{g(z)}\,z^n\frac{dz}{z}$$

$$+\frac{g(t)}{2\pi i}\int_{|z|=r}\left(t\sum_{k=1}^{n-1}\frac{ka_k}{z^{k+1}} + \sum_{k=2}^{n-1}\frac{(k-1)a_k}{z^k} - \frac{a_n}{z^n}\right)\sum_{j=0}^{\infty}\frac{\beta_j}{z^j}\,z^n\frac{dz}{z}$$

(c) We now estimate (3.13).

The lemma A with $\lambda_k = t^{-k}$ and (3.2) show that

(3.14)
$$\sum_{k=1}^{\infty} k|\alpha_k|^2 = \sum_{k=1}^{\infty} k\left|\sum_{\ell=1}^{\infty} b_{k\ell}t^{-\ell}\right|^2 \le \sum_{k=1}^{\infty}\frac{1}{k}r^{-2k} = \log\frac{1}{1-r^{-2}}$$

because $|t| = r > 1$.

(I) It follows from (3.14), lemma C and the fact that

$$\sum_{k=1}^{n}\frac{1}{k} > \log(n+1/2) + c, \text{ where } c = 0.57721\ldots$$

is Euler's constant, that

(3.15)
$$(n-1)|\beta_n| \le (n-1)e^{-c/2}(1-r^{-2})^{-1/2}(n+1/2)^{-1/2} < \;< \frac{3}{4}\sqrt{n-1}\;(1-r^{-2})^{-1/2}$$

because

$$e^{-c/2} \approx 0.7493083 < 0.75 = 3/4.$$

In the same ways we have

(3.16)
$$(n-1)|t|\,|\beta_{n-1}| \le (n-1)e^{-c/2}\,r(1-r^{-2})^{-1/2}\,(n-1/2)^{-1/2}.$$

(II) Let

$$K_1 = \frac{1}{2\pi i}\int_{|z|=r}\left[(z+t)\sum_{k=1}^{n-1}\frac{k\alpha_k}{z^{k+1}}\right]\left[\sum_{j=0}^{n-2}\frac{\beta_j}{z^j}\right]\frac{g(t)}{g(z)}\,z^n\frac{dz}{z}.$$

Let us determine a bound for K_1 by use of Schwarz's inequality and evaluate the resulting two integrals by Parseval's formula, noting that, on the circle $|z| = r$, $|g(t)/g(z)| \leq 1$ (by virtue of Eq. (3.9)). Thus putting $z = re^{i\theta}$, we have

$$|K_1| \leq r^n \left(\frac{1}{2\pi} \int_0^{2\pi} \left| (z+t) \sum_{k=1}^{n-1} \frac{k\alpha_k}{z^{k+1}} \right|^2 d\theta \times \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{j=0}^{n-2} \frac{\beta_j}{z^j} \right|^2 d\theta \right)^{1/2}$$

$$= r^n \left(2 \sum_{k=1}^{n-1} k^2 |\alpha_k|^2 r^{-2k} \sum_{j=0}^{n-2} |\beta_j|^2 r^{-2j} \right)^{1/2}$$

$$\leq r^{n-1} \sqrt{2} \left(\sum_{k=1}^{n-1} k^2 |\alpha_k|^2 \sum_{j=0}^{n-2} |\beta_j|^2 \right)^{1/2}$$

Now, using lemma B, we obtain

$$|K_1| \leq \sqrt{2}\, r^{n-1} \left(\sum_{k=1}^{n-1} k^2 |\alpha_k|^2 (n-1) \exp\left\{ \sum_{k=1}^{n-1} k |\alpha_k|^2 - \sum_{k=1}^{n-1} \frac{1}{k} + 1 - \frac{1}{n-1} \sum_{k=1}^{n-1} k^2 |\alpha_k|^2 \right\} \right)^{1/2}$$

If we apply the obvious inequality

$$x e^{1-x} \leq 1, \text{ with } x = \frac{1}{n-1} \sum_{k=1}^{n-1} k^2 |\alpha_k|^2,$$

we find that

$$K_1 \leq \sqrt{2}\, r^{n-1} (n-1)(1-r^{-2})^{-1/2} e^{-c/2} (n-1/2)^{-1/2} \tag{3.17}$$

$$\leq \frac{3}{4} \sqrt{2} \sqrt{n-1}\, r^{n-1} (1-r^{-2})^{-1/2}.$$

(III) Applying Schwarz's inequality and then Parseval's formula we see that last term, call it $K_2\, g(t)$, in (3.13) is bounded by

$$|K_2 g(t)| \leq |g(t)|\, r^n \left[\sum_{j=0}^{\infty} |\beta_j|^2 r^{-2j} \left(r^{-2} + \sum_{k=2}^{n-1} [k^2 + (k-1)^2] |a_k|^2 r^{-2k} + |a_n|^2 r^{-2n} \right) \right]^{1/2}$$

$$\leq \sqrt{2}\, |g(t)| \sqrt{n-1}\, r^n \left(\sum_{j=0}^{\infty} |\beta_j|^2 \sum_{k=1}^{\infty} k |a_k|^2 r^{-2k} \right)^{1/2} \tag{3.18}$$

Because $f(s)$ is univalent in $|s| < 1$ we have by the area theorem

$$\sum_{k=1}^{\infty} k |a_k|^2 r^{-2k} = \frac{1}{\pi} \iint_{|s| \leq \frac{1}{r}} |f'(s)|^2 d\Omega = \frac{1}{\pi} \text{area} \left\{ f(s) : |s| \leq \frac{1}{r} \right\} \tag{3.19}$$

$$\leq M(r)^2 = |g(t)|^{-2}$$

by (3.9) and (3.10). From (3.19) we obtain

$$\left(\sum_{k=1}^{\infty} k |a_k|^2 r^{-2k} \right)^{1/2} |g(t)| \leq 1. \tag{3.20}$$

Furthermore, by lemma D and (3.14)

$$\sum_{j=0}^{\infty} |\beta_j|^2 \leq \exp\left[\sum_{k=1}^{\infty} k |\alpha_k|^2 \right] \leq (1 - r^{-2})^{-1}. \tag{3.21}$$

Hence, by (3.18) (3.20 and (3.21) we obtain

$$|K_2 g(t)| \leq \sqrt{2} \sqrt{n-1}\, r^n (1 - r^{-2})^{-1/2}. \tag{3.22}$$

The estimates obtained in [(I), cf. (3.15) and (3.16)], [(II), cf. (3.17)] and [(III) cf. 3.22] for the terms on the right-hand side of (3.13) show that

$$(n-1) |a_{n+1} - t^2 a_{n-1}| \leq \frac{3}{4} \sqrt{n-1} (1 - r^{-2})^{-1/2} + (n-1) r |\beta_{n-1}| \tag{3.23}$$

$$+ \left(\frac{3}{4} \sqrt{2}\, r^{n-1} + \sqrt{2}\, r^n \right) \sqrt{n-1} (1 - r^{-2})^{-1/2}$$

(d) Since the assertion follows from $|a_2| \leq 2$ and $|a_3| \leq 3$ in the cases $n = 1$ and $n = 2$ respectively, we may assume that $n \geq 3$ and choose $r = \left\{ 1 - \frac{1}{n-1} \right\}^{-1/2}$. Since

$$r^{n-2} = \left\{ 1 - \frac{1}{n-1} \right\}^{-\frac{n-2}{2}} < e^{1/2}$$

and $(1 - r^{-2})^{-1/2} = \sqrt{n-1}$ it follows from (3.16) and (3.23) after multiplication by $(n-1)^{-1}\, r^{-2}$ that

$$\left| \left(1 - \frac{1}{n-1} \right) a_{n+1} - (r^{-2} t^2) a_{n-1} \right| \leq \frac{3}{2} r^{-2} + \frac{3}{4} \sqrt{2}\, r^{n-3} \tag{3.24}$$

$$+ \sqrt{2}\, r^{n-2} \leq \frac{3}{2} \left(1 - \frac{1}{n-1} \right) + \frac{3}{4} \sqrt{2}\, e^{1/2} \left(1 - \frac{1}{2(n-1)} \right) + \sqrt{2}\, e^{1/2} =$$

$$= \frac{6 + 7\sqrt{2e}}{4} - \frac{12 + 3\sqrt{2e}}{8(n-1)}.$$

Since $|t| = r$ we deduce from (3.24)

$$|a_{n+1}| - |a_{n-1}| > \left(1 - \frac{1}{n-1}\right)|a_{n+1}| - |a_{n-1}| \tag{3.25}$$

$$> -\frac{(6+7\sqrt{2e})}{4} + \frac{(12+3\sqrt{2e})}{8(n-1)} > -5.50 + \frac{(12+3\sqrt{2e})}{8(n-1)}$$

In the opposite direction we use that

$$|a_{n+1}| < 1.066\,(n+1)$$

and conclude from (3.24) that, for $n \geq 3$,

$$|a_{n+1}| - |a_{n-1}| \leq \left[1 - \left(1 - \frac{1}{n-1}\right)\right]|a_{n+1}| \tag{3.26}$$

$$+ \frac{(6+7\sqrt{2e})}{4} - \frac{(12+3\sqrt{2e})}{8(n-1)}$$

$$< 1.066\left(1 + \frac{2}{n-1}\right) + \frac{(6+7\sqrt{2e})}{4} - \frac{(12+3\sqrt{2e})}{8(n-1)}$$

$$< 6.60 + \frac{5.06 - 3\sqrt{2e}}{8(n-1)}.$$

Thus, from (3.25) and (3.26) we have proved theorem 1.

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