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Order of cyclicity of the singular point of Lineard's polynomial vector fields

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In this article, we give an estimate of the maximum number of limit cycles which appear from the singular point of

$$X_a = (f(x) - y)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{1}$$

where $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ and $f(x) = \sum_{i=1}^{d} a_i x^i$, under variations of the coefficients a_i. If $d = 2n+1$ or $d = 2n+2$, this number turns out to be equal do n.

We remark that it was conjectured in [2] that this is precisely the maximum number of limit cycles that X_a can have in the plane.

1. The origin is the only singularity of X_a. This singularity is an attractor if $a_1 < 0$ and a repellor if $a_1 > 0$. Therefore, if $a_1 \neq 0$, cycles can not appear from the origin under slight variation of a.

1.1 **Definition.** Let $a^* \in \mathbb{R}^d$. We shall say that the singularity of X_{a*} has cyclicity of order N_{a*} (N_{a*} integer ≥ 0) if:

a) it is possible to find numbers $\varepsilon_0 > 0$ and $\delta_0 > 0$, such that every X_a with a in the ε_0-neighborhood of a^* cannot have more than N_{a*} limit cycles within the δ_0-neighborhood of $0 \in \mathbb{R}^2$;

b) for any choice of positive numbers $\varepsilon > \varepsilon_0$ and $\delta < \delta_0$, there exists a in the ε-neighborhood of a^* such that X_a has N_{a*} limit cycles within the δ-neighborhood of $0 \in \mathbb{R}^2$.

We shall see that, for $a \in \mathbb{R}^d$, the number N_a only depends on the coefficients a_i with i odd. Therefore we shall split polynomials into their odd and even parts. Let $\mathbb{R}^d = \mathbb{R}^{n+1} \oplus \mathbb{R}^{s+1}$ with coordinates $(b_0, \ldots, b_n, c_0, \ldots, c_s)$, where $s = n-1$ if $d = 2n+1$ and $s = n$ if $d = 2n+2$.

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For $(b,c) \in \mathbb{R}^{n+1} \oplus \mathbb{R}^{s+1}$, we set

$$X_{(b,c)} = (g_b(x) + h_c(x) - y)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

where $g_b(x) = \sum_{i=0}^{n} b_i x^{2i+1}$ and $h_c(x) = \sum_{j=0}^{s} c_j x^{2j+2}$.

1.2 **Theorem.** *Fix $c \in \mathbb{R}^{s+1}$. Then $N_{(0,c)} = n$. If $b \in \mathbb{R}^{n+1}$ is not zero, then $N_{(b,c)} = m$, where b_m is the first nonzero coordinate of b.*

2. Consider a C^ω singularity of vector field $(R^2, 0, X)$ where

$$X = X_1\frac{\partial}{\partial x} + X_2\frac{\partial}{\partial y}.$$

Going over to polar coordinates $\psi : S^1 \times \mathbb{R} \to \mathbb{R}^2$, $\psi(\alpha, r) = (r\cos\alpha, r\,\mathrm{sen}\,\alpha)$, the blowing-up of X is the C^ω vector field

$$\overline{X} = \varphi_1\frac{\partial}{\partial\alpha} + r\varphi_2\frac{\partial}{\partial r}$$

where

$$\varphi_1(\alpha, r) = \frac{1}{r^2}[-r\,\mathrm{sen}\,\alpha\, X_1 + r\cos\alpha\, X_2]$$

$$\varphi_2(\alpha, r) = \frac{1}{r^2}[r\cos\alpha\, X_1 + r\,\mathrm{sen}\,\alpha\, X_2].$$

Suppose that $\varphi_1(\alpha, 0) = 1 - a_1\,\mathrm{sen}\,\alpha\cos\alpha \neq 0$ everywhere on $S^1 \times \{0\}$. Then $S^1 \times \{0\}$ is a cycle of $\overline{X}$ and we can consider the associated Poincaré mapping $P_{\overline{X}} : \Sigma \to \{0\} \times \mathbb{R}$, where Σ is a neighborhood of $r = 0$ in $\{0\} \times \mathbb{R}$.

In order to calculate the coefficients of the Taylor expansion of $P_{\overline{X}}$, we use the Poincaré's method explained in [3]. We associate to $\overline{X}$ the equation

$$\frac{dr}{d\alpha} = \frac{r\varphi_2}{\varphi_1} \tag{2}$$

which is defined in a neighborhood of $S^1 \times \{0\}$. The right hand side of (2) can be expanded in a series, arranged according to powers of r, $R_1 r + R_2 r^2 + \ldots$, where the R_k are periodic functions of α.

The solution $r' = r'(\alpha, r)$ of (2) satisfying the initial contidion $r'(0, r) = r$ can be represented as a series

$$r' = v_1(\alpha)r + v_2(\alpha)r^2 + \ldots \tag{3}$$

and the associated Poincaré mapping of $\overline{X}$ is

$$P_{\overline{X}}(r) = v_1(2\pi)r + v_2(2\pi)r^2 + \ldots.$$

By the initial condition imposed, we have $v_1(0) = 1$ and $v_k(0) = 0$ if $k \geq 2$. The coefficients v_k are determinated by the recursive system of differential equations:

$$\begin{cases} \dfrac{dv_1}{d\alpha} = v_1 R_1 \\ \dfrac{dv_2}{d\alpha} = v_2 R_1 + v_1^2 R_2 \\ \dfrac{dv_3}{d\alpha} = v_3 R_1 + 2v_1 v_2 R_2 + v_1^3 R^3. \end{cases} \tag{4}$$

So, in neighborhood of $S^1 \times \{0\}$, the cycles of $\overline{X}$ correspond with the zeros of the function

$$F(r) = \sum_{n=1}^{\infty} w_n r^n$$

where $w_1 = v_1(2\pi) - 1$ and $w_n = v_n(2\pi)$ for $n \geq 2$. If $\overline{X}$ depends analytically of a external parameter λ, then the coefficients w_n are also C^ω functions of λ.

Remark. Let $f : U \to S^1 \times \mathbb{R}$ be a C^ω diffeomorphism in a neighborhood of $S^1 \times \{0\}$ such that $f \mid S^1 \times \{0\} = id$, $f \mid (\{0\} \times \mathbb{R}) \cap U = id$ and set $\tilde{X} = f_*(\overline{X})$. It is clear that the associated Poincaré's mappings $P_{\overline{X}}$ and $P_{\tilde{X}}$ are equal. Therefore, in calculating the coefficients of the Taylor expansion of P, we can admit this type of coordinate change.

3. As in Bautin's work [1], the proof of the theorem depends essentially on the knowledge of all centers of the system (1).

3.1 **Proposition** [2]. *For every $c \in \mathbb{R}^{s+1}$, the singularity of $X_{(0,c)}$ is of center type.*

We shall see that the converse also holds.

3.2 **Proposition.** *For $1 \leq m \leq n$, let $H_m \subset \mathbb{R}^{n+1}$ be the subspace defined by $b_0 = 0, \ldots, b_{m-1} = 0$.*
Then

$$w_{2m+1} \mid H_m \oplus \mathbb{R}^{s+1} = b_m \int_0^{2\pi} \cos^{2m+2}\alpha \cdot d\alpha.$$

Proof. Fix $(b, c) \in H_m \oplus \mathbb{R}^{s+1}$, and write

$$X_{(b,c)} = X_{(0,c)} + Y_b \tag{6}$$

where $Y_b = g_b \cdot \frac{\partial}{\partial x}$. After blowing up, (6) becomes

$$\overline{X}_{(b,c)} = \overline{X}_{(0,c)} + \overline{Y}_b.$$

Since the singularity of $X_{(0,c)}$ is of center type, there exists a neighborhood U of $S^1 \times \{0\}$ and a C^ω diffeomorphism onto the image

$$\phi : U \to S^1 \times \mathbb{R}, \quad (\alpha, s) \mapsto (\alpha, s + s^2 \cdot \psi(\alpha, s))$$

with $\psi(0, s) \equiv 0$, such that

$$(\phi^{-1})_* \overline{X}_{(0,c)}(\alpha, s) = f(\alpha, s)\frac{\partial}{\partial\alpha}$$

with $f(\alpha, 0) \equiv 1$. Therefore

$$(\phi^{-1})_* \overline{X}_{(b,c)} = f\frac{\partial}{\partial\alpha} + (\phi^{-1})_* \overline{Y}_b.$$

Now, we shall investigate the form of $(\phi^{-1})_* \overline{Y}_b$. First we write

$$\overline{Y}_b(\alpha, r) = Q_1(\alpha, r)\frac{\partial}{\partial\alpha} + r \cdot Q_2(\alpha, r)\frac{\partial}{\partial r}$$

where

$$Q_1(\alpha, r) = -b_m r^{2m} \operatorname{sen}\alpha \cos^{2m+1}\alpha + r^{2m+1}\beta_1(\alpha, r)$$
$$Q_2(\alpha, r) = b_m r^{2m} \cos^{2(m+1)}\alpha + r^{2m+1}\beta_2(\alpha, r).$$

The diffeomorphism ϕ^{-1} must be of the form $\phi^{-1}(\alpha, r) = (\alpha, r + r^2\tilde{\psi}(\alpha, r))$, then

$$D\phi^{-1} = \begin{pmatrix} 1 & 0 \\ r^2\tilde{\psi}_\alpha & 1 + r\tilde{\psi}_r \end{pmatrix}$$

where $\tilde{\psi}_\alpha = \frac{\partial\tilde{\psi}}{\partial\alpha}$ and $\tilde{\psi}_r = 2\tilde{\psi} + r\frac{\partial\tilde{\psi}}{\partial r}$. It follows that

$$(\phi^{-1})_* \overline{Y}_b = (Q_1 \circ \phi)\frac{\partial}{\partial\alpha} + [r^2\tilde{\psi}_\alpha \cdot Q_1 + r(1 + r\tilde{\psi}_r)Q_2] \circ \phi\frac{\partial}{\partial s}$$

with

$$(Q_1 \circ \phi)(\alpha, s) = -b_m s^{2m} \operatorname{sen}\alpha \cdot \cos^{2m+1}\alpha + s^{2m+1}\tilde{\beta}_1(\alpha, s)$$

and

$$[r^2\tilde{\psi}_\alpha Q_1 + r(1 + r\tilde{\psi}_r)Q_2] \circ \phi(\alpha, s) = b_m s^{2m+1}\cos^{2m+2}\alpha + s^{2m+2}\tilde{\beta}_2(\alpha, s).$$

Finally, we can write

$$(\phi^{-1})_* \overline{X}_{(b,c)}(\alpha, s) = \tilde{f}(\alpha, s)\frac{\partial}{\partial\alpha} + (b_m s^{2m+1}\cos^{2m+2}\alpha + s^{2m+2}\beta_2(\alpha, s))\frac{\partial}{\partial s}$$

with $\tilde{f}(\alpha, 0) \equiv 1$. The associated equation (§2) of this vector field is

$$\frac{ds}{d\alpha} = b_m \cdot \cos^{2m+2}\alpha \cdot s^{2m+1} + R_{2m+2} \cdot s^{2m+2} + \dots$$

Since $v_1 \equiv 1$ and $R_1 = R_2 = \dots = R_{2m} \equiv 0$, we have

$$\frac{dv_{2m+1}}{d\alpha} = b_m \cdot \cos^{2m+2}\alpha$$

and

$$w_{2m+1}(b, c) = b_m \int_0^{2\pi} \cos^{2m+2}\alpha \, d\alpha.$$

This finishes the proof of the proposition.

3.3 **Corollary.** *The singularity of* $X_{(b,c)}$ *is of center type iff* $b = 0$.

Further, we know that $w_1((b, c)) = 0$ iff $b_0 = 0$ and $\frac{\partial w_1}{\partial b_0} \neq 0$ over $H_1 \oplus \mathbb{R}^{s+1}$. So we have

3.4 **Corollary.** *Let* $(b, c) \in H_m \oplus \mathbb{R}^{s+1}$, $1 \le m \le n$, *and* $I^m_{(b,c)}$ *be the ideal of germs at* (b, c) *of* C^ω *functions generated by the coordinates* b_i, $i = 0, \dots, m$. *Then* $I^m_{(b,c)}$ *is also generated by the functions* w_{2k+1}, $k = 0, \dots, m$.

§4. The proof of the theorem.

Let $(b, c) \in \mathbb{R}^{n+1} \oplus \mathbb{R}^{s+1}$. Choose $\delta > 0$ and a convex neighborhood V of $(\bar{b}, \bar{c})$ in $\mathbb{R}^{n+1} \oplus \mathbb{R}^{s+1}$ such that

$$F(r, (b, c)) = \sum_{k=1}^{\infty} w_k((b, c))r^k \tag{7}$$

is convergent in $(-\delta, \delta) \times V$. We suppose first that $\bar{b} = 0$. Then, the set

$$h(r, (b, c)) = \sum_{k=2n+2}^{\infty} w_k((b, c))r^k$$

is identically zero over $\{0\} \oplus \mathbb{R}^{s+1}$. By Corollary 3.4 we can write

$$h = r^{2n+2}\left(\sum_{k=0}^{n} h_k w_{2k+1}\right)$$

with the h_k, $k=0,\dots,n$, C^ω functions over $(-\delta,\delta)\times V$. Further, as $w_{2m}\equiv 0$ over $H_m\oplus R^{s+1}$ $(1\le m\le n)$, we have

$$w_{2m}=\sum_{k=0}^{m-1} g_{mk}\, w_{2k+1}$$

with the g_{mk} C^ω functions over V. Now, (7) becomes

$$F(r,(b,c))=r\left[\sum_{k=0}^{n} w_{2k+1}\,\psi_k r^{2k}\right]$$

where $\psi_k=1+r\tilde{\psi}_k$ and the $\tilde{\psi}_k$ are C^ω functions over $(-\delta,\delta)\times V$.

Reducing perhaps δ and V, we can suppose that

$$\psi_k\ge 1/2 \quad \text{on} \quad (-\delta,\delta)\times V \qquad (k=0,\dots,n)$$

and we represent F in the form

$$F=r\left[\sum_{k=0}^{n} w_{2k+1}\,\xi_k r^{2k}\right]$$

where $\xi_k=\psi_k/\psi_1$.

We must look for positive zeros of the function

$$F_0=\sum_{k=0}^{n} w_{2k+1}\,\xi_k\, r^{2k}.$$

Without loss we can suppose that

$$\xi_k\ge 1/2 \quad \text{on} \quad (-\delta,\delta)\times V \qquad (k=1,\dots,n).$$

We have

$$\frac{\partial F_0}{\partial r}=\sum_{k=1}^{n} w_{2k+1}\left(2k\,\xi_k\, r^{2k-1}+r^{2k}\frac{\partial \xi_k}{\partial r}\right)=$$

$$=r\left(\sum_{k=1}^{n} w_{2k+1}\,\eta_k r^{2(k-1)}\right)$$

where $\eta_k=2k\,\xi_k+r\dfrac{\partial\xi_k}{\partial r}$, $k=1,\dots,n$.

Now, on $(-\delta,\delta)$, the number of positive zeros of F_0 cannot exceed the number of positive zeros of the function

$$F_1=\sum_{k=1}^{n} w_k\,\eta_k\, r^{2(k-1)}$$

by more than unity. Continuing this process a further step we obtain a function F_2 such that the number of positive zeros of F_1 cannot exceed the number of positive zeros of F_2 by more than unity. So, the number of positive zeros of F_0 cannot exceed the number of positive zeros of F_2 by more than two.

This process stops at the n^{th} step when we obtain a function F_n which has no positive zeros. Therefore, for δ and V sufficiently small, F cannot have more than n positive zeros whitin $(0,\delta)$.

Now, we suppose $\overline{b}\ne 0$ and let m $(1\le m\le n)$ be such that $\overline{b}_m$ is the first non zero coordinate. Perhaps reducing δ and V, (7) becomes

$$F=r\left(\sum_{k=0}^{m-1} w_{2k+1}\,\psi_k\, r^{2k}+\alpha\, r^{2m}\right)$$

where $\psi_k=1+r\tilde{\psi}_k$ and α C^ω functions over $(-\delta,\delta)\times V$ and such that

$$\alpha(r,(b,c))\ne 0 \quad \text{if} \quad (r,(b,c))\in(-\delta,\delta)\times V$$

and it follows easily that the process stops at the m^{th} step. This proves that our estimate satisfies (a) of (1.1). The rest of the proof follows easily from properties of the functions w_{2k+1}, by playing adequately with the coefficients b_i.

References

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