On rigidity of isometric immersions with constant mean curvature

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Except when explicitly stated M and N will denote connected smooth Riemannian manifolds of dimensions n and $n+1$, respectively. Furthermore we will suppose that N is a complete simply-connected manifold of non-zero constant sectional curvature.

Let $x : M \to N$ be an isometric immersion. We will say that x is *rigid* if given any other isometric immersion $\bar{x} : M \to N$ there exists an isometry $T: N \to N$ such that $x = T \circ \bar{x}$.

The goal of this paper is to give a simple proof of the following result.

Theorem. – *Let $x : M \to N$ be an isometric immersion between connected smooth Riemannian manifolds of dimensions n and $n+1$ respectively. Assume that N is a complete simply-connected manifold with constant curvature $C \neq 0$. If $n \geq 3$ and x has non-zero constant mean curvature then x is rigid.*

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Remark 1. This theorem was first proved by Matsuyama in [5]. But he does not corretly apply Lemma 2.9 of [4]. However when this paper was ready he informed me how it is possible to correct his paper. On the other hand, the approach here is independent and simpler then Matusyama's.

Remark 2. This paper is also a correction of the proof that we presented in [2].

We will now describe some examples to show that the conditions of the theorem are actually necessary.

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Remark 3. Let $S^n=\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1};\ x_1^2+\dots+x_{n+1}^2=1\}$ be the Euclidean sphere of dimension n. Let $\alpha_H,\ \beta_H:\mathbb{R}\to S^3$ be curves given by

$$\alpha_H(t)=(b_1\cos a_1t,\ b_1 \text{ sen } a_1t,\ b_2\cos a_2t,\ b_2 \text{ sen } a_2t),$$

and

$$\beta_H(t)=(-b_1 \text{ sen } a_1t,\ b_1\cos a_1t,\ -b_2 \text{ sen } a_2t,\ b_2\cos a_2t),$$

where $a_1=\dfrac{H+\sqrt{H^2+4}}{2}$, $a_2=\dfrac{H-\sqrt{H^2+4}}{2}$, $b_1^2=\dfrac{\sqrt{H^2+4}-H}{2\sqrt{H^2+4}}$ and $b_2^2=\dfrac{\sqrt{H^2+4}+H}{2\sqrt{H^2+4}}$. Then $x_H:R^2\to S^3$ defined by

$$x_H(u,t)=\cos u\,\alpha_H(t)+\text{sen } u\,\beta_H(t)$$

is an isometric immersion from $\mathbb{R}^2$ to S^3 with constant mean curvature H. By varing H we obtain a family of isometric immersion from $\mathbb{R}^2$ to S^3 which shows that the theorem is not true when M has dimension two and the ambient space is the Euclidean sphere.

In what follows we consider $S^3\subseteq S^4$ as the set $\{(x_1,x_2,x_3,x_4,x_5)\in S^4;\ x_5=0\}$. Let $y_H:\mathbb{R}^2\times(-\varepsilon,\varepsilon)\to S^4$, $\varepsilon<\pi/2$, be an immersion given by

$$y_H(u,t,v)=exp_{x_H}(u,t)v\,e=\cos v\,x_H(u,t)+\text{sen } v\,e,$$

where *exp* is the exponential map of S^4 and $e=(0,0,0,0,1)$. By taking $\mathbb{R}^2\times(-\varepsilon,\varepsilon)$ with the induced metric we obtain an isometric immersion y_H with mean curvature $\tilde{H}=\dfrac{H}{\cos v}$. Since the induced metric does not depend on H, we have a counter-example for the theorem when the mean curvature is not constant or zero and the ambient space is the Euclidean sphere.

Remark 4. Let $H^n=\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1};\ -x_1^2+\dots+x_{n+1}^2=-1\}$ be the Hyperbolic space. Let $\mathbb{R}^2$ be the plane with the metrics $I=du^2+dv^2$ and $I_1=du^2+(\cosh^2v+\text{senh}^2v)dv^2$, respectively. Then

$x:(\mathbb{R}^2,I)\to H^3$

$$(u,v)\mapsto\left(\sqrt{1+r^2}\cosh\frac{u}{\sqrt{1+r^2}},\ \sqrt{1+r^2}\,\text{senh}\frac{u}{\sqrt{1+r^2}},\ r\cos\frac{v}{r},\ r\,\text{sen}\frac{v}{r}\right),$$

and

$y:(\mathbb{R}^2,I_1)\to H^3$

$$(u,v)\mapsto(\cosh u\cosh v,\ \text{senh}u\cosh v,\ \cos u\,\text{senh}v,\ \text{sen}u\,\text{senh}v)$$

are isometric immersions with constant mean curvature $H=\dfrac{1+2r^2}{r\sqrt{1+r^2}}$ and zero respectively. From R. Tribuzzi [8] we have that there exist non trivial deformations of x and y respectively. And so the theorem is not true when M has dimension two and the ambient manifold is the hyperbolic space. If the ambient manifold N is the hyperbolic space and has dimension 4, we can use the above non trivial deformations of x and y, respectively, and an analogous construction to the one we have made in Remark 3, to obtain a counter-example of the theorem, in the case that the mean curvature is not constant or zero.

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1. Notation and a sketch of the proof of the theorem.

We will denote ∇ and $\overline{\nabla}$ the covariant derivatives of M and N, respectively. Let $x: M\to N$ be an isometric immersion. We will identify, for each $p\in M$, T_pM with $dx_p(T_pM)$ and we will write

$$\overline{\nabla}_XY=\nabla_XY+\text{II}(X,Y)\xi,$$

where X, Y are tangent fields to M and ξ is a normal field to M. It is well known that, for every $p\in M$, II induces a symmetric bilinear form II_p on T_pM. This form is called the *second fundamental form* of x and its trace is known as the *mean curvature H* of the isometric immersion x.

Sketch of the proof of the theorem: Let π_p be the kernel of the second fundamental form II_p, that is,

$$\pi_p=\{v\in T_pM;\ \text{II}_p(v,w)=0 \text{ for all } w\in T_pM\}.$$

Assume that $\nu=\dim\pi_p$ is constant and greater than zero on an open set $U\subseteq M$. It is well known that through every $p\in U$ there passes a totally geodesic submanifold $M_p\subseteq M$ such that $x(M_p)$ is also a totally geodesic submanifold of N. Let $\gamma:(-\varepsilon,\varepsilon)\to M_p$ be a geodesic with $\gamma(0)=q\in M_p$. Let $N_\gamma M_p$ be the normal bundle of M_p along γ defined by

$$N_\gamma M_p=\{v\in T_{\gamma(t)}M,\ t\in(-\varepsilon,\varepsilon),\ \text{and } \langle v,w\rangle=0, \text{ for all } w\in T_{\gamma(t)}M_p\}.$$

Let $\perp$ denote the orthogonal projection on $N_\gamma M_p$; define

$$A: N_\gamma M_p\to N_\gamma M_p$$
$$X\mapsto(\nabla_XY)^\perp, \tag{1.1}$$

and

$$S: N_\gamma M_p \to N_\gamma M_p$$

$$X \mapsto (\overline{\nabla}_X \xi)^\perp, \tag{2.2}$$

where Y is an extension of γ' and ξ is a normal field to M. It can be shown (see Lemma 1) that

$$\frac{d}{dt}(\operatorname{tr} A) = -\operatorname{tr} A^2 - (n-\nu)C \tag{1.3}$$

$$\frac{d}{dt}(\det A) = -\operatorname{tr} A(\det A + C), \tag{1.4}$$

$$\frac{dH}{dt} = -\operatorname{tr} S\,A, \tag{1.5}$$

and that

$$\frac{d^2H}{dt^2} = C\,H + 2\,tr\,S\,A^2, \tag{1.6}$$

where C is the constant sectional curvature of N.

Now we assume that there is an open set $U \subseteq M$ where $\nu = \dim \pi_p$ is constant and greater than $n-3$. We then have three possibilities: i) $n = \nu$, ii) $n = \nu + 1$, and (iii) $n = \nu + 2$. i) is not possible because $H \neq 0$ and from (1.3), (1.5), $H \neq 0$ and $C \neq 0$, it follows that ii) is not possible either. From a linear algebra argument and (1.3) we can show that $\det A = \frac{C}{2}$ and thus $\operatorname{tr} A = 0$. In the case $C \neq 0$, the computation of the eigenvalues of A leads us to a contradiction of (iii) with both (1.3) and $\operatorname{tr} A = 0$. Then from the classical Beez's theorem [1] we can conclude that x is rigid.

2. Auxiliary lemmas

We begin by mentioning some facts on the index of the relative nullity which we will need in the proof of the auxiliary lemmas. Let $x: M \to N$ be an isometric immersion. A vector $v \in T_pM$ is called a relative nullity vector for x at p if $\mathrm{II}_p(v, w) = 0$ for all $w \in T_pM$, and the space π_p of the nullity relative vectors is known as the *nullity relative space* at p. The dimension of the nullity relative space π_p is called the *index of the relative nullity* of x at p. The proof of the following proposition can be found in [3].

Proposition. *Let $U \subseteq M$ be an open subset on which the index of the relative nullity of x is constant. Then the distribution π of the relative nullity spaces is integrable and its integral submanifolds are totally geodesic. Furthermore, if $\gamma: (-\varepsilon, \varepsilon) \to M$ is a geodesic on the integral submanifold $M_{\gamma(0)}$ then π is parallel along γ.*

In what follows $U \subseteq M$ is an open set of M, where the index of the relative nullity of x is a constant $\nu > 0$. Let M_p be an integral submanifold of the distribution π that passes through $p \in U$. Let $\gamma: (-\varepsilon, \varepsilon) \to M_p$ be a geodesic with $\gamma(0) = p \in M_p$. Let $N_\gamma M_p$ be defined by

$$N_\gamma M_p = \{v \in T_{\gamma(t)}M: t \in (-\varepsilon, \varepsilon),\ \langle v, w\rangle = 0 \text{ for all } w \in T_{\gamma(t)}M_p\}.$$

Then we can define A and S by (1.1) and (1.2) respectively. It is easy to see that A and S are operators on $N_\gamma M_p$, and if H denotes the mean curvature along γ then $H = tr\,S$.

Lemma 1. *Let $x: M \to N$ be an isometric immersion. Let γ, A, S, H be as defined above. Then we have*

i) $\frac{d}{dt} tr\,A = -tr\,A^2 - (n-\nu)C$,

ii) $\frac{dH}{dt} = -tr\,S\,A$,

iii) $\frac{d^2H}{dt^2} = C\,H + 2tr\,S\,A^2$,

iv) $\frac{d}{dt}(det\,A) = -tr\,A(det\,A + C)$, *if* $n - \nu = 2$.

Proof. i) Let X be a vector field in $N_\gamma M_p$ and Y an extension of γ' which is tangent to the integral submanifolds of π. Then

$$(\nabla_Y A)X = \nabla_Y AX - A((\nabla_Y X)) = (\nabla_Y \nabla_X Y - \nabla_{\nabla_Y X} Y)^\perp,$$

where $\perp$ means the orthogonal projection on $N_\gamma M_p$. By using the curvature tensor R of M, we have

$$(\nabla_Y A)X = (R(Y, X)Y + \nabla_X \nabla_Y Y + \nabla_{[Y,X]} Y - \nabla_{\nabla_Y X} Y)^\perp.$$

Since the integral submanifold M_p is a totally geodesic submanifold of N, we have

$$R(Y, X)Y = -CX,$$

where C is the sectional curvature of N. On the other hand if Z is a vector field in $N_\gamma M_p$, we have

$$\langle \nabla_X \nabla_Y Y, Z\rangle = X\langle \nabla_Y Y, Z\rangle - \langle \nabla_Y Y, \nabla_X Z\rangle.$$

Since Y is a extension which is tangent to the integral submanifolds of π and $Y(\gamma(t)) = \gamma'(t)$ for all $t \in (-\varepsilon, \varepsilon)$, we get

$$\langle \nabla_X \nabla_Y Y, Z \rangle = 0$$

along γ. And so

$$(\nabla_Y A)X = -A^2 X - CX. \tag{2.1}$$

Therefore, by taking the trace of $\nabla_Y A$, we obtain i).

ii) Let ξ be a normal field to M. Let X and Y be as in the proof of i). Then we have

$$(\nabla_Y S)X = \nabla_Y SX - S((\nabla_Y X)^\perp) = (\overline{\nabla}_Y \overline{\nabla}_X \xi - \overline{\nabla}_{\overline{\nabla}_Y X} \xi)^\perp,$$

where $\perp$ means the orthogonal projection on $N_\gamma M_p$. By using the curvature tensor $\overline{R}$ of N, it follows that

$$(\nabla_Y S)X = (\overline{R}(Y, X)\xi + \overline{\nabla}_X \overline{\nabla}_Y \xi + \overline{\nabla}_{[Y,X]}\xi - \overline{\nabla}_{\nabla_Y X}\xi)^\perp.$$

Since $\overline{R}(Y, X)\xi = 0$ and Y is tangent to the integral submanifolds of π, we get

$$(\nabla_Y S)X = -(\overline{\nabla}_{\nabla_X Y}\xi)^\perp = -(S\,A)X.$$

Then, by taking the trace of $\nabla_Y S$, we obtain ii).

iii) From ii) we have

$$\frac{d^2 H}{dt^2} = \frac{d}{dt} tr\, SA = -tr\left(\frac{dS}{dt} A + S \frac{dA}{dt}\right).$$

Then we use (2.1) to get

$$\frac{d^2 H}{dt^2} = 2\, tr(S A^2) + C\, trS = 2\, tr(S A^2) + C\, H.$$

iv) Let $\{e_1, e_2\}$ be an orthonormal basis of $N_\gamma M_p$ at $\gamma(t_0)$. From (2.1) we obtain

$$\frac{d}{dt} det\, A\,\big|_{t=t_0} = -det\begin{pmatrix}\langle A^2 e_1, e_1\rangle, & \langle A e_2, e_1\rangle\\ \langle A^2 e_1, e_2\rangle, & \langle A e_2, e_2\rangle\end{pmatrix}_{t=t_0} -$$

$$-det\begin{pmatrix}\langle A e_1, e_1\rangle, & \langle A^2 e_2, e_1\rangle\\ \langle A e_1, e_2\rangle, & \langle A^2 e_2, e_2\rangle\end{pmatrix}_{t=t_0} - C\, tr\, A(t_0).$$

At $t = t_0$ we can write

$$A^2 e_1 = \langle A e_1, e_1\rangle A e_1 + \langle A e_1, e_2\rangle A e_2,$$

and

$$A^2 e_2 = \langle A e_2, e_1\rangle A e_1 + \langle A e_2, e_2\rangle A e_2.$$

Therefore

$$\frac{d}{dt} det\, A\,\big|_{t=t_0} = -\langle A e_1, e_2\rangle\big|_{t=t_0} det\, A(t_0) - \langle A e_2, e_2\rangle\big|_{t=t_0} det\, A(t_0)$$

$$-\, c\, tr\, A(t_0) = -\, tr\, A(t_0)\, (det\, A(t_0) + C).$$

Since t_0 is arbitrary we have that

$$\frac{d}{dt} det\, A = -\, tr\, A(det\, A + C).$$

The proof of the following lemma is a simple computation with matrices.

Lemma 2. *Let B and C be 2×2 matrices over $\mathbb{R}$. If B is symmetric and $tr\ BC = 0$, then,*

$$tr\ BC^2 = -\, tr\, B\, det\, C.$$

3. Proof of the theorem.

We will first show that the subset $U \subseteq M$ where dim $\pi_p \geq n-2$ has no interior points. In fact if U has an interior point then there is an open set $V \subseteq U$ where one of three prossibilities takes place:

i) $x(V)$ is totally geodesic
ii) dim $\pi_p = n-1$ for all $p \in V$,
iii) dim $\pi_p = n-2$ for all $p \in V$.

The first case i) does not occur because $H \neq 0$. Suppose that ii) occurs. From Lemma 1-ii) and the fact that H is a non-zero constant we obtain that $tr\ A = 0$. We then use the Lemma 1-i) to show that $tr\ A = 0$ contradicts the hypothesis on sectional curvature of the ambient space. In what follows we will show that iii) is not possible. In fact, from Lemma 1-iii) and the fact that H is a constant, we have

$$tr\, SA^2 = \frac{C}{2} H.$$

By using Lemma 2, we obtain

$$H\ det\, A = tr\, S\, det\, A = \frac{C}{2} H.$$

Since $H \neq 0$, it follows that

(3.1) $$det\ A = \frac{C}{2}.$$

But from Lemma 1-iv) we have that

(3.2) $$tr\ A = 0.$$

Now we use (3.1) and (3.2) to conclude that the eigenvalues of A at $p \in V$ are $\pm \sqrt{-\frac{C}{2}}$ if $C < 0$ and $\pm \sqrt{\frac{C}{2}}$ if $C > 0$. But from Lemma 1-i) and (3.2) we obtain that $tr\ A^2 = -2C$ at $p \in V$. Therefore iii) cannot occur, that is, U has no interior points.

Now we use the following result (cf. [6] vol V, pg. 244) and an argument of continuity to conclude the proof of the theorem.

Let $x: M \to N$ be an isometric immersion. If the dimension of the complement of π_p in T_pM is greater then two for every $p \in M$, then x is rigid.

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