Uniqueness in the Cauchy Problem for first order operator with C^k coefficients

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1. Introduction.

The main purpose of this paper is to extend certain known results on uniqueness on the Cauchy problem for first order linear differential operators with smooth coefficients to the case of C^k coefficients. We shall consider an operator

$$L = \sum_{j=1}^{N} c^j(x) \frac{\partial}{\partial x_j} + c^0(x) \tag{1.1}$$

whose coefficients $c^j(x)$, $0 \leq j \leq N$, are complex valued C^k functions in an open subset Ω of $\mathbb{R}^N$, $N \geq 2$. We denote by

$$L_0 = \sum_{j=1}^{N} c^j \frac{\partial}{\partial x_j}$$

the principal part of L and assume that

(1.2) $\qquad L_0$ *does not vanish in* Ω

Let Σ be a C^2 hypersurface in Ω, $p \in \Sigma$ and consider a regular real valued C^2 function ϕ, such that in a neighborhood U of p, $U \cap \Sigma = \{\phi(x) = \phi(p)\}$. We shall assume that Σ is non-characteristic, i.e.

$$L_0(\phi) \neq 0 \quad \text{in} \quad \Sigma \cap U \tag{1.3}$$

We recall that L has property (P) if: *for all $x_0 \in \Omega$ and every complex number z such that $Re(zL_0)(x_0) \neq 0$, there is a neighborhood U of x_0 such that $Im(zL_0)$ does not change direction along any characteristic curve of $Re(zL_0)$ contained in U.*

In this article we prove

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Theorem 1.1. *Assume that L is given by (1.1) and satisfies (1.2) and (P) and that Σ is given by (1.1) $\phi(x)=0$ and satisfies (1.3). Assume that $k \geq 7$ and consider a non-negative integer ℓ such that $k \geq 2\ell+5$. Then every point p of Σ has a neighborhood U such that for all $u \in D'_\ell(U)$ such that*

$$\text{supp } u \subseteq \{\phi(x) \geq 0\} \quad \text{and} \quad Lu=0 \quad \text{in} \quad U$$

it follows that $u=0$ in U.

Here $D'_\ell(U)$ denotes the space of distributions of order ℓ, i.e. the dual of $C^\ell_c(U)$ with its usual topology. When the coefficients of L are smooth there is no restriction in the size of ℓ. Since, locally, every distribution has finite order, Theorem 1.1 generalizes the results of [2].

As in [2], the proof of Theorem 1.1 is divided into two parts described as "outside" and "inside the critical set". No new ideas are involved in the proof of the former so we omit details. On the other hand, the second part in [2] used the Weierstrass-Malgrange preparation theorem and an analogous treatment in the C^k case would yield a weaker result due to the loss of regularity in the coefficients of the Weierstrass polynomials ([1], [5] and [6]).

Instead, we use a different method based on the existence of ρ-flat solutions (see appendix for precise statements). It is used to prove uniqueness in the same way as Cauchy-Kovalevska's theorem is used to prove Holgrem's theorem. This unifies the techniques inside and outside the critical set, randering ρ-flat solutions as the main tool to treat uniqueness in the Cauchy problem once it is known for C^1 solutions.

In the C^∞ case, this approach simplifies the proof and sets it in a more "classical" framework.

2. Inside the critical set.

In this section L is given by (1.1) and (1.2) but may not verify (P). Let's write $X=ReL_0$, $Y=ImL_0$. Denote by U^* the subset of U where X and Y are linearly dependent, whenever $U \subseteq \Omega$ is open. *For p, q in U^* we write*

$$p \sim q \quad \text{in} \quad U^*$$

if and only if there exists a curve $\gamma:[a,b] \to U$ of class C^1 such that $\gamma(a)=p$, $\gamma(b)=q$ and X and Y are parallel to $\gamma'(s)$ along $\gamma(s)$. The equivalence classes will be denoted $[p]/U$. These equivalence classes are endowed with a natural topology which makes the projection $U \to U/\!\sim$ continuous. Each equivalence class is homeomorphic to one of the following:

i) $\mathbb{R}$;
ii) $\mathbb{R}^+ = [0,\infty)$;
iii) $[0,1]$;
iv) $S^1 = \{z \in \mathbb{C}, |z|=1\}$;
v) $\{0\}$.

This partition of U^* determined by L_0 is invariant under multiplication by non-vanishing complex factors of class C^k. On this subject we refer to [4].

Let p belong to a non-characteristic surface $\Sigma \subset \Omega$. Consider a connected neighborhood U of p which is divided by Σ into two components U^+ and U^- and denote by $\Gamma^+(U)$ the union of all the equivalence classes $[q]/U$, $q \in U$, such that either

a) *$[q]/U$ is homeomorphic to $\mathbb{R}$*

or

b) *$[q]/U$ is homeomorphic to $\mathbb{R}^+$ and whenever $[q]/U \cap \overline{U}^+ \neq \varnothing$, $[q]/U \cap \overline{U}^-$ is compact.*

Theorem 2.1. *Let L be given by (1.1) and satisfying (1.2), let Σ be a non--characteristic hypersurface of class C^2 and let ℓ be a non-negative integer. If $k \geq 2\ell+3$ every point $p \in \Sigma$ has a neighborhood U such that for every $u \in D'_\ell(U)$, supp $u \subseteq \Gamma^+(U) \backslash U^-$ and $Lu=0$ implies $u=0$.*

Proof. Since we may replace Σ by another hypersurface of class C^∞ which passes through p and leaves supp u on one side, there is no restriction in assuming that Σ is smooth. By appropriate choice of the coordinates and division by a non-vanishing factor of class C^k we may assume that

$$L = \frac{\partial}{\partial t} + i \sum_{j=1}^{n} b^j(x,t) \frac{\partial}{\partial x^j} + c(x,t) \quad (x,t) \in U,$$

$$U = \{(x,t), |x|<1, |t|<1\}$$

$$U^+ = \{(x,t) \in U, \ t>0\}$$

$$\Sigma = \{(x,0), |x|<1\},$$

where the coefficients of L are of class C^k, the functions $b^j(x,t)$ are real and $n+1=N$. Set

$$r(x) = \sup\left\{t \geq 0, \sum_{j=1}^{n} (b^j(x,t))^2 > 0\right\}$$

(if for a given x all the functions $b^j(x, t)$ vanish for all $0 \le t < 1$, then $r(x) = 0$). The function $r(x)$ is lower semicontinuous and

$$\Gamma^+(U) \backslash U^- = \{(x, t) \in U,\ t \ge r(x)\}.$$

Let's reason by contradiction and assume that there is a point $q = (x_0, t_0)$ in supp u. We may find a smooth real function $g(x)$, $|x| < 1$, such that $g(x_0) > t_0$ and

(2.1) $\{(x, t) \in \text{supp } u,\ g(x) \ge t\}$ is compact

Consider the change of variables $s = t - g(x), y = x$. The expression of L in the new variables is $L = h(y)Q$, with

$$Q = \frac{\partial}{\partial s} + ih^{-1}(y) \sum_{j=1}^{n} \tilde{b}_j \frac{\partial}{\partial y^j} + \tilde{c},$$

where

$$\tilde{b}^j(y, s) = b^j(y, s + g(y)),\ h(y) = 1 - i \sum_{j=1}^{n} \frac{\partial g}{\partial y^j} \neq 0$$

and the expression of $\tilde{c}$ is irrelevant.

By theorem A.1, the backwards Cauchy problem

(2.2)
$$\begin{cases} Q\phi = \psi, & s \le 0, \\ \phi|_{s=0} = 0 \end{cases}$$

has a $\ell + 1 - \rho -$flat solution of class $C^{\ell+1}$ where

$$\rho(y, s) = \left| \int_0^s (\Sigma \tilde{b}^j(y, \bar{s})^2)^{1/2} h^{-1}(y) \, |d\bar{s}| \right| = \left| h^{-1}(y) \int_{g(y)}^{g(y)+s} (\Sigma (b^j(y, t))^2)^{1/2} dt \right|$$

Note that the transpose ${}^tQ$ of Q and $-Q$ have the same principal part. If we take ψ smooth and supp $\psi \subseteq \{s < 0\}$, it follows recursively from (2.2) that all derivatives of ϕ up to order $\ell + 1$ vanish for $s = 0$. Thus we may extend ϕ by zero for $s > 0$. the extension is a $C^{\ell+1}$ function that we keep calling ϕ. It follows from (2.1) that supp $u \cap \{s \le 0\}$ is compact so

(2.3) $$\langle u, {}^tQ\phi \rangle = \langle Qu, \phi \rangle = \langle 0, \phi \rangle = 0$$

On the other hand, $\rho(y, s)$ vanishes on supp $u \cap \{s < 0\}$. Since supp $u \subseteq \Gamma^+(U)$, thus, ${}^tQ\phi - \psi$ vanishes of order $\ell + 1$ on supp u and in particular we have

(2.4) $$\langle u, {}^tQ\phi \rangle = \langle u, \psi \rangle$$

Combining (2.3) and (2.4) we conclude that u vanishes for $s < 0$. This contradicts the fact that $q = (y_0, s_0) = (x_0, t_0 - g(x_0))$, $s_0 < 0$, belongs to supp u.

Remarks. Since Theorem 2.1 is valid for a general first order operator of principal type, we believe that it may prove useful in other situations. Heuristically, this theorem says that in the study of uniqueness in the Cauchy problem for such an operator, one needs only care about the orbits of L_0 of dimensions ≥ 2, for once it is known that the support of the solution u is contained in the union of the one-dimensional orbits, u must vanish.

It is natural to ask wether the condition $k \ge 2\ell + 3$ may be relaxed to $k \ge \ell + 1$. This is an open question. An analogous comment can be made on Theorem 1.1.

3. Proof of Theorem 1.1.

As in the proof of theorem 2.1 we may assume that

$$L = \frac{\partial}{\partial t} + i \sum_{j=1}^{n} b^j(x, t) \frac{\partial}{\partial x_j} + c(x, t),\ (x, t) \in U$$

$$U = \{(x, t),\ |x| < 1,\ |t| < 1\}$$

and define

$$r(x) = \sup_{0 \le t \le 1} \left\{ t : \sum_{s=1}^{n} (b^j(x, t))^2 > 0 \right\}.$$

Set $V = \{(x, t) \in U, t < r(x)\}$. If we show that supp $u \cap V = \emptyset$ we will be able to apply theorem 2.1 to conclude that $u \equiv 0$ in U. Assume by contradiction that there is a point $q = (x_0, t_0)$ in supp $u \cap V$. Using condition (P) we may find a local diffeomorphism $y(x)$ in a neighborhood of x_0, such that $y(x_0) = 0$ and the expression in the coordinates (y, t) is

$$L = \frac{\partial}{\partial t} - ib(y, t) \frac{\partial}{\partial y^1} + \tilde{c}(y, t),\ |t| < 1, |y| < \delta$$

and furthermore, $b(y, t) \ge 0$, $|t| < 1$, $|y| < \delta$. It follows from the choice of q and the definition of V that there exist $t_1 > t_0$ such that $b(0, t_1) > 0$ (the $(y, t) -$ coordinates of q are $(0, t_0)$). Since u vanishes for $t < 0$, we may modify arbitrarily the coefficients of L for, say, $t < -1/2$ and we may assume that $b(y, t) > 0$ for $t < -1/2$. Consider a smooth function $\beta(y')$,

$y' = (y^2, \ldots, y^n)$, with compact support contained in $\{|y'| < 1\}$ and set $v = \langle u, \beta\rangle$. Then $v \in D'_\ell\{(y^1, t), |y^1| < 1, |y| < 1\}$ and satisfies

$$\frac{\partial v}{\partial t} + ib(y,t)\frac{\partial v}{\partial y^1} + \bar{c}(y,t)v = 0 \quad |t| < 1, |y| < \delta$$

where $y^2, \ldots, y^n$ are kept fixed. By Proposition 3.1 v is C^1 in a neighborhood of $y^1 = 0, -1 < t < t_1$. Then the results of [7] apply to prove that v vanishes in a neighborhood of $y^1 = 0, t = t_0$. Since β is arbitrary we conclude that u vanishes in a neighborhood of q. This is a contradiciction.

In the next proposition we consider an operator in two variables (y^1, t)

$$L = \frac{\partial}{\partial t} + ib(y^1, t)\frac{\partial}{\partial y^1} + c(y^1, t) \tag{3.1}$$

defined in an open subset Ω of $\mathbb{R}^2$, with b real and b, c of class C^k.

Proposition 3.1. *Let L be given by* (3.1) *and assume that it satisfies* (*P*). *If for every x, each component of $\{t; (x,t) \in \Omega, b(x,t) = 0\}$ is compact and $k \geq 2\ell + 5$, then, if $u \in D'_\ell(\Omega)$ and $Lu = 0$, it follows that $u \in C^1(\Omega)$.*

The proof is a straightforward modification of the methods of §3 of [3] and is left to the reader. The main step of the proof is the construction of a parametrix K so that $KL = \text{Identify} + R$ and the kernel distribution of R is given by a C^ℓ function. Essentialy, K is a Fourier integral operator with complex phase. Both amplitude and phase functions are taken to be $\ell + 1 - \rho -$ flat solutions with $\ell + 1$ continuous derivatives of suitable equations.

A. Appendix.

We shall consider approximate solutions of the Cauchy problem

$$u_t = \sum_{i=1}^{n} a_i(x,t)\frac{\partial u}{\partial x_i} + b(x,t)u + c(x,t) = f(x,t,u) \tag{A.1}$$

$$u(x,t') = u_0(x)$$

Here the functions a_i, $i = 1, \ldots, n$, b and c, are complex valued, with m continuous derivatives. They are defined for (x,t) in $\Omega \times (-T,T)$ where $\Omega \subseteq \mathbb{R}^n$ is open and $T > 0$. We shall consider the function

$$\rho(x,t,t') = \left|\int_{t'}^{t} (\Sigma|a_i(x,s)|^2)^{1/2} ds\right|$$

Definition A.1. *A function $u(x,t,t')$ of class C^p, $p \geq 1$ is said to be a global ℓ-ρ-flat solution of* (A.1) *if*

i) $u(x,t,t')$ is defined for all x in Ω, t, t' in $(-T,T)$;
ii) $u(x,t,t) = u_0(x,t)$ for all x in Ω, $-T < t < T$;
iii) for all $r + s + |\alpha| \leq \min(p-1, \ell)$ the function

$$[\rho(x,t,t')]^{r+s+|\alpha|-\ell}(D_t^r D_{t'}^s, D_x^\alpha(u_t - f(x,t,u)))$$

is continuous.

Theorem A.1. *If $m \geq p + \ell + 1$ there exist global ℓ-ρ-flat solutions of* (A.1) *of class C^p.*

The proof of Theorem A.1 follows from straightforward modifications of the results of [3].

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