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Second order elliptic operators with non smooth characteristics and the uniqueness of the Cauchy Problem

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The uniqueness of the Cauchy problem for elliptic operators has been very much investigated. See [3], [5], [9], [11] ... In all these works the complex characteristics are supposed to be of constant multiplicity or smooth functions. On the other hand A. Pliš [6] has constructed an elliptic fourth-order operator P in three variables with analytic coefficients (and non smooth characteristic roots) and a function $a \in C^{\infty}(\mathbb{R}^3)$ such that $P + a$ fails to have the uniqueness property (see [1] for more general non uniqueness theorems). However very few works are devoted to the case of non smooth roots (see [4], [10]). We consider here second order elliptic equations in two variables and we prove a result which implies that every such operator with analytic principal part has the uniqueness property for every bounded lower order terms. Our result improves those of [10] by a different proof based on Carleman estimates.

More precisely let us consider in a neighborhood V of the origin in $\mathbb{R}^2$, the *elliptic* differential operator

(1) $$P = D_t^2 + 2bD_xD_t + cD_x^2 + \alpha D_x + \beta D_t + \gamma$$

and let us put

(2) $$\Delta(x, t) = (b^2 - c)(x, t).$$

We can state the

Theorem 1. *Let P be defined by (1) where $b, c \in C^{\infty}(V)$ and $\alpha, \beta, \gamma \in L^{\infty}(V)$. Let us suppose that the function $t \mapsto \Delta(0, t)$ has, for $t = 0$, a zero of finite order. Then there exists a neighborhood W of the origin in which every $u \in C^{\infty}(V)$, such that $Pu = 0$ in V and $u_{|t<0} = 0$, vanishes.*

Corollary 2. *Let P be a second order elliptic differential operator, in a neighborhood V of a point $x_0 \in \mathbb{R}^2$, with analytic principal part and bounded*

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lower order terms. Let $S = \{x \in V : \phi(x) = \phi(x_0)\}$ *a* C^2 *hypersurface near* x_0. *Then there exists a neighborhood W of the origin such that every* $u \in C^\infty$ *satisfying*

$$\begin{cases} Pu = 0 \ \text{in} \ V \\ u = 0 \ \text{in} \ \{x \in V : \phi(x) < \phi(x_0)\} \end{cases}$$

vanishes in W.

Let us note that this result cannot be extended to fourth order elliptic operator in $\mathbb{R}^2$ according with the following example of Pliš

$$P = (\partial_t - i\partial_x)^4 + t^8 \partial_x^4 + i\partial_x^3$$

for which there exists $a \in C^\infty(\mathbb{R}^2)$ such that $P + a$ fails to have the uniqueness property.

Proof of Theorem 1.

Step 1. It is sufficient to prove theorem 1 assuming that

(3) $$\Delta(x, t) = t^\ell \Delta_1(x, t), \ \ell \in N, \ \Delta_1(0, 0) \neq 0.$$

We use an argument of [8] (see also [7]) slightly modified to take account of the fact that Δ may take complex values. Let us suppose Theorem 1 true under (3). By the Malgrange – Weierstrass theorem we can write for $|x| < r$ and $0 < t < T$

$$\Delta(x, t) = a(x, t) \left\{ t^k + \sum_{j=1}^{k} a_j(x)\, t^{k-j} \right\}$$

with $a(0, 0) \neq 0, \ a_j(0) = 0$.

We know (see [8]) that there exists k open sets $o_1, \ldots, o_k$, included in $]-r, r[$, whose union is dense, such that in each o_j, Δ/a has exactly j distinct roots which can be represented by j C^α functions $\rho_1(x), \ldots, \rho_j(x)$. Let u be a solution of $Pu = 0$ such that $u_{|t<0} = 0$. Suppose $\text{supp}\, u \cap]-r, r[\times]0, T[\neq \varnothing$; then there exist j and a connected component V of one o_j such that $\overset{\circ}{\widehat{\text{supp}\, u}} \cap V \times]0, T[\neq \varnothing$. Shrinking V we can assume that

(4) For every open set $W \subset V$, $\text{supp}\, u \cap W \times]0, T[\neq \varnothing$.

Let us write in V

$$\frac{\Delta}{a}(x, t) = \prod_{\ell=1}^{p} (t - \rho_\ell(x))^{\alpha_\ell} \prod_{\ell=p+1}^{j} (t - \rho_\ell(x))^{\alpha_\ell}$$

where $\rho_1, \ldots, \rho_p$ are the non real roots and $\rho_{p+1}, \ldots, \rho_j$ the real roots. Let $x_0 \in V$ be such that $\text{Im}\, \rho_1(x_0) \neq 0$. There exists $x_0 \in V_1 \subset V$ such that

$\text{Im}\, \rho_1(x) \neq 0$ for $x \in V_1$: now in $V_1 \times]0, T[$, $t - \rho_1(x) \neq 0$. We look at the second root ρ_2 in V_1. If $\text{Im}\, \rho_2 \equiv 0$ then ρ_2 is real in V_1, if not we shrink V_1 to V_2 where $\text{Im}\, \rho_2 \neq 0$ so that $t - \rho_2(x) \neq 0$ in V_2 etc... until ρ_p. We are now in the following situation: there exists an open set $\tilde{V} \subset V$ such that for $(x, t) \in \tilde{V} \times]0, T[$

$$\Delta(x, t) = b(x, t) \cdot \prod_{\ell \in J} (t - \rho_\ell(x))^{\alpha_\ell}$$

where $J \subset \{1, 2, \ldots, j\}$, the roots ρ_ℓ are real and b does not vanish. For simplicity we will take $J = \{1, 2, \ldots, q\}$ and will suppose that in $\tilde{V}$ we have $\rho_1(x) < \ldots < \rho_q(x)$. Note that by (4), $\text{supp}\, u \cap \tilde{V} \times]0, T[\neq \phi$. For $x \in \tilde{V}$ we define

$$\tilde{\rho}_0(x) = 0, \ \tilde{\rho}_\ell(x) = \sup(0, \inf(T, \rho_\ell(x))), \ 1 \leq \ell \leq q, \ \tilde{\rho}_{q+1}(x) = T$$

and

$$A_\ell = \{(x, t) \in \tilde{V} \times]0, T[: \tilde{\rho}_\ell(x) \leq t \leq \tilde{\rho}_{\ell+1}(x)\}, \ 0 \leq \ell \leq q.$$

Let $\ell_0 = Min\ \{\ell : \text{supp}\, u \cap A_\ell \neq \phi\}$

a) Suppose there exists $(x_0, t_0) \in A_{\ell_0} \cap \text{supp}\ u$ with $t_0 = \rho_{\ell_0}(x_0)$. We perform the following change of coordinates: $x' = x - x_0$, $t' = t - \rho_{\ell_0}(x)$. In these coordinates, near origin $\Delta = t'^k \tilde{\Delta}$ with $\tilde{\Delta}(0, 0) \neq 0$, and the new function $\tilde{u}$ vanishes for $t' < 0$. By hypothesis $u = 0$ near (x_0, t_0) which is a contradiction.

b) Suppose $\text{supp}\, u \cap A_{\ell_0} \cap \{t = \rho_{\ell_0}(x)\} = \varnothing$. We perform the same change of coordinates as before with x_0 the mid point of $\tilde{V}$. The set $\in \{(x, t)\ V \times]0, T[\ \ t = \rho_{\ell_0}(x)\}$ is transformed in $\{(x, t), t' = 0, |x'| < r\}$. Let $\varepsilon_0 = \text{Min}\ \{\varepsilon : \{\text{Graphe of } t' - \varepsilon = (-\varepsilon/r^2)\, x'^2\} \cap \tilde{A}_{\ell_0} \cap \text{supp}\, \tilde{u} \neq 0\}$. By hypothesis $\varepsilon_0 > 0$. We take $(x'_0, t'_0) \in A_{\ell_0} \cap \text{supp}\tilde{u}$ such that $t'_0 = (-\varepsilon_0/r^2)\, x_0'^2 + \varepsilon_0$. Near this point the operator P has simple roots since Δ is different from zero and the support of u is from one side of the parabola $t' = (-\varepsilon_0/r^2)\, x'^2 + \varepsilon_0$. We conclude that $\tilde{u}$ vanishes near (x'_0, t'_0) which is a contradiction.

Step 2. Theorem 1 is true under the assumption (3).

Since the result is well known for $\ell = 0$, let us suppose that $\ell \geq 1$. Let $P = \partial_t^2 + 2b\, \partial_x \partial_t + c\, \partial_x^2 + \alpha\, \partial_x + \beta\, \partial_t + \gamma$. Following [2] we make near the origin the singular change of coordinates

$$X = x, \ t = (\delta - X^2)T, \ \delta > 0 \text{ small.}$$

It is easy to see that P transforms to an elliptic operator $\tilde{P}$ such that

$$(\delta - X^2)\tilde{P} = A\, \partial_T^2 + 2(\delta - X^2)B\, \partial_X \partial_T + C(\delta - X^2)^2\, \partial_X^2 + f\, \partial_T + g(\delta - X^2)\partial_X + h$$

where $f, g, h \in L^\infty$ and A, B, C are C^∞ functions of (X, T) such that

$$(5)\quad \begin{cases} A \text{ is real}, \ A(0,0) = 1, \ \operatorname{Im} B(0,0) \neq 0 \\ (B^2 - AC)(X, T) = (\delta - X^2)^{\ell+2}\, T^\ell\, D(X, T), \ D(0,0) \neq 0. \end{cases}$$

Writing for simplicity (x, t) instead of (X, T), it follows that we have

$$(6)\quad (\delta - x^2)\tilde{P} = A \cdot P_1 P_2 + \alpha \frac{\partial}{\partial t} + (\delta - x^2)(\beta + \gamma t^{\ell/2-1})\frac{\partial}{\partial x} + \lambda$$

where $\alpha, \beta, \gamma, \lambda \in L^\infty$ and

$$P_j = \frac{\partial}{\partial t} + (\delta - x^2) b \frac{\partial}{\partial x} + (-1)^j (\delta - x^2)^{\ell/2+1}\, t^{\ell/2}\, c \frac{\partial}{\partial x} \quad j = 1, 2$$

Here $b = \dfrac{B}{A}$ so that $\operatorname{Im} b(0,0) \neq 0$ and $c = \dfrac{D^{1/2}}{A}$. We prove now a Carleman estimate for P_j.

Proposition 3. *There exist positive constants C, γ_0, T_0, r such that for $\gamma \geq \gamma_0$ and every $u \in C^\infty$ near the origin such that supp $u \subset \{(x, t) : 0 \leq t \leq T_0, |x| \leq r\}$ we have*

$$(7)\quad \gamma \| t^{-\gamma-1} u \|^2 + \frac{1}{\gamma} \| t^{-\gamma} (\delta - x^2) \partial_x u \|^2 + \frac{1}{\gamma} \| t^{-\gamma} \partial_t u \|^2 \leq C \| t^{-\gamma} P_j u \|^2$$

where $\| \ \|^2$ is the L^2 norm.

Proof. Let us put $u = t^\gamma v$. It follows that

$$t^{-\gamma} P_j u = P_j v + \gamma t^{-1} v = Xv + Yv$$

where

$$(8)\quad \begin{cases} X = \partial_t + (\delta - x^2) b_1 \partial_x ; \ b_1 = \operatorname{Re} b + (-1)^j (\delta - x^2)^{\ell/2}\, t^{\ell/2} \operatorname{Re} c \\ Y = \gamma t^{-1} + i(\delta - x^2) b_2 \partial_x ; \ b_2 = \operatorname{Im} b + (-1)^j (\delta - x^2)^{\ell/2}\, t^{\ell/2} \operatorname{Im} c. \end{cases}$$

Now

$$(9)\quad \| t^{-\gamma} P_j u \|^2 = \| Xv \|^2 + \| Yv \|^2 + 2\operatorname{Re}(Xv, Yv)$$

It is easy to see that

$$(10)\quad 2 \operatorname{Re}(\partial_t v, \gamma t^{-1} v) = \gamma \| t^{-1} v \|^2$$

$$(11)\quad | 2 \operatorname{Re}((\delta - x^2) b_1 \partial_x v, \gamma t^{-1} v) | \leq c\gamma \| t^{-1/2} v \|^2$$

$$(12)\quad 2 \operatorname{Re}((\delta - x^2) b_1 \partial_x v, i(\delta - x^2) b_2 \partial_x v) = 0$$

On the other hand

$$(13)\quad \begin{aligned} &2 \operatorname{Re}(\partial_t v, \ i(\delta - x^2) b_2 \partial_x v) = \\ &= -\underbrace{\left(v, \ i(\delta - x^2) \frac{\partial b_2}{\partial t} \partial_x v\right)}_{1/} + \underbrace{(v, i\, \partial_x [(\delta - x^2) b_2] \partial_t v)}_{2/} \end{aligned}$$

Since $b_2(0,0) \neq 0$ we can write:

$$1/ = -\left(\frac{\partial_t b_2}{b_2} v, i(\delta - x^2) b_2 \partial_x v\right) = -\left(\frac{\partial_t b_2}{b_2} v, Yv\right) + \gamma \left(\frac{\partial_t b_2}{b_2} v, \ t^{-1} v\right)$$

It follows that for every $\varepsilon > 0$

$$(14)\quad | 1/ | \leq \varepsilon \| Yv \|^2 + c_\varepsilon \| t^{-1} v \|^2 + c\gamma (\| t^{-1/2} v \|^2 + \| t^{\ell/4-1} v \|^2)$$

By (8) we have

$$2/ = (v, i\, \partial_x [(\delta - x^2) b_2] Xv) - (v, i\, \partial_x [(\delta - x^2) b_2] (\delta - x^2) b_1 \partial_x v) = 3/ + 4/$$

Since $a = \partial_x [(\delta - x^2) b_2] \in L^\infty$ it is easy to see that

$$| 3/ | \leq \varepsilon \| Xv \|^2 + c_\varepsilon \| v \|^2$$

Now

$$4/ = -\left(\frac{a b_1}{b_2} v, \ i(\delta - x^2) b_2 \partial_x v\right)$$

and by the same argument which we used for 1/ we get

$$4/ \leq \varepsilon \| Yv \|^2 + c_\varepsilon \| t^{-1/2} v \|^2$$

It follows that

$$(15)\quad | 2/ | \leq \varepsilon (\| Xv \|^2 + \| Yv \|^2) + c_\varepsilon \| t^{-1/2} v \|^2$$

Using (9) ... (15) it is easy to see that for k and T_0^{-1} big enough

$$(16)\quad \gamma \| t^{-1} v \|^2 + \| Xv \|^2 + \| Yv \|^2 \leq c \| t^{-\gamma} P_j u \|^2$$

Now by (8)

$$(17)\quad \| \partial_t v \|^2 + \| (\delta - x^2) \partial_x v \| \leq \| Xv \|^2 + \| Yv \|^2 + \gamma^2 \| t^{-1} v \|^2$$

From (16) and (17) we deduce (7).

Proposition 4. *There exist positive constants C, γ_0, T_0, r such that $\gamma \geq \gamma_0$ and every u C^∞ near the origin such that*

$$\text{supp}\, u \ \{(x, t) : 0 \leq t < T_0, \ |x| \leq r\}$$

we have

$$(18)\quad \gamma^2 \| t^{-\gamma-2} u \|^2 \leq C \| (\delta - x^2) t^{-\gamma} \tilde{P} u \|^2$$

Before we give the proof of Proposition 4, let us remark that (18) implies by a classical argument uniqueness of the Cauchy Problem for $(\delta - x^2)\tilde{P}$ and then for the original operator P.

Proof of Proposition 4. If u is flat on $t=0$ it follows that $P_2 u$ is C^∞ and flat on $t=0$. By Proposition 3, applied twice, we can write

$$\gamma^2 \| t^{-\gamma-2} u \|^2 + \| t^{-\gamma-1}(\delta - x^2)\partial_x u \|^2 + \| t^{-\gamma-1}\partial_t u \|^2 \leq \leq c\gamma \| t^{-\gamma-1} P_2 u \|^2 \leq c' \| t^{-\gamma} P_1 P_2 u \|^2 \tag{19}$$

By (6)

$$\| t^{-\gamma} P_1 P_2 u \|^2 \leq C \{ \| t^{-\gamma}(\delta - x^2)\tilde{P} u \|^2 + \| t^{-\gamma}\partial_t v \|^2 + + \| (\delta - x^2) t^{-\gamma}\partial_x v \|^2 + \| t^{-\gamma-1+\ell/2}(\delta - x^2)\partial_x v \|^2 + \| t^{-\gamma} v \|^2 \} \tag{20}$$

The inequality (18) then follows from (19) and (20) taking γ and T_0^{-1} big enough.

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