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Some results on local growth of two-parameter Lévy processes

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Summary

In this article we consider some problems on local growth of 2-parameter Lévy processes, i.e., processes with independent and stationary increments, indexed by $[0, +\infty) \times [0, +\infty)$. The results are for the upper growth of these processes, at a fixed "time" z_0.

0. Introduction. In this article we consider some problems on local growth of two-parameter Lévy processes, i.e., processes with stationary and independent increments, and indexed by $\mathbb{R}^2_+ \stackrel{df}{=} \{(s, t) : s \geq 0,\ t \geq 0\}$. At this stage we only attempt to study upper growth, and at a fixed "time" z_0.

Section 1 contains the basic definitions and a short summary of preliminary results concerning the construction of such processes. More details can be found in [5] or [6].

In Section 2, we present the two-parameter analogue of Khinchin's theorem. The behavior at the origin is different from that at a point z_0 away from the axes. The basic idea for the proofs is to use twice the classical inequality of Skorohod; firstly for $D[0, 1]$ – valued random vectors, and then for usual real-valued random variables.

Section 3 contains and "integral test" for some quite general situations.

1. Preliminaries.

Here we consider processes that are indexed by $\mathbb{R}^2_+ = \{(s, t) : s \geq 0, t \geq 0\}$, and constitute a two-parameter analogue of processes with stationary and independent increments. More precisely, we work with stochastic processes $(X_z : z \in \mathbb{R}^2_+)$, defined on some probability space $(\Omega, \mathscr{F}, P)$, and such that:

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(a) $X_{s,0} = X_{0,t} = 0$ for all $s \geq 0, t \geq 0$.

(b) If $A =]s, s'] \times (t, t'] \subseteq \mathbb{R}^2_+$, set $X(A) = X_{s',t'} - X_{s,t'} - X_{s',t} + X_{s,t}$. Then, for A as above and $z_0 \in \mathbb{R}^2_+$, we have law of $X(A)$ = law of $X(A + z_0)$. (Here $A + z_0 = \{u + z_0 : u \in A\}$)

(c) If $n > 1$ and $A_1, \ldots, A_n$ are rectangles as above, and disjoint, then $X(A_1), \ldots, X(A_n)$ are independent random variables.

Remark. If we were to take $[0, s_0) \times [0, t_0)$ as parameter set, the modification would be completely obvious $(s_0 > 0, t_0 > 0)$.

The question of *existence and characterization* of such processes is answered in an article by Straf [5]. For the sake of completeness we give a brief summary of what it will be needed here from his results. But before doing this, we introduce some notation.

Definition 1.1. The following orderings on $\mathbb{R}^2_+$ are used: if $z = (s, t), z' = (s', t')$ in rectangular coordinates, we set

$$
\begin{aligned}
&z \leq_1 z' \quad \text{if} \quad s \leq s', t \leq t';\\
&z \leq_2 z' \quad \text{if} \quad s \leq s', t > t';\\
&z \leq_3 z' \quad \text{if} \quad s > s', t \leq t'; \quad \text{and}\\
&z \leq_4 z' \quad \text{if} \quad s > s', t > t'.
\end{aligned}
$$

Also, set $Q_i(z) = \{z \in \mathbb{R}^2_+ : z \leq_i z'\}$, $i = 1, \ldots, 4$.

Definition 1.2. Let $\emptyset \neq E \subseteq \mathbb{R}^2_+$. Following the notation of Straf, a function $f: E \to \mathbb{R}$ is called a *lamp function* on E (limit along monotone paths) if, for any sequence $(z_n)_n$ in E, $z_n \to z \in E$ monotonically according to some $\leq_i$ implies the existence of $\lim\limits_{n \to \infty} f(z_n)$ in $\mathbb{R}$. When $E = \mathbb{R}^2_+$ we simply say that f is a lamp function. We say that f is continuous from above if, for each z and $z_n \to z$ with $z_{n+1} \leq_1 z_n$ (all $n \geq 1$), we have $f(z_n) \to f(z)$ as $n \to \infty$.

Clearly, if f is a lamp function and continuous from above, then for each $z_0 \in \mathbb{R}^2_+$ such that $Q_i(z_0) \neq \emptyset$, we must have

$$L^i_f(z_0) = \lim_{\substack{z \to z_0 \\ z \in Q_i(z_0)}} f(z) \text{ exists and } L^1_f(z_0) = f(z_0).$$

$(i = 1, \ldots, 4)$

Definition 1.3. Set

$$D[0, 1)^2 = \{\text{functions that are lamp on } [0, 1]^2 \text{ and continuous from above, restricted to } [0, 1)^2\}$$

$$D = D(\mathbb{R}^2_+) = \{f : \mathbb{R}^2_+ \to \mathbb{R} \mid f \text{ is a lamp function and continuous from above}\}$$

$$D_0 = \{f \in D \mid f \text{ is zero on the axes}\}.$$

If the functions take values in some other metric space V, instead of $\mathbb{R}$, we write $D([0, 1)^2, V)$, $D(\mathbb{R}^2_+, V)$ and $D_0(\mathbb{R}^2_+, V)$, respectively.

Straf defines, on $D[0, 1)^2$, an analogue of Skorohod topology on $D[0, 1]$. With this topology $D[0, 1)^2$ can be made into a complete and separable metric space. Then, using the classical theory of convergence in distribution, and starting by a result of Wichura [7] – multiparameter analogue of Donsker's theorem – Straf has shown:

Theorem 1.1 (Straf). *Let F be an infinitely divisible distribution on $\mathbb{R}^d$ $(d \geq 1)$, and let ϕ denote its characteristic function.*

Then, it is possible to define a stochastic process $X = (X_z : z \in \mathbb{R}^2_+)$ verifying (a), (b) and (c) above, and continuous in probability, such that $\phi_{s,t} = \phi^{st}$, where $\phi_{s,t}$ denotes the characteristic function of $X_{s,t}$ $(s, t \geq 0)$.

Moreover, X can be constructed with paths in $D_0(\mathbb{R}^2_+, \mathbb{R}^d)$.(1)

Conversely, if $(X_z\ z \in \mathbb{R}^2_+)$ verifies (a), (b) and (c), and is continuous in probability, we have that law of $X_{s,t}$ = law of $X_{st,1}$. Thus, since $(X_{t,1})_{t \geq 0}$ is a 1-parameter process with stationary and independent increments, the characteristic function of $X_{s,t}$ is of the above form. In particular X could be assumed to have paths is D_0.

Definition 1.4. By a 2-parameter Lévy process we mean a stochastic process as described in Theorem 1.1.

For other characterizations of $D[0, 1)^2$, as well as more details on these preliminaries, the reader might consult Straf [5].

The Lévy system formula and Lévy decomposition can be found in [6], where other problems related to these processes are studied.

(1) The uniqueness in law is obvious.

2. An analogue of Khinchin's theorem on local growth.

Let $X = (X_z : z \in \mathbb{R}^2_+)$ be a two-parameter Lévy process, with values in $\mathbb{R}$. Without loss of generality we assume X to be defined on some complete probability space $(\Omega, \mathscr{F}, P)$, and to have paths in D_0. We may as well assume $\mathscr{F} = P-$ completion of $\sigma(X_z : z \in \mathbb{R}^2_+)$.

Notations.

(1) For $z \in \mathbb{R}^2_+$, $\mathscr{F}_z$ denotes the $P-$completion of $\sigma(X_\tau : \tau \leq_1 z)$ relatively to $\mathscr{F}$, where $\sigma(X_\tau : \tau \leq_1 z)$ denotes the $\sigma-$field generated by X_τ, $\tau \leq_1 z$.

(2) If $z = (s, t)$, $z' = (s', t')$ in $\mathbb{R}^2_+$, we write $z <_1 z'$ when $s < s'$ and $t < t'$. In this case, $(z, z']$ denotes the set $\{\tau : z <_1 \tau \leq_1 z'\}$.

(3) Let $(\Delta f)_{s,t} = L^1_f(s,t) - L^2_f(s,t) - L^3_f(s,t) + L^4_f(s,t)$
$$= f(s,t) - f(s,t-) - f(s-,t) + f(s-,t-)$$

for $s > 0$, $t > 0$ and $f \in D$.

Let's first consider local growth at $z_0 = 0$. (Here 0 denotes the origin $(0, 0)$.)

Let $g : \mathbb{R}^2_+ \to [0, +\infty)$ be such that $g = 0$ on the axes, $g > 0$ otherwise, and such that $z \leq_1 z' \Rightarrow g(z) \leq g(z')$, for all $z, z' \in \mathbb{R}^2_+$. We want to study $|X_z|/g(z)$ as $z \to 0$ $(0 <_1 z)$. Since the law of $X_{s,t}$ depends on (s, t) only through st, it is natural to take $g(s,t) = f(st)$, where $f : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function with $f(0) = 0$ and $f(x) > 0$ for $x > 0$.

Remark 2.1. It is clear that, for each z,

$\mathscr{F}_z = \bigcap_{z' : z <_1 z'} \mathscr{F}_{z'}$. Also $\bigcap_{s>0, t>0} F_{s,1} \vee F_{1,t}$ is P-trivial.

In fact, more general 0-1 laws hold for the filtration $(\mathscr{F}_z)_z$. Consequently, there exist constants $c_1 \leq c_2$ in $[0, +\infty]$, such that

$$\lim_{\varepsilon \downarrow 0} \sup_{0 < s, t < \varepsilon} \frac{|X_{s,t}|}{f(st)} = c_1 \text{ a.s. and}$$

$$\lim_{\varepsilon \downarrow 0} \sup_{(s,t) \in B_\varepsilon} \frac{|X_{s,t}|}{f(st)} = c_2 \text{ a.s.,}$$

where $B_\varepsilon = \{(s, t) : 0 < s,\ t < 1,\ st < \varepsilon\}$.

Theorem 2.1. *Let f be a function as above.*

(A) (i) *If* (*) $\overline{\lim_{t \downarrow 0}}\ P[|X_{t,1}| > 2f(t)] < 1/2$, *and*
$$\sum_{n \geq 1} nP[|X_{2^{-n},1}| > f(2^{-n})] < +\infty,$$
then $c_2 < +\infty$.

(ii) *If* $\sum_{n \geq 1} nP[|X_{2^{-n},1}| > f(2^{-n})] = +\infty$,

then $c_1 > 0$.

(B) *Writing in terms of an "integral test", we have:*

(i)′ *If* $\int_0^1 t^{-1} |\log t|\ P[|X_{t,1}| > f(t)]\,dt < +\infty$, *then* $c_2 < +\infty$.

(ii)′ *If the integral in* (i)′ *is equal to* $+\infty$, *then* $c_1 > 0$.

Proof. (i) Let's assume condition (*) and that
$$\sum_{n \geq 1} nP[|X_{a^n,1}| > f(a^n)] < +\infty$$
for some $a \in [1/2, 1)$. In what follows n, k, j will always be in $\{1, 2, \ldots\}$. Since $P[|X_{a^k, a^j}| > f(a^{k+j})] = P[|X_{a^{k+j},1}| > f(a^{k+j})]$, we have:
$$\sum_{k,j \geq 1} P[|X_{a^k,a^j}| > f(a^{k+j})] = \sum_n (n-1)\,P[|X_{a^n,1}| > f(a^n)] < +\infty.$$
By Borel-Cantelli, it follows that
$$P[|X_{a^k,a^j}| > f(a^{k+j}) \text{ for infinitely many pairs } (k,j)] = 0,$$
$$\text{i.e., } \lim_{n\to\infty} \sup_{k+j \geq n} \frac{|X_{a^k,a^j}|}{f(a^{k+j})} \leq 1 \text{ a.s.}\,.$$

Let $B(k,j) = [a^k, a^{k-1}) \times [a^j, a^{j-1})$. Next we show there exists some finite constant A such that

$$(2.1)\quad P[\sup_{z \in B(k,j)} |X_z - X_{a^k,a^j}| > Af(a^{k+j}) \text{ for infinitely many } (k,j)] = 0.$$

But

$$P[\sup_{z \in B(k,j)} |X_z - X_{a^k,a^j}| > 13\,f(a^{k+j})]$$
$$\leq P[\sup_{(s,t) \in B(k,j)} |X((a^k, s] \times (a^j, t])| > 7\,f(a^{k+j})] +$$
$$+ P[\sup_{a^k \leq s < a^{k+1}} |X_{s,a^j} - X_{a^k,a^j}| > 3\,f(a^{k+j})]$$

$$(2.2)\qquad + P\Big[\sup_{a^j \le t < a^{j+1}} |X_{a^k,t} - X_{a^k,a^j}| > 3f(a^{k+j})\Big]$$

$$\le P\Big[\sup_{z <_1 (a^k,a^j)} |X_z| > 7f(a^{k+j})\Big]$$

$$+ P\Big[\sup_{s<a^k} |X_{s,a^j}| > 3f(a^{k+j})\Big] + P\Big[\sup_{t<a^j} |X_{a^k,t}| > 3f(a^{k+j})\Big]$$

since $a \in [1/2, 1)$ and X has stationary increments.

Let's start by the first term of this sum, since the others are just as in 1-parameter situation.

The idea is to use Skorohod's inequality, as mentioned in the introduction.

Here $D[0,1]$ denotes, as usually, the Skorohod space of functions on $[0,1]$ that are rigth-continuous on $[0,1)$ and with left limits on $(0,1]$. $\mathbb{B}'$ is the Borel σ-field for the Skorohod topology, i.e., the σ-field generated by the coordinate maps.

We use the classical result (See, e.g., [3]):

Lemma 2.1. (Skorohod) *Let $Y_1, \dots, Y_n$ be independent random vectors defined on some $(\Omega', \mathscr{F}', P)$ and with values in $(E, \mathscr{E})$, where E is a vector space and $\mathscr{E}$ is a σ-field on E. Let $\rho : E \to R$ be a function such that $\rho(x+y) \le \rho(x) + \rho(y)$ for all $x, y \in E$. Assume that $\rho(Y_j + \dots + Y_k)$, $\rho(-(Y_j + \dots + Y_k))$ are $\sigma(Y_u : j \le u \le k)$ – measurable, for all $1 \le j \le k \le n$. Let $S_k = Y_1 + \dots + Y_k$, $1 \le k \le n$.*

For each $a > 0$, $b > 0$

$$P\Big[\sup_{1\le k\le n} \rho(S_k) > a+b\Big] \le \frac{P[\rho(S_n) > a]}{\min_{\le k \le n} P[\rho(S_k - S_n) \le b]}$$

provided $0 < \min_{1\le k\le n} P[\rho(S_k - S_n) \le b]$.

Let $Y_s = (X_{s,t} : 0 \le t \le 1)$, for each $s \ge 0$; then $Y_s : (\Omega, \mathscr{F}, P) \to (D[0,1], \mathbb{B}')$ is measurable. Moreover

(a) $0 \equiv Y_0$, $Y_{s_1} - Y_0$, $Y_{s_2} - Y_{s_1}, \dots, Y_{s_n} - Y_{s_{n-1}}$ are independent if $0 < s_1 < \dots < s_n$ and the distribution of $Y_{s+h} - Y_s$ on $\mathbb{B}'$ is that of Y_h, for each $s, h \ge 0$.

(b) If $s_n \downarrow s$, $Y_{s_n}(\omega) \to Y_s(\omega)$ pointwise on $[0,1]$ (each ω).

So $\| Y_s \| \le \lim_{n\to\infty} \| Y_{s_n} \|$, where $\| \cdot \|$ is the sup norm in $D[0,1]$.

Applying the lemma, with $\rho(x) = \| x \|$, to

$$Y_{s_0} = \sum_{k=1}^{2n} (Y_{k2^{-n}s_0} - Y_{(k-1)2^{-n}s_0}),$$

and using (b) we get that $[\sup_{s\le s_0} \| Y_s \| > a+b] \in \mathscr{F}$ and

$$P\Big[\sup_{s\le s_0} \| Y_s \| > a+b\Big] \le \frac{P[\| Y_{s_0} \| > a]}{1 - \sup_{s\le s_0} P[\| Y_s \| > b]}$$

for each $s_0 \ge 0$ (Convention: RHS $= +\infty$ if the denominator is zero).

Let $k \ge 1$, $j \ge 1$ be fixed. Arguing as above, with $Y_s = (X_{s,t} : t \le a^j)$, and $\| \cdot \| =$ sup norm on $D[0, a^j]$, we find that

$$P\left[\sup_{\substack{s\le a^k \\ t\le a^j}} |X_{s,t}| > 7f(a^{k+j})\right] = P\Big[\sup_{s\le a^k} \| Y_s \| > 7f(a^{k+j})\Big]$$

$$\le \frac{P[\| Y_{a^k} \| > 3f(a^{k+j})]}{1 - \sup_{s\le a^k} P[\| Y_s \| > 4f(a^{k+j})]}.$$

Using Skorohod's lemma again (now, for $\mathbb{R}$-valued random variables), we get

$$P[\| Y_s \| > 4f(a^{k+j})] \le \frac{P[| X_{s,a^j} | > 2f(a^{k+j})]}{1 - \sup_{t\le a^j} P[| X_{s,t} | > 2f(a^{k+j})]}$$

and

$$P[\| Y_{a^k} \| > 3f(a^{k+j})] \le \frac{P[| X_{a^k,a^j} | > f(a^{k+j})]}{1 - \sup_{t\le a^j} P[| X_{a^k,t} | > 2f(a^{k+j})]}$$

Thus,

$P\Big[\sup_{z\le_1 (a^k,a^j)} |X_z| > 7f(a^{k+j})\Big] \le c_{k,j} P[| X_{a^k,a^j} | > f(a^{k+j})]$, where

$$c_{k,j} = \frac{1}{1 - \sup_{t\le a^j} P[| X_{a^k,t} | > 2f(a^{k+j})]} \cdot \frac{1}{1 - \sup_{s\le a^k} \dfrac{P[| X_{s,a^j} | > 2f(a^{k+j})]}{1 - \sup_{t\le a^j} P[| X_{s,t} | > 2f(a^{k+j})]}}$$

provided we don't have zero in the denominator; $c_{k,j} = +\infty$ otherwise.

From (*), there exists $\varepsilon > 0$ and $0 < \delta < 1/2$ such that $P[| X_{t,1} | > 2f(t)] < \delta$ for all $0 < t < \varepsilon$.

If n is large enough so that $a^n < \varepsilon$, it is easy to seethat $k+j \ge n$ implies

$$c_{k,j} \le (1-\delta)^{-1} \left(1 - \frac{\delta}{1-\delta}\right)^{-1}.$$

Consequently,

$$\sum_{k,j} P\Big[\sup_{z\le_1 (a^k,a^j)} |X_z| > 7f(a^{k+j})\Big] < +\infty.$$

For the other two terms in (2.2) the estimates are immediate:

$$P[\sup_{s \le a^k} |X_{s,a^j}| > 3 f(a^{k+j})] \le \frac{1}{1-\delta} P[|X_{a^k,a^j}| > f(a^{k+j})],$$

if $k+j \ge n$, n large enough, and similarly for the third term in (2.2).

From this, (2.2), and Borel-Cantelli, (2.1) follows with $A = 13$. Since f is nondecreasing we can indeed say that

(2.3) $\quad c_2 \le 14.$ (Collecting above facts.)

If we assume $\int_0^1 t^{-1} |\log t| \, P[|X_{t,1}| > f(t)] \, dt < +\infty$, it is easily seen that $\sum_{n \ge 1} n P[|X_{a^n,1}| > f(a^n)] < +\infty$ for "a" in a dense subset of $(0,1)$. Also $\int_0^1 t^{-1} P[|X_{t,1}| > f(t)] \, dt < +\infty$, and since this implies $P[|X_{t,1}| > 2f(t)] \to 0$ as $t \downarrow 0$, (*) also holds. So, (i)′ follows from the proof of (i).

Next we prove (ii). We assume

$$\sum_{n \ge 1} n P[|X_{2^{-n},1}| > f(2^{-n})] = +\infty \text{ or, equivalently,}$$

$$\sum_{k,j} P[|X_{2^{-k},2^{-j}}| > f(2^{-k-j})] = +\infty.$$

Let

$$A_{k,j} = [\sup_{\substack{2^{-k-1} \le s < 2^{-k} \\ 2^{-j-1} \le t < 2^{-j}}} |X((2^{-k-1}, s] \times (2^{-j-1}, t])| > \frac{1}{16} f(2^{-k-j})].$$

The $A_{k,j}$ are independent; since $P[|X_{s,t}| > a] \le 16 P\left[|X_{s/4,t/4}| > \frac{a}{16}\right]$ we find,

(2.4)
$$P(A_{k,j}) = P[\sup_{z <_1 (2^{-k-1}, 2^{-j-1})} |X_z| > \frac{1}{16} f(2^{-k-j})]$$
$$\ge P[|X_{2^{-k-2},2^{-j-2}}| > \frac{1}{16} f(2^{-k-j})]$$
$$\ge \frac{1}{16} P[|X_{2^{-k},2^{-j}}| > f(2^{-k-j})],$$

implying that $\sum_{k,j} P(A_{k,j}) = +\infty$. In view of their independence, Borel--Cantelli implies: $1 = P(A_{k,j}$ for infinitely many pairs $(k, j)) = P\left(\bigcap_{n \ge 1} \bigcup_{k+j \ge n} A_{k,j}\right)$.

But, from (2.4)

$$\sum_{\substack{k \ge n \\ j \ge n}} P(A_{k,j}) \ge \frac{1}{16} \sum_{p \ge 2n} (p - 2n + 1) P[|X_{2^{-p},1}| > f(2^{-p})] = +\infty, \text{ for each } n \ge 1.$$

Thus,

$$P\left(\bigcap_{n \ge 1} \bigcup_{\substack{k \ge n \\ j \ge n}} A_{k,j}\right) = 1$$

From this, and because f is nondecreasing, one can easily see that

(2.5)
$$c_1 = \lim_{\varepsilon \downarrow 0} \sup_{0 < s,t < \varepsilon} \frac{|X_{s,t}|}{f(st)} \ge \frac{1}{64} \quad \text{(a.s.)}.$$

For (ii)′ we can argue as follows: it is easy to see that for $2^{-k} < s < 2^{-k+1}$, $2^{-j} < t < 2^{-j+1}$

$$P(A_{k,j}) \ge \frac{1}{16} P[|X_{s,t}| > f(st)]. \text{ Thus,}$$

$$P(A_{k,j}) (\log 2)^2 = P(A_{k,j}) \iint_{R(k,j)} \frac{ds\,dt}{st} \ge \frac{1}{16} \iint_{R(k,j)} \frac{1}{st} P[|X_{s,t}| > f(st)] \, ds\,dt,$$

where $R(k,j) = (2^{-k}, 2^{-k+1}] \times (2^{-j}, 2^{-j+1}]$.

So, the assumption in (ii)′ implies $\sum_{\substack{k \ge n \\ j \ge n}} P(A_{k,j}) = +\infty$. We conclude the proof as above.

Corollary 2.1. *Let $X = (X_z : z \in \mathbb{R}^2_+)$ be strictly stable with index α, $0 < \alpha < 2$. Suppose that X is not a deterministic drift, and let f be as in Theorem 2.1, Then:*

$\int_0^1 |\log t| \, [f(t)]^{-\alpha} dt < +\infty$ $(= +\infty)$ implies

$$\overline{\lim_{s \downarrow 0, t \downarrow 0}} \frac{|X_{s,t}|}{f(st)} = \lim_{\varepsilon \downarrow 0} \sup_{(s,t) \in B_\varepsilon} \frac{|X_{s,t}|}{f(st)} = 0 \ (= +\infty \text{ resp.}),$$

with probability one.

Proof. It follows easily from (2.3) and (2.5), since law of $X_{s,t}$ = law of $(st)^{1/\alpha} X_{1,1}$, and because there exists a and $A (0 < a < A < \infty)$ such that

$$a x^{-\alpha} \le P[|X_{1,1}| \ge x] \le A x^{-\alpha}, \text{ for } x \ge 1.$$

Remark 2.2. (a) Let X be as in previous corollary. Thus,

$$\overline{\lim_{s\downarrow 0,\, t\downarrow 0}}\ (st)^{-1/\alpha}\,|\log st\,|^{-b}\,|\,X_{s,t}\,| = \lim_{\varepsilon\downarrow 0}\ \sup_{B_\varepsilon}\,(st)^{-1/\alpha}\,|\log st\,|^{-b}\,|\,X_{s,t}\,|$$

is equal to 0 or $+\infty$ a.s., according as b is $>$ or $\le 2/\alpha$.

Let's recall that for 1-parameter processes the change (from 0 to $+\infty$) happens at $b = 1/\alpha$. From the proof of Theorem 2.1, it is clear that the difference is due to what happens "near the axes". For "non tangential lim sup" the situation would be just as in the 1-parameter case. More precisely, in the situation of Theorem 2.1 we set

$C(r) = \{(s,t) \in (0,1]^2 : r \le t/s \le 1/r\}$, for $0 < r < 1$. Then

$$\int_0^1 t^{-1} P[\,|\,X_{t,1}\,| > f(t)]\,dt \,(\overset{<}{=})\, +\infty \Rightarrow \lim_{\varepsilon\to 0}\ \sup_{\substack{(s,t)\in C(r)\\ s,t<\varepsilon}} \frac{|\,X_{s,t}\,|}{f(st)}\,(\overset{<+\infty}{>0}) \text{ a.s.}$$

(b) For strictly stable $\mathbb{R}$-valued processes, one can prove the one-sided version of Corollary 2.1 (as in the case of one parameter): assume X strictly stable (index α) and $\nu(0,+\infty) > 0$, where ν = Lévy measure of X. Then

$$\lim_{\varepsilon\downarrow 0}\ \sup_{0<s,t<\varepsilon} \frac{X_{s,t}}{f(st)} = \lim_{\varepsilon\downarrow 0}\ \sup_{(s,t)\in B_\varepsilon} \frac{X_{s,t}}{f(st)} \text{ is 0 or } +\infty \text{ a.s.,}$$

according as

$$\sum_{n\ge 1} n2^{-n}[f(2^{-n})]^{-\alpha} \text{ is } <+\infty \text{ or } =+\infty.$$

For the proof we only consider the divergent part, since the other follows from Corollary 2.1. By our assumption, $0 < b = P[X_z > 0]$ for all $0 <_1 z$. Also, as $x \to +\infty$, $x^\alpha P[X_{1,1} \ge x] \to a$, for some $a > 0$. Let's assume $\Sigma\, n2^{-n}[f(2^{-n})]^{-\alpha} = +\infty$. It is enough to consider the case of $2^n[f(2^{-n})]^\alpha$ away from zero ($n \ge 1$). Then if, $k, j \ge 1$, $n = k + j$, $c > 0$, and $A_{k,j} = [X((2^{-k-1}, 2^{-k}] \times (2^{-j-1}, 2^{-j}]) > cf(2^{-n})]$, we have

$$P(A_{k,j}) = P[X_{2^{-n-2},1} > cf(2^{-n})] \ge A(c)2^{-n-2}[f(2^{-n})]^{-\alpha}$$

for some $A(c) > 0$. So $\sum_{k,j} P(A_{k,j}) = +\infty$, and since they are independent $P(A_{k,j}\text{ i.o.}) = 1$.

If $C_{k,j} = [X_{2^{-k},2^{-j-1}} + X_{2^{-k-1},2^{-j}} - X_{2^{-k-1},2^{-j-1}} > 0]$,

for $k, j \ge 1$,

$$P(C_{k,j}) = P(Y_1 + Y_2 + Y_3 > 0) = b > 0, \text{ where}$$

$Y_1 = X((0, 2^{-k-1}] \times (0, 2^{-j-1}])$, $Y_2 = X((2^{-k-1}, 2^{-k}] \times (0, 2^{-j-1}])$

and $Y_3 = X((0, 2^{-k-1}] \times (2^{-j-1}, 2^{-j}])$.

Moreower, $C_{k,j}$ is independent of $A_{k,j}$. Letting $B_{k,j} = C_{k,j} \cap A_{k,j}$, we get:

(i) $\sum_{k,j} P(B_{k,j}) = +\infty$;

(ii) $P(B_{k,j} \cap B_{k',j'}) \le P(A_{k,j} \cap A_{k',j'}) = P(A_{k,j})\,P(A_{k',j'})$
$\le \frac{1}{b^2} P(B_{k,j})\,P(B_{k',j'})$ if $(k,j) \ne (k',j')$.

By the "refined version of Borel Cantelli", $P(B_{k,j}\text{ i.o.}) > 0$. Thus, Remark 2.1 implies $P(B_{k,j}\text{ i.o.}) = 1$. Moreover, as before we can indeed get $P(B_{k,j}\text{ i.o. for } k \ge n, j \ge n) = 1$ for each n, and since $B_{k,j} \subseteq [X_{2^{-k},2^{-j}} > cf(2^{-k-j})]$ and c is arbitrary in $(0,+\infty)$, we conclude that

$$\lim_{\varepsilon\downarrow 0}\ \sup_{0<s,t<\varepsilon} \frac{X_{s,t}}{f(st)} = +\infty \text{ a.s. when } \sum_{n\ge 1} n2^{-n}[f(2^{-n})]^{-\alpha} = +\infty.$$

(c) In the example of (a), when considering $\overline{\lim}\,\frac{|\,X_{s,t}\,|}{f(st)}$ as $s\downarrow 0$, $t\downarrow 0$, with $f(x) = x^{1/\alpha}|\log x\,|^b$, the "jump" from 0 to $+\infty$ occurs at different order than in the 1-parameter case. But, the difference is not detected by $f(x) = x^\beta$. This may suggest that the characterization of Blumenthal-Gettor upper index in terms of local growth [1] continues to hold.

Indeed, if $X = (X_z : z \in \mathbb{R}_+^2)$ has no Gaussian component and $\beta(\nu)$ is the Blumenthal-Getoor upper of ν(ν = Lévy measure of X); and we also assume no drift when $\int(|\,x\,|\wedge 1)\nu(dx) < +\infty$, then:

$$\beta(\nu) < \gamma \text{ implies } (st)^{-1/\gamma}\ |\,X_{s,t}\,| \to 0 \text{ as } s\downarrow 0,\ t\downarrow 0 \text{ (a.s.),}$$

$$\beta(\nu) > \gamma \text{ implies } \overline{\lim_{s\downarrow 0,\, t\downarrow 0}}\ (st)^{-1/\gamma}|\,X_{s,t}\,| = +\infty \text{ a.s.}$$

(divergent part is trivial from the 1-parameter case; for $\beta(\nu) < \gamma$, one uses the ideas of [4]).

But, for 1-parameter Lévy processes (X_t), it is also well known that, if X has no Gaussian component, no drift, $\gamma \le 1$, and $\int(|\,x\,|^\gamma \wedge 1)\,\nu(dx) < +\infty$, then $t^{-1/\gamma}\,|\,X_t\,| \underset{a.s.}{\to} 0$ as $t\downarrow 0$.

This does *not* hold anymore for 2-parameter Lévy processes. Indeed, one can give examples of 2-parameter increasing Lévy processes such that

$$\overline{\lim_{s\downarrow 0,\, t\downarrow 0}}\ (st)^{-1}X_{s,t} = +\infty \text{ a.s.} \tag{2.6}$$

(X is said to be increasing if $X(z,z'] \ge 0$ for $z <_1 z'$. Then, X is increasing iff it has no Gaussian component, the drift if any, is ≥ 0, and the Lévy measure ν is concentrated on $(0,+\infty)$, with $\int(|\,x\,|\wedge 1)\ \nu(dx) < +\infty$)

(d) Examples mentioned in (c): For $0<\alpha<1$, let ν be given by $\nu(dx)=$ $=dx/x^{1+\alpha}$ on $(0,+\infty)$, and $\nu(dx)=0$, otherwise. Let (X_z) be the corresponding strictly stable increasing Lévy process and $h_b(t)=t^{1/\alpha}\,|\log t|^b$ $(b>0)$, for $0<t<1$.

(2.7) By part (a) $\overline{\lim}_{s\downarrow 0,\,t\downarrow 0}\dfrac{X_{s,t}}{h_b(st)}=0$ or $+\infty$ a.s.

according $b>$ or $\leq 2/\alpha$. Also h_b is strictly increasing and convex on some $(0,a)$, $a=a(b,\alpha)>0$. Let $g_b(t)=h_b^{-1}(t)$ for $0<t<h_b(a)$, $g_b(t)=0$ otherwise. *If* $b>1/\alpha$ $\int g_b(x)\,\nu(dx)<+\infty$; thus, we can consider $Y_z^b=$ $=\sum_{\tau\leq_1 z} g_b(\Delta X_\tau)$, $z\in\mathbb{R}_+^2$, which is an increasing Lévy process with Lévy measure μ on $(0,+\infty)$, given by

$$\int_{(0,+\infty)} g\,d\mu=\int_{(0,h_b(a))}(g\circ g_b)\,d\nu,\ g\geq 0 \text{ Borel.}$$

Since g_b is concave, strictly increasing on $[0,h_b(a))$, $g_b(0)=0$, and (2.7) holds, we see that

$$\overline{\lim}_{s\downarrow 0,\,t\downarrow 0}(st)^{-1}Y_{s,t}^b=+\infty \text{ a.s. when } 1/\alpha<b\leq 2/\alpha.$$

Making proper choice of b, one can also see that for each $0<\varepsilon<1$, there exists $(Y_z: z\in\mathbb{R}_+^2)$ increasing process, such that

$$\overline{\lim}_{s\downarrow 0,\,t\downarrow 0}(st\,|\log st\,|^{\varepsilon})^{-1}\,Y_{s,t}=+\infty \text{ a.s.}$$

$[1/\alpha<b\leq(2-\varepsilon)/\alpha$ in previous construction$]$.

(e) Another way of finding these examples is simply by observing the following:

Let $h:[0,+\infty)\to[0,+\infty)$ be convex, strictly increasing, $h(0)=0$ and let (X_z) be an increasing Lévy process.

If $\displaystyle\int_0^1\int_0^1 \nu(h(uv),+\infty)\,du\,dv=+\infty$, then $N_{s,t}(h)=+\infty$ a.s.

for each $s,t>0$, where $N_z(h)=\sum_{(u,v)\leq_1 z} I[\Delta X_{u,v}\geq h(uv)]$. So,

$$\overline{\lim}_{s\downarrow 0,\,t\downarrow 0}\frac{X_{s,t}}{h(st)}\geq\overline{\lim}_{s\downarrow 0,\,t\downarrow 0}\frac{\Delta X_{s,t}}{h(s,t)}\geq 1 \text{ a.s.}$$

But, the convexity of h allows us to substitute h by ch (any $c>0$) and so $\overline{\lim}_{s\downarrow 0;\,t\downarrow 0}\dfrac{X_{s,t}}{h(st)}=+\infty$ a.s.

A question for which I don't know the answer would be the validadity or not of a certain "converse" of this.

Theorem 2.2. (Local growth at points z_0 away from the axes).
Let $X=(X_z: z\in\mathbb{R}_+^2)$ and f be as in Theorem 2.1. Take $z_0=(s_0,t_0)\in\mathbb{R}_+^2$ with $s_0>0$, $t_0>0$.
Then

$$\int_0^1 t^{-1}P[\,|X_{t,1}|>f(t)]\,dt<+\infty\ (=+\infty) \text{ implies that}$$

$$\lim_{\varepsilon\downarrow 0}\ \sup_{A(z_0,\varepsilon)}\frac{|X_{s,t}-X_{z_0}|}{f(st-s_0t_0)}<+\infty\ (>0,\ resp.) \text{ a.s.}$$

where $A(z_0,\varepsilon)=\{z: z_0\leq_1 z,\ 0<|z-z_0|<\varepsilon\}$. ($|\cdot|$ is euclidean norm.)

Proof. Remarks: (a) Recall that the above *lim sup* is indeed, with probability one, equal to some constant $c\in[0,+\infty]$. (Remark 2.1.).

(b) The normalization by $f(st-s_0t_0)$ is the natural one, since the law of $X_{s,t}-X_{s_0,t_0}$ depends on (s,t) only through $st-s_0t_0$, for $s\geq s_0$, $t\geq t_0$.

(c) It is enough to consider $z_0=(1,1)$. Trivial modification will give the general case.

Let's assume $\int_0^1 t^{-1}P[\,|X_{t,1}|>f(t)]\,dt<+\infty$. Then one knows:
(i) $P[\,|X_{t,1}|>2f(t)]\to 0$ as $t\downarrow 0$;
(ii) $\sum_n P[\,|X_{a^n,1}|>f(a^n)]<+\infty$ for a dense set of a's is $(0,1)$.

So, we take some $a\in[1/2,1)$ such that (ii) holds. Let $B_n=[1,1+a^n)\times$ $\times[1,1+a^n)$, $n\geq 1$ $(B_n\downarrow)$, and let $A_n=B_{n-1}\backslash B_n$, $n\geq 2$.

For $(s,t)\in A_n$ we have $st-1\geq a^n$. So, by Borel-Cantelli, it is enough to show that

$\sum_{n\geq 1}P[\sup_{(s,t)\in A_n}|X_{s,t}-X_{1,1}|\geq Af(a^n)]<+\infty$, for some constant $A\in(0,+\infty)$. But

$$P[\sup_{(s,t)\in A_n}|X_{s,t}-X_{1,1}|>15\,f(a^n)]$$

(2.8) $$\leq P[\sup_{A_n}|X((1,s]\times(1,t])|>7f(a^n)]$$

$$+\,2P[\sup_{a^n\leq s-1\leq a^{n-1}}|X_{s,1}-X_{1,1}|>4f(a^n)]$$

$$\leq P[\sup_{s,t\leq a^{n-1}}|X_{s,t}|>7\,f(a^n)]+2P[\sup_{a^n\leq s<a^{n-1}}|X_{s,1}|>4f(a^n)].$$

As in the proof of Theorem 2.1,

$$P[\sup_{s,t\le a^{n-1}} |X_{s,t}| > 7f(a^n)] \le c_n P[|X_{a^{n-1},a^{n-1}}| > f(a^n)]$$

$$\le c_n P[|X_{a^{2n-2},1}| > f(a^{2n-2})]$$

for $n \ge 2$, where

$$c_n = \left[\left(1 - \sup_{t\le a^{n-1}} P[|X_{ta^{n-1},1}| > 2f(a^n)]\right)\left(1 - \sup_{t\le a^{n-1}} \frac{P[|X_{ta^{n-1},1}| > 2f(a^n)]}{1-\sup_{s\le a^{n-1}} P[|X_{s,t}| > 2f(a^n)]}\right)\right]^{-1}$$

Since $P[|X_{t,1}| > 2f(t)] \to 0$ we can take some $\delta \in (0, 1/2)$ and $n_0 \ge 2$ large enough so that

$c_n \le (1-\delta)^{-1}\left(1 - \dfrac{\delta}{1-\delta}\right)^{-1}$ for $n \ge n_0$. Thus,

(2.9) $$\sum_{n\ge 1} P[\sup_{s,t\le a^{n-1}} |X_{s,t}| > 7f(a^n)] < +\infty.$$

On the other side:

$$P[\sup_{a^n\le s<a^{n-1}} |X_{s,1}| > 4f(a^n)]$$

$$\le P[|X_{a^n,1}| > f(a^n)] + P[\sup_{a^n\le s<a^{n-1}} |X_{s,1} - X_{a^n,1}| > 3f(a^n)]$$

$$\le P[|X_{a^n,1}| > f(a^n)] + P[\sup_{s\le a^n} |X_{s,1}| > 3f(a^n)]$$

$$\le [1 + (1-\delta)^{-1}]\, P[|X_{a^n,1}| > f(a^n)] \text{ if } n \ge n_0$$

(since $a \in [1/2, 1)$, and so $a^{n-1} - a^n \le a^n$). Therefore,

(2.10) $$\sum_{n\ge 1} P[\sup_{a^n\le s<a^{n-1}} |X_{s,1}|| > 4f(a^n)] < +\infty.$$

From (2.8), (2.9) and (2.10) it follows that

$$P\left[\sup_{(s,t)\in A_n} \frac{|X_{s,t} - X_{1,1}|}{f(st-1)} \ge 15 \text{ i.o.}\right] = 0, \text{ which entails } c \le 15.$$

The divergent part does not require proof. It is trivial from 1-parameter result.

Remark 2.3. For points away from the axes, the situation described by Remark 2.2 ((d), (e)) does not happen.

If X and h are as in Remark 2.2 (e) and $s_0 > 0$, $t_0 > 0$, from Theorem 2.2 and Fristed's results for subordinators [2], it follows that

$$\overline{\lim}_{s\downarrow s_0,\, t\downarrow t_0} \frac{X_{s,t} - X_{s_0,t_0}}{h(st - s_0t_0)} = 0 \quad \text{or} \quad +\infty \quad \text{a.s.}$$

according as $\int_0^1 t^{-1}P[X_{t,1} > h(t)]\,dt$ is finite or not, or equivalently, according as $\int_0^1 \nu(h(t),\infty)dt < +\infty$ or not.

3. An "integral test" for upper growt.

Obviously, the estimates obtained in the proof of Theorem 2.1 ((2.3) and (2.5)) are very crude. The purpose there, loosely speaking, was only to investigate the order of growth. For example, if $(X_z : z \in \mathbb{R}^2_+)$ is a standard Wiener process and $g(t) = (t \log|\log t|)^{1/2}$, Theorem 2.1 will just give us that

$\overline{\lim}_{s\downarrow 0,\, t\downarrow 0} |X_{s,t}|/g(st) = c$ for some $0 < c < +\infty$, but not that c is indeed equal to 2 (cf. [8]), a well known result.

One question that I wanted to answer is the following: in some other situations – e.g., if X is stable with only negative jumps, and index $\alpha \in [1, 2)$ – where one has $\overline{\lim}_{t\downarrow 0} X_t/f(t) = c$ for some f as before and some $0 < c < +\infty$, which would be the constant "c" for the 2-parameter process? (Is the order still the same?) To answer this, we need a theorem with a kind of "integral test" (as in [3], where the above fact for 1-parameter processes is proved), that will give: under certain conditions on X and f, the convergence or divergence of

$$\int_0^1 t^{-1}|\log t|\, P[X_{t,1} > f(t)]\,dt \text{ implies } \overline{\lim}_{s\downarrow 0,\, t\downarrow 0} X_{s,t}/f(st) \le \text{ or } \ge 1, \text{ respectively.}$$

These are the object of Theorems 3.1 and 3.2 below.

Theorem 3.1. (the "convergent" part) *Let $X = (X_z : z \in \mathbb{R}^2_+)$ be an $\mathbb{R}$-valued 2-parameter Lévy process, and $f : [0,1] \to [0, +\infty)$ a continuous and non-decreasing function, with $f(0) = 0$, $f > 0$ otherwise.*

(A) *Assume*: (a) $\lim_{u\downarrow 1} \sup_{0<t<1} \left|\dfrac{f(ut)}{f(t)} - 1\right| = 0$, *as in* [16];

(b) $\underline{\lim}_{t\downarrow 0} P[|X_{t,1}| > \varepsilon f(t)] < 1/2$, *for each* $\varepsilon > 0$.

Then, if $\int_0^1 t^{-1} |\log t| P[X_{t,1} > f(t)]\,dt < +\infty$ *we have*

$$\lim_{\varepsilon \downarrow 0} \sup_{(s,t)\in B_\varepsilon} \frac{X_{s,t}}{f(st)} \le 1 \text{ a.s., where } B_\varepsilon = \{(s,t)\in(0,1]^2 : st < \varepsilon\}.$$

(B) *Same conclusion follows if, instead, we assume*:

(b) *as above*;

(c) *for each* $\varepsilon > 0$ *there exists* $a = a(\varepsilon) \in (0,1)$ *such that*

$$\sum_{n\ge 1} n\, P[X_{a^n,1} > (1+\varepsilon) f(a^n)] < +\infty.$$

The idea for the proof is, as in Theorem 2.1, to use Skorohod's inequality for the $D[0,1]$-valued random vectors $(X_{s,t} : 0 \le t \le 1)$. But now we use a one-sided inequality.

Proof of (*B*). Let $\varepsilon > 0$ and $0 < a < 1$. Also, let $k, j \ge 1$ and $n = k + j$. Applying Skorohod's inequality for $Y_s = (X_{s,t} : 0 \le t \le 1)$, $s \ge 0$ and $\rho^j(x) \overset{Df}{=} \sup\{x(t) : 0 \le t \le a^j\}$ in $D[0,1]$, we find

$$P[\sup_{s\le a^k} \rho^j(Y_s) > (1+3\varepsilon) f(a^n)] \le \frac{1}{\alpha_n^\varepsilon} P[\rho^j(Y_{a^k}) > (1+2\varepsilon) f(a^n)]$$

provided $0 < \alpha_n^\varepsilon \overset{Df}{=} P[\inf_{u\le a^n} X_{u,1} > -\varepsilon f(a^n)]$.

But $1 - \alpha_n^\varepsilon \le P[\sup_{u\le a^n} |X_{u,1}| \ge \varepsilon f(a^n)] \le$

$$\le \frac{P[|X_{a^n,1}| \ge (\varepsilon/2) f(a^n)]}{1 - \sup_{s\le a^n} P[|X_{s,1}| \ge (\varepsilon/2) f(a^n)]}$$

By (b), take $n_0 = n_0(\varepsilon, a)$ such that

$$\text{(3.1)} \qquad c_{n_0} \overset{df}{=} \sup_{s\le a^{n_0}} P[|X_{s,1}| \ge (\varepsilon/2) f(s)] < 1/2,$$

giving $1 - \alpha_n^\varepsilon \le c_{n_0}(1 - c_{n_0})^{-1} < 1$ for $n \ge n_0$.
So, if $n \ge n_0(\varepsilon, a)$ and $k + j = n (k, j \ge 1)$,

$$P[\sup_{\substack{s\le a^k\\ t\le a^j}} X_{s,t} > (1+3\varepsilon) f(a^n)] \le \frac{1-c_{n_0}}{1-2c_{n_0}} P[\sup_{t\le a^j} X_{a^k,t} > (1+2\varepsilon) f(a^n)]$$

$$\le \frac{1-c_{n_0}}{1-2c_{n_0}} \frac{P[X_{a^n,1} > (1+\varepsilon) f(a^n)]}{\inf_{t\le a^j} P[X_{a^k t,1} > -\varepsilon f(a^n)]}$$

$$\le \frac{1}{1-2c_{n_0}} P[X_{a^n,1} > (1+\varepsilon) f(a^n)],$$

from (3.1) and because f is nondecreasing.

For the given $\varepsilon > 0$, we could have chosen $a = a(\varepsilon) \in (0,1)$ as in (*c*). If

$$A_n = \Big[\sup_{\substack{s\le a^k,\, t\le a^j\\ k+j=n}} X_{s,t} > (1+3\varepsilon) f(a^n)\Big], \ (k, j \ge 1)$$

Borel-Cantelli implies $P(A_n \text{ i.o.}) = 0$. This entails

$$\text{(3.2)} \qquad \lim_{\delta\downarrow 0} \sup_{B_\delta} \frac{X_{s,t}}{f(st)} \le 1 + 3\varepsilon \text{ a.s.}.$$

Since the argument applies to any $\varepsilon > 0$, the verification of (*B*) is now completed.

Proof of (*A*). Let $\varepsilon > 0$. We want to show that for $a \in (0,1)$ sufficiently close to 1 and large n,

$$P[\sup_{R(k,j)} X_z > (1+3\varepsilon) f(a^n)] \le B P[X_{u,1} > f(u)] \text{ for all } u \in [a^n, a^{n-1}),$$

where $R(k,j) = [a^{k+1}, a^k] \times [a^{j+1}, a^j]$, $n = k + j$, $k, j \ge 1$ and B is some positive constant.

Let $k, j \ge 1$, $n = k + j$, and let $a \in (0,1)$ be such that $1 - a < a^2$. Setting $I(r) = [a^{r-1}(1-a), a^r]$ $(r \ge 1)$ and arguing as in the proof of (*B*), but with $\rho^j(g) \overset{Df}{=} \sup\{g(s) : s \in I(j)\}$ for $g \in D[0,1]$, we obtain: if $n_0 = n_0(\varepsilon, a)$ and c_{n_0} are as in that proof, if $n \ge n_0(\varepsilon, a)$ and $a^n \le u < a^{n-1}$, since $I(k) \subseteq [a^{k-1}(1-a), ua^{-j}]$ we have

$$P[\sup_{s\in I(k)} \rho^j(Y_s) > (1+3\varepsilon) f(a^n)] \le$$

$$\le \frac{P[\rho^j(Y_{ua^{-j}}) > (1+2\varepsilon) f(a^n)]}{\inf_{a^{k-1}(1-a)\le s\le ua^{-j}} P[\inf_{t\in I(j)} (X_{ua^{-j},t} - X_{s,t}) > -\varepsilon f(a^n)]}$$

$$\le (1 - 2c_{n_0})^{-1} P[X_{u,1} > (1+\varepsilon) f(a^n)].$$

Here we used

$$P[\inf_{t\in I(j)} (X_{ua^{-j},t} - X_{s,t}) \le -\varepsilon f(a^n)] \le$$

$$\le \frac{P[|X_{u-sa^j,1}| \ge 2^{-1}\varepsilon f(a^n)]}{1 - \sup_{t\le a^j} P[|X_{(ua^{-j}-s)t,1}| \ge 2^{-1}\varepsilon f(a^n)]} \le \frac{c_{n_0}}{1-c_{n_0}} < 1$$

if $a^{k-1}(1-a) \le s \le ua^{-j}$ $(n \ge n_0(\varepsilon, a))$, and

$$P[\rho^j(Y_{ua^{-j}}) > (1+2\varepsilon) f(a^n)] \le \frac{1}{1-c_{n_0}} P[X_{u,1} > (1+\varepsilon) f(a^n)]$$

(since $ua^{-j}(a^j - t) \le a^n$ for $t \in I(j)$, and so

$$P[|X_{ua^{-j}(a^j-t),1}| \ge \varepsilon f(a^n)] \le c_{n_0} \quad (n \ge n_0(\varepsilon, a)).$$

Using (a), we take $a = a(\varepsilon)$ sufficiently close to 1, so that $(1+\varepsilon) f(a^n) \ge f(a^{n-1})$ for all $n \ge 2$, besides $1 - a < a^2$. For $n \ge n_0(\varepsilon, a(\varepsilon)) = n_0(\varepsilon)$ and $k + j = n$, $k, j \ge 1$, we let

$$A_{k,j} = [\sup_{R(k,j)} X_z > (1+3\varepsilon) f(a^n)];$$

$$A'_n = \bigcup_{k+j=n} A_{k,j};$$

by (3.3) and because $a^{k+1} > a^{k-1}(1-a)$ we find

$$P(A'_n) \le (1-2c_{n_0})^{-1}(n-1)\, P[X_{u,1} > f(u)],$$

for any $u \in [a^n, a^{n-1})$. Hence

$$P(A'_n) \le (1-2c_{n_0})^{-1}\, |\log a|^{-1}\, (1-a)^{-1} \int_{a^n}^{a^{n-1}} \frac{1}{u} |\log u|\, P[X_{u,1} > f(u)]\, du,$$

and we get $P(A'_n \text{ i.o.}) = 0$, by Borel-Cantelli and the hypothesis. Clearly, this implies (3.2) and the conclusion follows.

Theorem 3.2. (The "divergent part")

Let $X = (X_z : z \in \mathbb{R}^2_+)$ be an $\mathbb{R}$-valued Lévy process and let $f : [0,1] \to [0, +\infty)$ be a nondecreasing and continuous function with $f(0) = 0, f > 0$ otherwise.

Assume:

(a) $\lim_{u \downarrow 1} \sup_{0<t<1} \left| \frac{f(ut)}{f(t)} - 1 \right| = 0;$

(b) $\lim_{t \downarrow 0} P[X_{t,1} \ge 0] > 0.$

Then if $\Sigma\, nP[X_{a^n,1} > f(a^n)] = +\infty$ for all $a \in A$, where A is a set $\subseteq (0,1)$ with $0 \in \bar{A}$, we have:

$$\lim_{\varepsilon \downarrow 0} \sup_{0<s,t<\varepsilon} \frac{X_{s,t}}{f(st)} \ge 1 \quad \text{a.s.}$$

In particular, the conclusion follows if (a) and (b) hold and

$$\int_0^1 t^{-1} |\log t|\, P[X_{t,1} > f(t)]\, dt = +\infty$$

Proof. Last statement follows easily from the first. Let $a \in A$ fixed. For $k, j \ge 1$, and writing $n = k + j$, let $z(k,j) = \left(\frac{a^k}{1-a}, \frac{a^j}{1-a} \right)$;

$R_{k,j} = (z(k+1, j+1), z(k,j)]$ and $A_{k,j} = [X(R_{k,j}) > f(a^n)]$

Then

(i) the $A_{k,j}$, $k \ge 1$, $j \ge 1$ are independent;
(ii) $P(A_{k,j}) = P(X_{a^n,1} > f(a^n))$ if $n = k + j$.

Thus, we have

$$\sum_{k,j \ge 1} P(A_{k,j}) = +\infty.$$

If $B_{k,j} = [X_{z(k+1,j)} + X_{z(k,j+1)} - X_{z(k+1,j+1)} \ge 0]$ we have:

(i)′ $B_{k,j}$ is independent of $A_{k,j}$, for each $k, j \ge 1$
(ii)′ $P(B_{k,j}) = P(X_{\alpha_n,1} \ge 0)$ where $n = k + j$ and $\alpha_n = a^n[(1-a)^{-2} - 1]$, again by independence and stationarity of increments of X, and because $X \equiv 0$ on the axes.

By assumption (b) we can find $A > 0$ and $n_0 = n_0(a)$ such that if $k + j = n \ge n_0(a)$, $P(B_{k,j}) \ge A$.

Then, if $C_{k,j} = B_{k,j} \cap A_{k,j}$ we have:

$$C_{k,j} \subseteq [X_{z(k,j)} > f(a^{k+j})] \text{ and if } n \ge n_0(a),\ P(C_{k,j}) \ge A\ P(A_{k,j}).$$

So $\sum_{k,j} P(C_{k,j}) = +\infty$; if $(k,j) \ne (k',j')$, $P(C_{k,j} \cap C_{k',j'}) \le P(A_{k,j} \cap A_{k',j'}) = P(A_{k,j}) P(A_{k',j'}) \le \frac{1}{A^2} P(C_{k,j}) P(C_{k',j'})$. And so, by "refined Borel-Cantelli":

$P(X_{z(k,j)} > f(a^{k+j})$ for infinitely many $(k,j)) > 0$.

But above event has probability 0 or 1. So, its probability is 1, i.e.,

$$\text{(3.4)} \qquad \lim_{n \uparrow \infty} \sup_{k+j \ge n} \frac{1}{f(a^{k+j})} X_{z(k,j)} \ge 1 \quad \text{a.s.}$$

Now, given $\varepsilon > 0$ can take $a \in A$ sufficiently small s.t.

$$f(a^n) \ge (1-\varepsilon)\, f(a^n/(1-a)^2) \text{ (by (a)).}$$

From (3.4) we get $\lim_{n \uparrow +\infty} \sup_{k+j \ge n} [f(a^{k+j}(1-a)^{-2})]^{-1} X_{z(k,j)} \ge 1 - \varepsilon$ a.s. and so

$$\lim_{\delta \downarrow 0} \sup_{\substack{s,t \in (0,1] \\ st < \delta}} \frac{X_{s,t}}{f(st)} \ge 1 \text{ a.s. (since } \varepsilon > 0 \text{ is arbitrary).}$$

Moreover, $P(A_{k,j})$ depends on (k,j) only through $k+j$. So we also have

$$\sum_{\substack{k \geq n \\ j \geq n}} P(A_{k,j}) = +\infty \text{ for each } n.$$

Same argument will then give us

$$\lim_{\delta \downarrow 0} \sup_{0<s,t<\delta} \frac{X_{s,t}}{f(st)} \geq 1 \text{ a.s. and we conclude the proof.}$$

Example. Let $(X_z : z \in \mathbb{R}_+^2)$ be a stable Lévy process with only negative jumps and index $\alpha \in [1,2)$. The exponent of X is of the form (taking X strictly stable if $\alpha \in (1,2)$):

$$\psi(\lambda) = -c\,|\lambda|^{\alpha} \left(1 - i \frac{\lambda}{|\lambda|} \beta(\lambda, \alpha)\right), \ \lambda \neq 0$$

$$= 0, \ \lambda = 0$$

for some $c > 0$, where

$$\beta(\lambda, \alpha) = tg\, \frac{\pi}{2} a \text{ if } 1 < \alpha < 2$$

$$= \frac{2}{\pi} \log|\lambda| \text{ if } \alpha = 1$$

Gihman and Skorohod have shown that for the 1-parameter Lévy process $(X_t : t \geq 0)$ $(X_0 = 0)$ with this exponent,

$$\overline{\lim_{t \downarrow 0}} \frac{X_t}{f(t)} = 1 \text{ a.s., where}$$

$$f(t) = \alpha(\alpha-1)^{\frac{1-\alpha}{\alpha}} (c_1 t)^{\frac{1}{\alpha}} (\log|\log t|)^{\frac{\alpha-1}{\alpha}} \quad \text{if } 1 < \alpha < 2,$$

and $c_1 = c / |\cos(\pi/2)\alpha|$;

$$f(t) = \frac{2ct}{\pi} |\log t| \text{ for } \alpha = 1$$

One obvious question is then: does f as above still give right order for upper growth of the corresponding 2-parameter Lévy process? Equivalently, let $A \in [0, +\infty]$ be such that $\overline{\lim\limits_{s\downarrow 0, t\downarrow 0}} \dfrac{X_{s,t}}{f(st)} = A$ a.s.; the question is: does $A \in (0, +\infty)$?

Claim. $A = 2^{\frac{\alpha-1}{\alpha}}$

In particular, for the Cauchy process without positive jumps, there is no change at all.

Proof of "Claim".

Having at hand the (crucial) estimates of Gihman and Skorohod [3], it will be an easy application of Theorems 3.1 and 3.2.

For the case $1 < \alpha < 2$, one has the following [3]: there exists $c_2 > 0$ such that for $\omega(t,a)$ sufficiently large $(a > 0)$

$$c_2[\omega(t,a)]^{-1/2} \exp[-\omega(t,a)] \leq P[X_{t,1} > a] \leq \exp[-\omega(t,a)],$$

where $\omega(t,a) = \dfrac{\alpha-1}{\alpha} c_1 t a \left(\dfrac{a}{\alpha c_1 t}\right)^{1/\alpha - 1}$

Moreover, $f(t)$ was chosen to verify $\omega(t \cdot f(t)) = \log|\log t|$. Since $\omega(t, ba) = b^{\alpha/\alpha-1} \cdot \omega(t,a)$ $(b > 0)$, we have

$$\omega(t, 2^{\alpha-1/\alpha}(1-\varepsilon) f(t)) = 2(1-\varepsilon)^{\alpha/\alpha-1} \omega(t, f(t))$$

$$= 2(1-\varepsilon)^{\alpha/\alpha-1} \log|\log t| \underset{(t\downarrow 0)}{\rightarrow} \infty$$

So, for t sufficiently small and some $c_2 > 0$,

$$P[X_{t,1} > 2^{\alpha-1/\alpha}(1-\varepsilon) f(t)] \geq$$

$$\geq \exp[-2(1-\varepsilon)^{\alpha/\alpha-1} \log|\log t|] \cdot [2(1-\varepsilon)^{\alpha/\alpha-1} \log|\log t|]^{-1/2} c_2 \geq$$

$$\geq B\,|\log t|^{-2(1-\varepsilon)^{\alpha/\alpha-1}} \quad \text{for some } B > 0$$

So $\int_0^1 t^{-1} |\log t| P[X_{t,1} > 2^{\alpha-1/\alpha}(1-\varepsilon) f(t)] dt = +\infty$ and since $P(X_{t,1} \geq 0) = P(X_{1,1} \geq 0) > 0$, and f clearly verifies conditions (a) of Theorem 3.2, it follows (by Theorem 3.2) (since $\varepsilon > 0$ is arbitrary):

$$\overline{\lim_{s\downarrow 0, t\downarrow 0}} \frac{X_{s,t}}{f(st)} \geq 2^{\alpha-1/\alpha} \text{ a.s.}.$$

On the other side, for t small $(t > 0)$, $\varepsilon > 0$

$P[X_{t,1} > 2^{\alpha-1/\alpha}(1+\varepsilon) f(t)] \leq |\log t|^{-2(1+\varepsilon)^{\alpha/\alpha-1}}$. So, if $0 < a < 1$:
$P[X_{a^n,1} > 2^{\alpha-1/\alpha}(1+\varepsilon) f(a^n)] \leq B(a) n^{-2(1+\varepsilon)^{\alpha/\alpha-1}}$ if n is large enough ($B(a)$ some positive number).

So $\sum nP[X_{a^n,1} > 2^{\alpha-1/\alpha}(1+\varepsilon) f(a^n)] < +\infty$ for all $a \in (0,1)$. By Theorem 3.1 (B) it follows (since $\varepsilon > 0$ is arbitrary):

$$\lim_{\delta\downarrow 0} \sup_{(s,t)\in B_\delta} \frac{X_{s,t}}{f(st)} \leq 2^{\alpha-1/\alpha} \text{ a.s.}$$

Could also use part (A), since

$$P[|X_{t,1}| > \varepsilon f(t)] = P[|X_{1,1}| > \varepsilon t^{-1/\alpha} f(t)] \to 0 \text{ as } t \downarrow 0.$$

(B_δ is as in the theorem).

Now, let $\alpha = 1$. It is enough to show that for all $\varepsilon > 0$

$$\lim_{\delta \downarrow 0} \sup_{(s,t) \in B_\delta} \frac{X_{s,t}}{f(st)} \le 1 + \varepsilon \text{ a.s.,}$$

since converse is trivial from the 1-parameter case.

Condition (b) of Theorem 3.1 is easily verified here. Also f clearly verifies (a) of the same theorem.

Gihman and Skorohod have shown: if $\omega(t, a)$ is sufficiently large $(a > 0)$

$$P[X_{t,1} > a] \le exp[-\omega(t, a)], \text{ where } \omega(t, a) = \frac{2ct}{\pi} \, exp\left[\frac{\pi a}{2ct} - 1\right].$$

For $f(t) = \frac{2ct}{\pi} \, |log\, t|$, $0 < t < 1$, notice that

$$\omega(t, (1+\varepsilon) f(t)) = \frac{2ct}{\pi} \, exp[(1+\varepsilon)|\, log\, t\,| - 1] =$$

$$= A(c) t^{-\varepsilon} \underset{t \downarrow 0}{\to} +\infty (\text{for some } A(c) > 0).$$

So, for t sufficiently small $P[X_{t,1} > (1+\varepsilon) f(t)] \le exp[-A(c) t^{-\varepsilon}]$; hence

$$\textstyle\int_0^1 t^{-1} |\, log\, t\,| \, P[X_{t,1} > (1+\varepsilon) f(t)] dt < +\infty.$$

The conclusion is

$$\lim_{\delta \downarrow 0} \sup_{(s,t) \in B_\delta} \frac{X_{s,t}}{f(st)} = 2^{\alpha - 1/\alpha} = \overline{\lim_{s \downarrow 0, t \downarrow 0}} \frac{X_{s,t}}{f(st)} \text{ a.s.}$$

for X and f as above.

Remark.

This factor $2^{\alpha - 1/\alpha}$ agrees with what happens in the case of Brownian Motion $(\alpha = 2)$ where the factor is $\sqrt{2}$, as it is well known, and $f(t) = \sqrt{2t \, log\, |\, log\, t\,|}$. This well known result can obviously, be deduced from the theorems above.

Note. It was very recently – sometime later than the completion of this work – that I had a chance to see the work of N.M. Zinchenko [9]. In his article similar results are presented, concerning upper growth in a neighborhood of (0,0). Both are completely independent works. The results of Zinchenko correspond to Theorems 3.1 and 3.2 of our paper, but the conditions are different. Moreover, Theorems 2.1 and 2.2 of this present article are of a more general nature; they constitute the analogue of the classical Khinchin's theorem on local growth of Lévy processes. These two do not have a counterpart in Zinchenko's paper.

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