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A note on isomorphic group rings

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Let p be a rational prime. It has been conjectured that finite p-groups are determined by their group rings over F_p, the field with p elements. It is easy to see that this would imply that p-groups are also determined by their integral group rings.

Also, the following proposition is easy to prove using [7, Theorem III. 4,11]:

Proposition. *Let $G_1 = H_1 \times A_1$ be a finite groups with gcd $(|H_1|, |A_1|) = 1$ and let R be an integral domain of characteristic 0 such that $U(R) \cap \{o(g) | g \in G_1\} = 1$. Then $RG_1 \cong RG_2$ if and only if G_2 can be written in the form $G_2 = H_2 \times A_2$ and $RA_1 \cong RA_2$, $RH_1 \cong RH_2$.*

Thus, if the conjecture turns out to be true it would follow that finite nilpotent groups are determined by their integral group rings. In partial support of this conjecture, we shall give certain isomorphism invariants of p-groups.

Definition. Let G be a finite p-group. Then ℓ_i will denote the number of class sums K such that there exists a class sum L satisfying $L^{p^i} = K$ in F_pG.

Equivalently, ℓ_i is the number of conjugacy classes K such that:

(i) if $x \in K$, there exists $y \in G$ such that $y^{p^i} = x$ and

(ii) no conjugate z of y satisfies $z^{p^i} = x$.

We will prove that if G, H are finite p-groups such that $F_pG \cong F_pH$, then the numbers ℓ_i are the same for both G and H.

We recall the following result

Lemma [6]. *Let K be a conjugacy class of a finite p-group G and set $b \in G$ fixed. Then, for $t > 0$, the number of solutions $(x_1, x_2, \ldots, x_{p^t})$ of $x_1 x_2 \ldots x_{p_t} = b$, $x_i \in K$, not all x_i equal, is a multiple of p.*

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Proposition. *Let G be a finite p-group. Denote by $Z(F_pG)$ and $Z(G)$ the centers of F_pG and G respectively, and let $\Gamma_i(G) = \{y \in Z(F_pG) \mid y^{p^i} = 1\}$ for $i > 0$. Then,*

$$|\Gamma_i(G)| = p^{\alpha - (Z(G):G_i) - \ell_i}$$

where α is the number of conjugacy classes in G and $G_i = \{g \in Z(G) \mid g^{p^i} = 1\}$.

Proof. Let $\{w_1 = 1, w_2, \dots, w_m\}$ be a transversal of G_i in $Z(G)$. Writing $G_i = \{g_1, g_2, \dots, g_n\}$, an element $y \in Z(F_pG)$ can be written in the form:

$$y = \sum_{s,j} \alpha_{sj} w_s g_j + \Sigma \beta_h K_h,$$

where $\alpha_{sj}, \beta_h \in F$ and K_h are conjugacy classes with more than one element. Thus:

$$y^{p^i} = (\sum_{s,j} \alpha_{sj} w_s g_j)^{p^i} + \beta_1^{p^i} K_1^{p^i} + \beta_r^{p^i} K_r^{p^i}.$$

We note that the Lemma implies that $K_h^{p^i}$ is either zero or a class sum itself and, consequently, $K_h^{p^i} \neq 0$ if and only if $K_h^{p^i}$ satisfies the rules defining the numbers ℓ_i.

Hence, $y^{p^i} = 1$ if and only if $(\sum_{s,j} \alpha_{sj} w_s g_j)^{p^i} = 1$ and, for a given class sum L, $\sum_h \beta_h^{p^i} = 0$, summed over all h such that $K_h^{p^i} = L$.

The first condition amounts to $\sum_j a_{sj}^{p^i} = 1$ if $s = 1$, and 0 otherwise. This represents a loss of $(Z(G) : G_i)$ degrees of freedom. The second condition amounts to an additional loss of ℓ_i degress of freedom. Hence

$$|\Gamma_i(G)| = p^{\alpha - (Z(G):G_i) - \ell_i}.$$

Theorem. *The ℓ_i are isomorphism invariants.*

Proof. If $F_pG \cong F_pH$, then $|\Gamma_i(G)|$, $Z(G) \cong Z(H)$ [7] and α is the same in both G and H (since class sums form a basis of the centers).

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